Communications in Combinatorics and Optimization
Vol. ??, No. ?? (????), pp.1-15
DOI: 10.22049/CCO.2022.27728.1346

CCO
Commun. Comb. Optim.

Research Article

Chromatic transversal Roman domination in graphs

P. Roushini Leely Pushpam

Department of Mathematics, D.B. Jain College, Chennai - 600 097, India
roushinip@yahoo.com

Received: 9 April 2022; Accepted: 25 October 2022
Published Online: 28 October 2022

Abstract: For a graph G with chromatic number k, a dominating set S of G is called a chromatic-transversal dominating set (ctd-set) if S intersects every color class of any k-coloring of G. The minimum cardinality of a ctd-set of G is called the *chromatic transversal domination number* of G and is denoted by $\gamma_{ct}(G)$. A *Roman dominating function* (RDF) in a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The concept of *chromatic transversal domination* is extended to Roman domination as follows: For a graph G with chromatic number k, a *Roman dominating function* f is called a *chromatic-transversal Roman dominating function* (CTRDF) if the set of all vertices v with $f(v) > 0$ intersects every color class of any k-coloring of G. The minimum weight of a chromatic-transversal Roman dominating function of a graph G is called the *chromatic-transversal Roman domination number* of G and is denoted by $\gamma_{ctR}(G)$. In this paper a study of this parameter is initiated.

Keywords: Domination, Coloring, Chromatic transversal Roman domination

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$ we mean a finite, connected, undirected and simple graph. The order of G is denoted by n. For graph theoretic terminology we in general follow [3].
One of the fastest growing areas within graph theory is the study of domination and related problems. A comprehensive treatment of fundamentals of domination is given in the book of Haynes et al. [12]. Surveys of several advanced topics in domination can be seen in the book edited by Haynes et al. [11]. Another area of research which has received much attention within graph theory is graph colorings which deals with the

fundamental problem of partitioning a set of objects into classes according to certain conditions. Benedict Michael et al. [20] combined these two concepts to obtain a new variant of domination called the *chromatic transversal domination*. One more variant which combines domination and graph colorings known as dominator coloring is also well studied in literature [1, 7, 10, 18, 19].
A set $S \subseteq V$ is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set in G is called the *domination number* of G and is dentoed by $\gamma(G)$. The *chromatic number* of a graph G is the minimum number of colors needed to color the vertices of G so that no two adjacent vertices share the same color and is denoted by $\chi(G)$. As defined by Benedict Michael et al. [20], for a graph G with chromatic number k, a dominating set S of G is called a chromatic-transversal dominating set (ctd-set) if S intersects every color class of any k-coloring of G. The minimum cardinality of a ctd-set of G is called the *chromatic transversal domination number* of G and is denoted by $\gamma_{ct}(G)$. E.J. Cockayne et al. [8] intoduced the concept of Roman domination. A *Roman dominating function* (RDF) in a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. An RDF of weight $\gamma_R(G)$ is called a γ_R-function of G or $\gamma_R(G)$-function. If V_0, V_1, V_2 are the sets of vertices assigned the values 0, 1 and 2 respectively under f, then there is a 1-1 correspondence between the function $f : V(G) \to \{0, 1, 2\}$ and the sets V_0, V_1, V_2 of $V(G)$. Thus f can be written as $f = (V_0, V_1, V_2)$. For a detailed study in Roman domination, one can refer to [2, 4–6, 8, 9, 13–17, 21–27]. The concept of *chromatic-transversal domination* is extended to Roman domination as follows: For a graph G with chromatic number k, a *Roman dominating function* f is called a *chromatic-transversal Roman dominating function* (CTRDF) if the set of all vertices v with $f(v) > 0$ intersects every color class of any k-coloring of G. The minimum weight of a chromatic-transversal Roman dominating function of a graph G is called the *chromatic-transversal Roman domination number* of G and is denoted by $\gamma_{ctR}(G)$. A CTRDF of weight $\gamma_{ctR}(G)$ is called a γ_{ctR}-function of G or a $\gamma_{ctR}(G)$-function. In this paper a study of this parameter is initiated.

2. Notation

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by n. A subgraph of G is a graph having all its vertices and edges in G. For any set $S \subseteq V$, the induced subgraph $G[S]$ is the maximal subgraph of G with respect to S. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *diameter* of a graph G is the maximum distance between the pair of vertices in G. The *degree* of a vertex v in a graph G is the number of edges that are incident to the vertex v and is denoted by $deg(v)$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. A vertex of degree zero is called an *isolated* vertex, while a vertex of degree one is called a *leaf* vertex or a *pendant* vertex of G. An edge incident to a leaf is called a *pendant edge*. A *strong support* is a vertex that is adjacent to at least two leaf vertices. A set S of vertices is called *independent* if no two vertices in S are adjacent. A simple graph in which every pair of distinct vertices are adjacent is called a *complete graph*. A *clique* of a simple graph G is a subset S of V such that $G[S]$ is complete. The *clique number* of a graph G, denoted by $\omega(G)$ is the number of vertices in a maximum clique of G. For $n \geq 4$, the *wheel* W_n is defined to be the graph obtained by connecting a single vertex to all the vertices of C_{n-1}, where C_{n-1} is a cycle on $n-1$ vertices and is called the *rim* of the wheel. For two positive integers r, s, the *complete bipartite* graph $K_{r,s}$ is the graph with partition $V(G) = X \cup Y$ such that $|X| = r$, $|Y| = s$, X and Y are independent and every two vertices belonging to different partite sets are adjacent to each other. A complete bipartite graph of the form $K_{1,n}$ is called a star graph. A connected graph without any cycle is called a tree and if G has exactly one cycle, then G is called a *unicyclic graph*. The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

3. Some Standard Graphs

In this section γ_{ctR} values for paths, cycles and complete bipartite graphs are determined. To begin with we state the following theorem proved in [8].

Theorem 1. *[8] For the classes of paths P_n and cycles C_n, $\gamma_r(P_n) = \gamma_r(C_n) = \lceil \frac{2n}{3} \rceil$.*

Theorem 2. *For paths P_n,*

$$\gamma_{ctR}(P_n) = \begin{cases} n & \text{if } n \leq 4 \\ \lceil \frac{2n}{3} \rceil & \text{if } n \geq 5. \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. It is clear that $\chi(P_n) = 2$ and $\gamma_{ctR}(P_n) \geq \gamma_R(P_n)$. When $n = 2$, choose a γ_R-function of P_2 which assigns 1 to both the vertices of P_2. Then clearly $\gamma_{ctR}(P_2) = 2$. When $n = 3$, there is a unique γ_R-function of P_3 which assigns 2 to the central vertex and 0 to the end vertices. Thus $\gamma_{ctR}(P_3) = 3$. When $n = 4$, any γ_R-function of P_4 will assign either 2 to v_2, 1 to v_4 and 0 elsewhere or 2 to v_3, 1 to v_1 and 0 elsewhere. In both the cases either $\{v_1, v_3\}$ or $\{v_2, v_4\}$ form a color class of any χ-coloring of P_4. Hence $\gamma_{ctR}(P_4) = 4$. For $n \geq 5$, let f be a

γ_R-function of P_n defined as

$$f(v_i) = \begin{cases} 2, & i = 3j-1, 1 \le j \le \lfloor \frac{n+1}{3} \rfloor \\ 1, & i = n \text{ and } n \equiv 1 \pmod 3 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\{v_2, v_5\}$ intersects both the color classes of any χ-coloring of P_n. Hence $\gamma_{ctR}(P_n) \le \lceil \frac{2n}{3} \rceil$. Thus $\gamma_{ctR}(P_n) = \lceil \frac{2n}{3} \rceil$. □

Corollary 1. *For paths P_n, $\gamma_{ctR}(P_n) = \gamma_R(P_n)$ if and only if $n \neq 3, 4$.*

A similar proof can be given for cycles C_n. Hence the following theorem is stated without proof.

Theorem 3. *For cycles C_n,*

$$\gamma_{ctR}(C_n) = \begin{cases} n & \text{if } n = 4 \text{ and } n \text{ is odd} \\ \lceil \frac{2n}{3} \rceil & \text{otherwise.} \end{cases}$$

Corollary 2. *For cycles C_n, $\gamma_{ctR}(C_n) = \gamma_R(C_n)$ if and only if $n \neq 3, 4, 5$.*

Theorem 4. *For wheels $G = W_n$,*

$$\gamma_{ctR}(W_n) = \begin{cases} n & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. When n is even, $\chi(G) = 4$. Hence for every $v \in V(G)$. $\{v\}$ is a color class of a χ-partition of G. Thus $\gamma_{ctR}(G) = n$. When n is odd, $\chi(G) = 3$. Let $f : V(G) \to \{0, 1, 2\}$ be a function defined by $f(w) = 2$, $f(x) = f(y) = 1$, $f(z) = 0$ for every $z \in V(G) \backslash \{x, y, w\}$, where w is the central vertex and x, y are two adjacent vertices on the rim of the wheel. Clearly $\{w, x, y\}$ intersects every color class of any χ-coloring of G. Hence $\gamma_{ctR}(G) \le 4$. Further since $\chi(G) = 3$, $|V_2 \cup V_1| \ge 3$. But $|V_1| = 3$ is not possible. Thus $|V_2| = 1$ and $|V_1| = 2$ which implies that $\gamma_{ctR}(G) \ge 4$. Hence $\gamma_{ctR}(G) = 4$. (Refer Figure 1). □

4. Bipartite Graphs

In the following theorem we prove that for any bipartite graph G, $\gamma_{ctR}(G)$ lies between $\gamma_R(G)$ and $\gamma_R(G) + 1$.

Theorem 5. *For bipartite graphs G,*

$$\gamma_R(G) \le \gamma_{ctR}(G) \le \gamma_R(G) + 1.$$

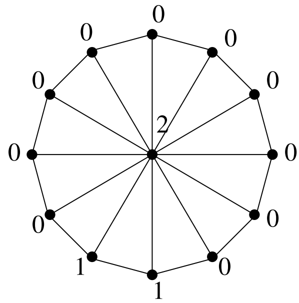

Figure 1. **The wheel W_{13} with $\gamma_{ctR}(W_{13}) = 4$**

Proof. Let (X, Y) be the bipartition of $V(G)$. Clearly $\chi(G) = 2$. If for every γ_R-function $f = (V_0, V_1, V_2)$ of G, the distance between any 2 vertices of $V_1 \cup V_2$ is even, then $V_1 \cup V_2$ is either X or Y. Thus, either X or Y is a color class of a χ-partition which does not intersect $V_1 \cup V_2$ in which case $\gamma_{ctR}(G) > \gamma_R(G)$. Now define $g : V(G) \to \{0, 1, 2\}$ by $g(x) = 1$ for some $x \in V_0$ and $g(x) = f(x)$ otherwise. Then g is a γ_{ctR}-function of G. Thus $\gamma_{ctR}(G) = \gamma_R(G) + 1$.
If for some γ_R-function of G say $f = (V_0, V_1, V_2)$, there is a pair of vertices $x, y \in V_1 \cup V_2$ such that $d(x, y)$ is odd, then $V_1 \cup V_2$ intersects both the color classes X and Y. Hence $\gamma_{ctR}(G) = \gamma_R(G)$. Thus $\gamma_R(G) \leq \gamma_{ctR}(G) \leq \gamma_R(G) + 1$. □

Corollary 3. *For a bipartite graph G, $\gamma_{ctR}(G) = \gamma_R(G)$ if and only if there exists a γ_R-function $f = (V_0, V_1, V_2)$ of G such that there are at least 2 vertices u, v in $V_1 \cup V_2$ with $d(u, v)$ as an odd number.*

Theorem 6. *For complete bipartite graphs $G = K_{r,s}$, $r \leq s$, $s \geq 2$*

$$\gamma_{ctR}(G) = \begin{cases} 3 & \text{if } r = 1 \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let (X, Y) be the bipartition of G with $|X| = r$, $|Y| = s$. If $r = 1$, then $G = K_{1,s}$ and clearly $\gamma_{ctR}(G) = 3$. If $r = 2$, $\gamma_R(G) = 3$ and $V_2 \cup V_1 = X$, where $f = (V_0, V_1, V_2)$ is a γ_R-function of G. Such an assignment is unique. But $V_2 \cup V_1$ does not intersect Y which forms a color class of any χ-coloring of G. Thus $\gamma_{ctR}(G) \geq 4$. Now by assigining 2 to a vertex in X and a vertex in Y, it is evident that $\gamma_{ctR}(G) \leq 4$. Thus $\gamma_{ctR}(G) = 4$. When $r \geq 3$, it is clear that $\gamma_{ctR}(G) = \gamma_R(G) = 4$. □

Corollary 4. *For complete bipartite graphs $G = K_{r,s}$, $r \leq s$, $s \geq 2$, $\gamma_{ctR}(G) = \gamma_R(G)$ if and only if $r \neq 1, 2$.*

5. Split Graphs

A graph G is said to be a *split graph* if $V(G)$ can be partitioned into two sets X and Y such that X induces a complete graph and Y is independent. In this section we determine $\gamma_{ctR}(G)$, where G is a split graph. For this purpose we consider $k \leq |X|$ vertices in X as follows: Let $G = G_1$ and $v_1 \in X$ such that $deg_{G_1}(v_1) = \Delta(G_1)$. Remove all the neighbors of v_1 in Y. Let G_2 be the resulting graph and $v_2 \in X$ such that $deg_{G_2}(v_2) = \Delta(G_2)$. Remove the neighbors of v_2 in Y. Repeat the process until all the vertices in Y are removed. Let $v_1, v_2, \ldots, v_k$ be the vertices in X whose neighbors in Y were removed successively. Then k is called the *split number* of G.
In all the results that follow in this section, a split graph G means a graph G with partition (X, Y) where X induces a complete graph and Y is independent.

Theorem 7. *For a split graph G, $\gamma_{ctR}(G) = |X| + k$, where k is the split number of G.*

Proof. Since every vertex in Y is not adjacent to at least one vertex in X; $\chi(G) = |X|$ and $\gamma_{ctR}(G) > |X|$. Let k be the split number of G and let $v_1, v_2, \ldots, v_k$ be the corresponding vertices in X as described above. Now any γ_{ctR}-function of G will assign a total weight of 2 to each $N[v_i]$, $1 \leq i \leq k$ and 1 to the vertices in $X - \{v_1, v_2, \ldots, v_k\}$. Hence $\gamma_{ctR}(G) \geq |X| - k + 2k \geq |X| + k$.
Now define $f : V(G) \to \{0, 1, 2\}$ by

$$f(v) = \begin{cases} 2 & \text{if } v = v_i, 1 \leq i \leq k \\ 1 & \text{if } v \in X \setminus \{v_1, v_2, \ldots, v_k\} \\ 0 & \text{if } v \in Y. \end{cases}$$

Then clearly f is a CTRDF of G as X intersects every color class of any χ-coloring of G. Hence $\gamma_{ctR}(G) \leq |X| + k$. Thus, $\gamma_{ctR}(G) = |X| + k$. □

Corollary 5. *For a split graph G, $\gamma_{ctR}(G) = \gamma_R(G)$ if and only if every vertex in X is a strong support.*

Corollary 6. *For a split graph G, $\gamma_{ctR}(G) = n$ if and only if every vertex in X is of degree at most $|X|$.*

6. Realization

Theorem 8. *Given two positive integers a, b with $2 \leq a \leq b$, there exists a graph G such that $\gamma_{ctR}(G) = b$ and $\gamma_R(G) = a$.*

Proof. If $a = b = 2$, then for the graph K_2, $\gamma_{ctR}(K_2) = \gamma_R(K_2) = 2$. Hence, we assume that $3 \leq a \leq b$. Consider the graph $H \circ 2K_1$ where H is a tree and

take a copy of K_{b-a+2}. If $a < b$, a is even, then join a vertex of K_{b-a+2} to a vertex of H in $H \circ 2K_1$, where $|V(H)| = \frac{a-2}{2}$. For the resulting graph G, clearly $\gamma_{ctR}(G) = b - a + 2 + 2\left(\frac{a-2}{2}\right) = b$ and $\gamma_R(G) = 2 + 2\left(\frac{a-2}{2}\right) = a$.
If $a \leq b$, a is odd, then join a vertex of K_{b-a+2} to a vertex of H in $H \circ 2K_1$, where $|V(H)| = \frac{a-3}{2}$ and in turn join a K_2 to one of the vertices of H. For the resulting graph G, $\gamma_{ctR}(G) = b - a + 2 + 2\left(\frac{a-3}{2}\right) + 1 = b$ and $\gamma_R(G) = 2 + 2\left(\frac{a-3}{2}\right) + 1 = a$.
If $a = b$ and a is even, then consider G to be the graph $H \circ 2K_1$ where $|V(H)| = \frac{a}{2}$. Then $\gamma_{ctR}(G) = 2 \times \frac{a}{2} = a$ and $\gamma_R(G) = a$. Hence, the theorem holds. □

7. Bounds

For K_2, $\gamma_{ctR}(K_2) = 2$ and $\gamma_{ctR}(K_1) = 1$. Thus one can easily observe that for $n \geq 3$, $3 \leq \gamma_{ctR}(G) \leq n$.

Theorem 9. *For any graph G, $\gamma_{ctR}(G) = 3$ if and only if G is either a K_3 or a star.*

Proof. Suppose $\gamma_{ctR}(G) = 3$. Then there exists a γ_{ctR}-function $f = (V_0, V_1, V_2)$ of G such that either $|V_1| = 3$, $|V_2| = 0$ or, $|V_1| = 1$ and $|V_2| = 1$. In the first case, clearly $G = K_3$. In the latter case, $\chi(G) \leq 2$. Since G is connected, G is bipartite. Thus the vertex in V_2 say w is adjacent to every vertex in $V(G)$. Hence G is a star. □

Next we prove that, for any tree T, $\gamma_{ctR}(T)$ is bounded above by $\frac{4n}{5}$ and characterize those trees which attain this bound. For this purpose we state the following theorems proved in [2].

Theorem 10. *[2] If T is an n-vertex tree with $n \geq 3$, then $\gamma_R(T) \leq \frac{4n}{5}$.*

Theorem 11. *[2] If T is an n-vertex tree, then $\gamma_R(T) = \frac{4n}{5}$ if and only if $V(T)$ can be partitioned into sets inducing P_5 such that the subgraph induced by the central vertices of these paths are connected.*

Theorem 12. *For any tree T with $n \geq 5$, $\gamma_{ctR}(T) \leq \frac{4n}{5}$ and equality holds if and only if either $T = T_1$ (as given in Figure 2) or $V(T)$ can be partitioned into sets inducing P_5 such that the subgraph induced by the central vertices of these paths are connected.*

Proof. Since T is a tree, $\gamma_R(T) \leq \gamma_{ctR}(T) \leq \gamma_R(T) + 1$. If $\gamma_R(T) < \frac{4n}{5}$, then $\gamma_{ctR}(T) < \frac{4n}{5} + 1$. Thus $\gamma_{ctR}(T) \leq \frac{4n}{5}$. If $\gamma_R(T) = \frac{4n}{5}$, then by Theorem 11, T is as described in the statement of the theorem. If $T = P_5$, then $\gamma_{ctR}(T) = 4$. Otherwise, define $f : \{0, 1, 2\} \to \gamma(T)$ by

$$f(v) = \begin{cases} 0, & \text{if } v \text{ is a support vertex} \\ 1, & \text{if } v \text{ is a leaf} \\ 2, & \text{otherwise.} \end{cases}$$

It is clear that f is $\gamma_{ctR}(T)$-function with weight $\frac{4n}{5}$. Thus, $\gamma_{ctR}(T) = \frac{4n}{5}$. Thus, in all the cases $\gamma_{ctR}(T) \leq \frac{4n}{5}$.
Now suppose that $\gamma_{ctR}(T) = \frac{4n}{5}$. If $\gamma_{ctR}(T) = \gamma_R(T) = \frac{4n}{5}$, then by Theorem 11, T is of the required type as mentioned in the statement. If $\gamma_{ctR}(T) = \gamma_R(T) + 1$, then $\gamma_R(T) = \frac{4n}{5} - 1$. Hence, $V(T)$ will be partitioned into sets $W_1, W_2, \ldots, W_{n/5}$ such that $|W_i| = 5$, $1 \leq i \leq n/5$ and any γ_R-function of T will assign a total weight of 4 to each of the sets W_i except one say W_1 and W_1 will be assigned a total weight of 3. Clearly each W_i, $2 \leq i \leq n/5$, will induce a P_5. Let v_1, v_2, v_3, v_4, v_5 be vertices in W_2 which form a P_5 in that order. Let f be a γ_R-function of T which assigns 2 to v_2, zero to v_1, v_3, 1 to v_4, v_5, a total weight of 4 to the vertices in W_i, $3 \leq i \leq \frac{n}{5}$ and a total weight of 3 to the vertices in W_1. Clearly, v_4 and v_5 belong to different color classes of any χ-coloring of T. Hence, $f(v_4) = f(v_5) = 1$ implies that $\gamma_{ctr}(T) = \gamma_R(T)$, which is not the case. Hence, $\bigcup_{i=2}^{n/5} W_i = \emptyset$ and $V(T) = W_1$ and $|W_1| = 5$. Hence T is either P_5 or $K_{1,4}$ or T_1 as given in Figure 2. Further $\gamma_{ctR}(T) = 4$. If $T = K_{1,4}$, then $\gamma_{ctR}(T) = 3$ which is not the case. Hence, T is either P_5 or T_1 as given in Figure 2 (Refer Figure 3).
Converse part is straightforward. □

Theorem 13. *For any graph G, $\gamma_{ctR}(G) \geq \omega(G)$ and equality holds if and only if $G = K_n$.*

Proof. Let $f = (V_0, V_1, V_2)$ be a γ_{ctR}-function of G and H be a maximum complete subgraph in G. Then, $|V(H)| = \omega(G)$. Further, $\chi(G) \geq \omega(G)$ and $|V_2 \cup V_1| \geq \chi(G)$ which implies that $|V_2 \cup V_1| \geq \omega(G)$. That is, $\gamma_{ctR}(G) \geq \omega(G)$.
Suppose that $\gamma_{ctR}(G) = \omega(G)$. Then $|V_2 \cup V_1| \geq \omega(G)$ implies that $|V_2 \cup V_1| \geq \gamma_{ctR}(G)$. That is $|V_2 \cup V_1| \geq 2|V_2| + |V_1|$. But $|V_2 \cup V_1| \leq 2|V_2| + |V_1|$. Hence $|V_2 \cup V_1| = 2|V_2| + |V_1|$. Thus $|V_2| = 0$ and $|V_1| = n = \gamma_{ctR}(G) = \omega(G)$. Hence, G is a complete graph.
Conversely if $G = K_n$, then clearly $\gamma_{ctR}(G) = \omega(G)$. □

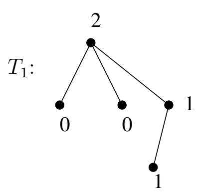

Figure 2. **The tree T_1 with $\gamma_{ctR}(T_1) = 4$**

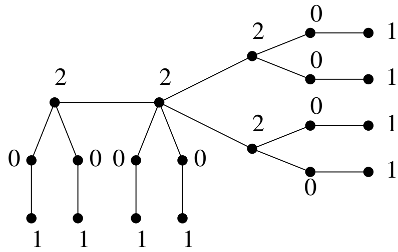

Figure 3. **A tree** T **with** $\gamma_{ctR}(T) = \frac{4n}{5}$

8. Graphs with $\gamma_{ctR}(G) = n$

In this section, graphs with $\gamma_{ctR}(G) = n$ are investigated.

Theorem 14. *If G is a bipartite graph with* $\gamma_{ctR}(G) = n$*, then* $diam(G) \leq 3$.

Proof. Since G is a bipartite graph, $\chi(G) = 2$. Suppose that $diam(G) \geq 4$. Let $Q = (v_1, v_2, v_3, \ldots, v_{diam(G)+1})$ be a diametral path in G. Define $f : V(G) \to \{0, 1, 2\}$ by $f(v_2) = 2$, $f(v_1) = f(v_3) = 0$, $f(v) = 1$ for every $v \in V(G) \backslash \{v_1, v_2, v_3\}$. Since v_4, v_5 are in different color classes, f is a CTRDF with $f(V) < n$, a contradiction. Thus $diam(G) \leq 3$. □

Theorem 15. *Let G be a bipartite graph. Then* $\gamma_{ctR}(G) = n$*, if and only if* $G = P_2, P_3, P_4$ *or* C_4.

Proof. Suppose that G is a tree. If $diam(G) = 3$ and $G \neq P_4$, then G is a bistar. Now by assigning 2 to the support vertices and zero to the leaf vertices, a CTRDF is obtained of weight lesser than n, a contradiction. Hence, $G = P_4$. If $diam(G) = 2$ and $G \neq P_3$, then G is a star. Clearly $\gamma_{ctR}(G) = 3 < n$, a contradiction. Hence, $G = P_3$. If $diam(G) = 1$, then $G = P_2$.
Suppose that G is not a tree. Then G has only even cycles. If G has a cycle $C_k = (v_1, v_2, \ldots, v_k)$, $k \geq 6$, then by assigning 2 to v_1, zero to v_2 and v_k and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Hence any cycle in G is C_4.
Next we claim that $G = C_4$. Suppose there exists a vertex $w \in V(G) \backslash V(C_4)$ which is adjacent to a vertex in C_4. Without loss of generality let w be adjacent to v_1, then by assigning 2 to v_1, zero to v_2, w and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Thus, $G = C_4$.
Converse is obvious. □

Theorem 16. *Let G be a unicyclic graph with cycle* C_k*. Then* $\gamma_{ctR}(G) = n$*, if and only if either* $G = C_4$ *or the following holds* *(i) k is odd.*

(ii) Every vertex not in C_k is at a distance at most 2 from C_k.

(iii) Every vertex not in C_k is of degree at most 2.

(iv) Every vertex in C_k is of degree at most 3.

Proof. If G is bipartite, then by Theorem 15, $G = C_4$. Suppose that G is not bipartite. Then G contains an odd cycle which proves (i). Further $\chi(G) = 3$ and all the three colors are used to color the vertices of the odd cycle in G by any χ-coloring of G. Suppose that there is a vertex not in C_k at a distance at least 3 from C_k. Then there exists at least 3 vertices say a_1, a_2, a_3 not in C_k and form a P_3 in that order. Now by assigning 2 to a_2, zero to a_1, a_3 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Hence, (ii) is proved.
To prove (iii), suppose that there is a vertex w not in C_k of degree more than 2. Let w_1, w_2 be 2 neighbors of w not in C_k. Then by assigning 2 to w, zero to w_1, w_2 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Hence, (iii) is proved. A similar contradiction can be arrived if there is a vertex in C_k of degree more than 3 which proves (iv).
Conversely suppose G is of the given type. If $G = C_4$, then $\gamma_{ctR}(G) = 4$. Suppose that G satisfies the given conditions. Since k is odd, $\chi(G) = 3$. Now no vertex in C_k can be assigned zero by any γ_{ctR} function of G. For, otherwise the vertex which is assigned zero can be colored with a unique color by some χ-coloring of G. The other 2 colors can be used to color the rest of the vertices. Further by conditions (ii), (iii) and (iv), one can infer that if some vertex not in C_k is assigned zero, then the corresponding neighbor which is assigned 2 is adjacent to exactly one vertex assigned zero. Thus, $\gamma_{ctR}(G) = n$. □

Theorem 17. *Let G be a non-bipartite graph with $\chi(G) = w(G)$. Then $\gamma_{ctR}(G) = n$ if and only if there exists a maximum clique H in G such that the following holds.*

(i) Each component of the subgraph induced by $V(G)\backslash V(H)$ is a K_2 or a K_1.

(ii) Every vertex in H has at most one neighbor not in H.

Proof. Let H be a maximum clique in G. As in the proof of Theorem 16, one can prove that every vertex not in H is at a distance at most 2 from H. Next we claim that if w is a vertex not in H at a distance 2 from H, then $deg(w) = 1$. Suppose to the contrary that $deg(w) > 1$. Then there exist two vertices $w_1, w_2 \in N(w)$ such that $w_1, w_2 \notin V(H)$. Now by assigning 2 to w, zero to w_1, w_2 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction.
Again as in the proof of Theorem 16, it can be proved that every vertex not in H is of degree at most 2. Thus each component of the subgraph induced by $V(G)\backslash V(H)$ is a K_2 or a K_1 which proves (i).

Suppose there is a vertex in H say w which has 2 neighbours w_1, w_2 not in H. Then by assigning 2 to w, zero to w_1, w_2 and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction.
Converse is straightforward. □

Remark 1. Characterization of split graphs G, with $\gamma_{ctR}(G) = n$ can also be derived using Theorem 17.

In the following theorems, graphs with $\chi(G) = w(G) + 1$ and $\gamma_{ctR}(G) = n$ are characterized. For this purpose we define two families $\mathcal{G}_1, \mathcal{G}_2$ of graphs as follows.
A graph $G \in \mathcal{G}_1$ if G satisfies the following conditions.

(i) G is non bipartite

(ii) No two odd cycles in G are disjoint.

(iii) If B is the set of all vertices in G which lie in every odd cycle, then each component of the subgraph induced by $V(G)\backslash B$ is a K_2 or a K_1.

(iv) Every vertex in B has at most two neighbors not in B.

(v) If a vertex in B has two neighbors x, y not in B, then every odd cycle in G contains either x or y (Refer Figure 4).

For the graph G given in Figure 4, one can infer that G contains 4 odd cycles and $B = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$. The vertex w_1 has two neighbors x, y not in B and every odd cycle in G contains either x or y. Further G satisfies all the conditions of $\mathcal{G}_1$. Hence $G \in \mathcal{G}_1$.
A graph $G \in \mathcal{G}_2$ if $V(G)$ can be partitioned into two sets such that one set induces a complete subgraph H_1 of order $\omega(G)-2$ and the other set induces a subgraph $H_2 \in \mathcal{G}_1$ such that the following holds.

(i) If there is an odd cycle say C in H_2 such that every vertex in C is adjacent to every vertex in H_1, then every vertex in H_1 is adjacent to at most one vertex not in C (with respect to H_2). (Refer Figure 5).

(ii) If no such odd cycle exists, then every vertex in B (as mentioned in the definition of $\mathcal{G}_1$) is adjacent to every vertex in H_1 and in turn every vertex in H_1 is adjacent to at most two vertices not in B. (with respect to H_2). If a vertex in H_1 is adjacent to two vertices not in B, then both the vertices have a common neighbor in B.

For the graph G given in Figure 5, clearly $H_2 \in \mathcal{G}_1$ and there is an odd cycle C in H_2 in which every vertex of C is adjacent to every vertex of H_1 and no vertex in H_1 has a neighbor in $V(H_2)\backslash V(C)$. Hence, $G \in \mathcal{G}_2$.

Theorem 18. *Let G be a graph with $\chi(G) = w(G)+1$ and $w(G) = 2$. Then $\gamma_{ctR}(G) = n$ if and only in $G \in \mathcal{G}_1$.*

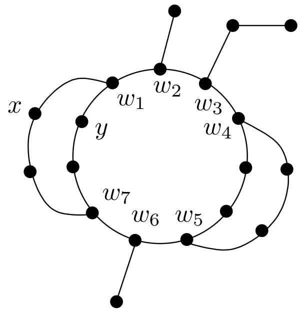

Figure 4. **A graph $G \in \mathcal{G}_1$ with $\gamma_{ctR}(G) = n$**

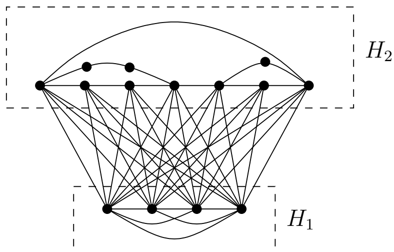

Figure 5. **A graph $G \in \mathcal{G}_2$ with $\gamma_{ctR}(G) = n$**

Proof. Let $\gamma_{ctR}(G) = n$. Since $\chi(G) = 3$, G is not bipartite. Suppose that G has two odd cycles which does not have a vertex in common. Let v_1, v_2, v_3 be three vertices in that order in one of the odd cycles. Then define $f : V(G) \to \{0, 1, 2\}$ by $f(v_2) = 2$, $f(v_1) = f(v_3) = 0$ and $f(v) = 1$ for every $v \in V(G)\backslash\{v_1, v_2, v_3\}$. Now it is clear that f is a CTRDF of G of weight lesser than n, a contradiction. Thus (ii) is proved.
To prove (iii), suppose to the contrary that some component of the subgraph induced by $V(G)\backslash B$ is neither a K_2 nor a K_1. Then there exists vertices v_1, v_2, v_3 which form a path in that order. As before we get a CTRDF of weight lesser than n, a contradiction. Thus (iii) is proved.
To prove (iv), suppose that there is a vertex w in B which has at least three neighbors not in B. Choose 2 vertices $x, y \notin B$ which are neighbors of w such that either x, y belong to the same odd cycle or x is in one odd cycle and y not in any odd cycle or both x, y does not belong to any odd cycle, or x, y belong to different odd cycles. In the first three cases by assigning 2 to w and zero to x, y and 1 elsewhere, will give a CTRDF of weight lesser than n, as in each case all the three colors will be used to the vertices assigned the value 1 by any χ-coloring of G. Hence, we get a contradiction. If x, y belong to different odd cycles, then choose a vertex $z \notin B$ which is adjacent to w and different from x and y. Now by assigning 2 to w, zero to y, z and 1 elsewhere, will give a CTRDF of weight lesser than n, as all three colors will be used to color the vertices in the odd cycle containing x by any χ-coloring of G. Thus, a contradiction

is obtained and (iv) is proved.
To prove (v), let w be a vertex in B which has two neighbors x, y not in B. We claim that every odd cycle in G contains either x or y. Suppose to the contrary that some odd cycle does not contain both x and y, then by assigning 2 to w, zero to x, y and 1 elsewhere, will give a CTRDF of weight lesser than n, a contradiction. Thus (iv) is proved and hence, $G \in \mathcal{G}_1$.
Conversely suppose G is a graph satisfying the given conditions. No vertex in B can be assigned zero by any γ_{ctr}-function of G, as the vertices in B lie in every odd cycle and $\{v\}$ is color class for every $v \in B$ in some χ-coloring of G. By conditions (iii), (iv) and (v), if any γ_{ctR}-function assigns zero to a vertex not in B, then the corresponding vertex which is assigned 2 is adjacent to exactly one vertex assigned zero. Thus $\gamma_{ctR}(G) = n$. □

Remark 2. For odd cycles C_n, $\gamma_{ctR}(C_n) = n$ can also be derived from Theorem 18.

Theorem 19. *Let G be a graph with $\chi(G) = w(G)+1$ and $w(G) \geq 3$. Then $\gamma_{ctR}(G) = n$ if and only in $G \in \mathcal{G}_2$.*

Proof. Let H_1 be a complete subgraph of order $\omega(G) - 2$. Let H_2 be the subgraph induced by $V(G)\backslash V(H)$. First we claim that $H_2 \in \mathcal{G}_1$. Let $w(G) = r$. Since $w(H_2) \geq 3$, H_2 is not bipartite, which proves (i) of the definition of $\mathcal{G}_1$. To prove (ii) of $\mathcal{G}_1$, suppose to the contrary that there are two odd cycles in H_2 which are disjoint. Since $\chi(G) = w(G) + 1$, the $(r + 1)^{th}$ color say c is used to color some vertex in H_2. In any χ-coloring of G, we have the following possibilities. The color c will be used in

(a) None of the two cycles

(b) Both the cycles

(c) Exactly one cycle.

Let v_1, v_2, v_3 be a path in that order in one of the cycles (in case (c), choose them to be in the cycle which does not use the color c). Now by assigning 2 to v_2, zero to v_1, v_3 and 1 elsewhere, a CTRDF is obtained of weight lesser than n. Thus $\gamma_{ctR}(G) < n$, a contradiction. Hence, (ii) of $\mathcal{G}_1$ is proved. Now to prove every component of the subgraph induced $V(H_2)\backslash B$ is a K_2 or K_1, suppose that there are vertices v_1, v_2, v_3 which form a path in that order exist in $V(H_2)\backslash B$. Then as discussed earlier, a CTRDF is obtained of weight lesser than n, as some vertex in B will be assigned the color c by every χ-coloring of G. Thus, (iii) of $\mathcal{G}_1$ is proved. As in the proof of Theorem 18, conditions (iv) and (v) can be proved. Thus $H_2 \in \mathcal{G}_1$.
Now to prove condition (i) of $\mathcal{G}_2$, suppose there is an odd cycle C in H_2 such that every vertex in C is adjacent to every vertex in H_1. Then we claim that every vertex in H_1 is adjacent to at most one vertex not in C (with respect to H_2). For otherwise, if there are 2 vertices x, y not in C adjacent to a vertex w in H_1. Then by assigning 2 to w, zero to x, y and 1 elsewhere, a CTRDF is obtained of weight lesser than n,

as all the three colors, other than the $r-2$ colors used in H_1 are used to color the vertices of C. Thus we get a contradiction. Hence, condition (i) of $\mathcal{G}_2$ is proved.
To prove condition (ii) of $\mathcal{G}_2$, suppose that no such odd cycle (as mentioned above) exists. We claim that every vertex in B is adjacent to every vertex of H_1. Suppose to the contrary that some vertex w in B is not adjacent to a vertex in H_1. Then clearly the $(r-2)$ colors used to color the vertices of H_1 and 2 colors used to color the vertices of H_2 are sufficient for the entire graph G which implies that $\chi(G) = w(G)$ which is not the case. Hence our claim holds. Next we claim that every vertex in H_1 is adjacent to at most 2 vertices not in B (with respect to H_2). This fact can be proved in a way similar to the proof of condition (iv) of Theorem 18. Finally we claim that if a vertex in H_1 is adjacent to two vertices not in B, then both the vertices have a common neighbor in B. Suppose to the contrary that a vertex w in H_1 is adjacent to two vertices x, y not in B and both x, y does not have a common neighbor in B, then by assigning 2 to w, zero to x, y and 1 elsewhere, a CTRDF is obtained of weight lesser than n, a contradiction. Summing the above arguments, condition (ii) of $\mathcal{G}_2$ holds and thus, $G \in \mathcal{G}_2$.
As in the proof of Theorem 18, the converse part is proved. □

Remark 3. For wheels W_n with even order, $\gamma_{ctR}(G) = n$, can also be derived from Theorem 19.

Acknowledgement
The author wishes to thank the referees for their valuable suggestions which were instrumental in transforming the paper to its present form.

References

[1] S. Askari, D.A. Mojdeh, and E. Nazari, *Total global dominator chromatic number of graphs*, TWMS J. App. and Eng. Math. **12** (2022), no. 2, 650–661.
[2] E.W. Chambers, B. Kinnersley, N. Prince, and D.B. West, *Extremal problems for Roman domination*, SIAM J. Discrete Math. **23** (2009), no. 3, 1575–1586.
[3] G. Chartrand and L. Lesniak, *Graphs and Digraphs*, Chapman and Hall, CRC, 2005.
[4] M. Chellali and N. Jafari Rad, *Trees with unique Roman dominating functions of minimum weight*, Discrete Math. Algorithms Appl. **6** (2014), no. 3, ID: 1450038.
[5] ______, *Roman domination stable graphs upon edge-addition*, Util. Math. **96** (2015), 165–178.
[6] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Varieties of Roman domination II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984.

[7] F. Choopani, A. Jafarzadeh, A. Erfanian, and D.A. Mojdeh, *On dominated coloring of graphs and some Nordhaus–Gaddum-type relations*, Turkish J. Math. **42** (2018), no. 5, 2148–2156.

[8] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.

[9] E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh, J. Munganga, and J.H. Van Vuuren, *Protection of a graph*, Util. Math. **67** (2005), 19–32.

[10] R. Gera, *On dominator colorings in graphs*, Graph Theory Notes of New York **52** (2007), 25–30.

[11] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs: Advanced Topics*, Marcel Dekker, Inc., New York, 1998.

[12] ———, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[13] M.A. Henning, *A characterization of Roman trees*, Discuss. Math. Graph Theory **22** (2002), no. 2, 325–334.

[14] ———, *Defending the Roman empire from multiple attacks*, Discrete Math. **271** (2003), no. 1-3, 101–115.

[15] N. Jafari Rad and H. Rahbani, *A Nordhaus–Gaddum bound for Roman domination*, Discrete Math. Algorithms Appl. **11** (2019), no. 5, ID: 1950055.

[16] N. Jafari Rad and L. Volkmann, *Roman bondage in graphs*, Discuss. Math. Graph Theory **31** (2011), no. 4, 763–773.

[17] ———, *Changing and unchanging the Roman domination number of a graph*, Util. Math. **89** (2012), 79–95.

[18] H. Merouane and M. Chellali, *On the dominator colorings in trees*, Discuss. Math. Graph Theory **32** (2012), no. 4, 677–683.

[19] H.B. Merouane, M. Haddad, M. Chellali, and H. Kheddouci, *Dominated colorings of graphs*, Graphs Combin. **31** (2015), no. 3, 713–727.

[20] L.B. Michaelraj, S.K. Ayyaswamy, and S. Arumugam, *Chromatic transversal domination in graphs*, J. Comb. Math. Comb. Comput. **75** (2010), 33–40.

[21] C.S. ReVelle, *Can you protect the Roman Empire?*, Johns Hopkins Magazine **49** (1997), no. 2, 40.

[22] P. Roushini Leely Pushpam and T.N.M. Malini Mai, *On efficient Roman dominatable graphs*, J. Combin. Math. Combin. Comput. **67** (2008), 49–58.

[23] ———, *Edge Roman domination in graphs*, J. Combin. Math. Combin. Comput. **69** (2009), 175–182.

[24] ———, *Roman domination in unicyclic graphs*, J. Discrete Math. Sci. Crypt. **15** (2012), no. 4-5, 237–257.

[25] P. Roushini Leely Pushpam and S. Padmapriea, *Restrained Roman domination in graphs*, Trans. Comb. **4** (2015), no. 1, 1–17.

[26] ———, *Global Roman domination in graphs*, Discrete Appl. Math. **200** (2016), 176–185.

[27] I. Stewart, *Defend the Roman empire!*, Scientific American **281** (1999), no. 6, 136–138.