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Short Note

A note on odd facial total-coloring of cacti

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Abstract: A facial total-coloring of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color. A facial total-coloring of G is odd if for every face f and every color c, either no element or an odd number of elements incident with f is colored by c. In this note we prove that every cactus forest with an outerplane embedding admits an odd facial total-coloring with at most 16 colors. Moreover, this bound is tight.

Keywords: Facial coloring, Odd facial coloring, Plane graph

AMS Subject classification: 05C10, 05C15

1. Introduction

In this paper we consider finite simple graphs. A graph is *planar*, if it can be drawn in the Euclidean plane $\mathbb{R}^2$ so that its edges intersect only at their endvertices. A *plane graph* is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect except at their endvertices. Let G be a plane graph with vertex set $V(G)$ and edge set $E(G)$. The regions of $\mathbb{R}^2 \setminus G$ form the set $F(G)$ of faces of G. The *boundary* of a face f is the boundary in the usual topological sense. It is the collection of all edges and vertices contained in the closure of f. A face is said to be *incident* with the vertices and edges in its boundary. An *element* incident with f is a vertex or an edge incident with f. Each plane graph has exactly one unbounded face, called the *outer face*, all other faces are *inner faces*. *Outerplane* graphs are plane graphs such that every vertex is incident with the outer face. An edge of a plane graph not incident with the outer face is called *inner edge*. A *cactus* is a connected outerplane graph with no inner edges (i.e., a connected outerplane graph in which any two cycles have at most one vertex in common). A *cactus forest* is a graph such that every component is a cactus. A vertex v of G is a *cut-vertex* if $G-v$ has more components than G. An edge e of G is a *cut-edge* if $G-e$ has more components than G. Two edges are *facially adjacent* in a plane graph G if they are adjacent and consecutive in the cyclic order around their common endvertex.

The concept of facial total-coloring of plane graphs was introduced by Fabrici, Jendroľ, and Vrbjarová [7]. A *facial total-coloring* of a plane graph G is a coloring of the vertices and edges such that no facially adjacent edges, no adjacent vertices, and no edge and its endvertices are assigned the same color. The authors of [6] showed that every connected plane graph admits a facial total-coloring with at most 6 colors. Moreover, they improved this bound to 5 for plane triangulations and connected outerplane graphs. The conjecture, that every connected plane graph has a facial total-coloring with at most five colors [7], is still open. Recently, Fabrici, Horňák, and Rindošová [5] investigated facial total-coloring with an additional requirement. A *facial unique-maximum total-coloring* of a plane graph G is a facial total-coloring with positive integers such that, for each face f, the maximum color on the boundary of f is used exactly once (i.e., the maximum color on the elements incident with f is unique). In [5], it is proven that every 2-edge-connected plane graph admits a facial unique-maximum total-coloring with at most 6 colors. Moreover, this bound is tight.

Another type of facial total-coloring was introduced in [4]. An *odd facial total-coloring* of a plane graph is a facial total-coloring such that, for every face f and every color c, either no element or an odd number of elements incident with f is colored by c. Czap and Šugerek [4] showed that every connected acyclic plane graph admits such a coloring with at most 5 colors, moreover, the bound is tight. Recently, Czap [1] proved that every connected unicyclic plane graph admits an odd facial total-coloring with at most 10 colors. This bound is also tight.

In this note we investigate odd facial total-coloring of plane graphs with many cycles. We prove that every cactus forest with an outerplane embedding has an odd facial total-coloring with at most 16 colors. Moreover, this bound is tight.

2. Results

Let $\chi''(G)$ denote the minimum number of colors required in a facial total-coloring of a plane graph G.

Lemma 1. *If C_n is the cycle on $n \geq 3$ vertices, then*

$$\chi''(C_n) = \begin{cases} 3 & \text{if} \quad n = 3k, \text{ where } k \text{ is a positive integer,} \\ 4 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n = v_1e_1v_2e_2 \dots v_ne_nv_1$, where $v_1, v_2, \dots, v_n$ are the vertices, e_1, $e_2, \dots, e_n$ are the edges, $e_i = v_iv_{i+1}$ for $i = 1, 2, \dots, n-1$ and $e_n = v_nv_1$.

In the following, we define a facial total-coloring of C_n with colors a, b, c, d. We color the vertices and edges in order $v_1, e_1, v_2, e_2, \dots, v_n, e_n$.

If $n = 3k$, then we use $2k$ times the pattern a, b, c.

If $n = 3k + 1$, then we use two times the pattern a, b, c, d and then $2k - 2$ times the pattern a, b, c.
If $n = 3k + 2$, then we use once the pattern a, b, c, d and then $2k$ times the pattern a, b, c.
Clearly, v_1, e_1, v_2 receive distinct colors in every facial total-coloring of C_n. Consequently, $\chi''(C_n) \geq 3$ for every $n \geq 3$. Moreover, if C_n has a facial total-coloring with three colors, then this coloring is unique. This implies that $\chi''(C_n) = 3$ if and only if $2n$ is a multiple of 3. □

Lemma 2. *If G is an outerplane graph with no inner edge, then $\chi''(G) \leq 4$. Moreover, this bound is tight.*

Proof. Let G be a counterexample with minimal number of vertices. The minimality of G implies that G is connected. First assume that G contains a vertex v of degree one. Let u be the neighbor of v and let e_1, e_2 be the edges facially adjacent to uv ($e_1 = e_2$ if u has degree two). The graph $G - v$ has fewer vertices than G, so it has a facial total-coloring with at most four colors. This coloring can be extended to a facial total-coloring of G in the following way. First we color the edge uv with a color which does not appear on e_1, e_2, u, then we color the vertex v with a color which does not appear on u, uv.
Now assume that G contains no vertex of degree one. By Lemma 1, G is not a cycle. Consequently, G contains a cycle $C = v_1e_1v_2e_2\ldots$ which is incident with exactly one cut-vertex, say u, in G. Without loss of generality assume that $u = v_2$. Let H be the graph obtained from G by removing the vertices end edges incident with C except for v_2. Let r be the edge of $G - E(C)$ facially adjacent to v_2v_3 in G and let s be the edge of $G - E(C)$ facially adjacent to v_2v_1 in G, see Figure 1 for illustration.

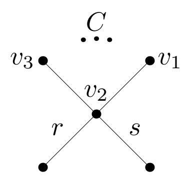

Figure 1. **A configuration in G.**

The graph H has fewer vertices than G so it has a facial total-coloring φ with at most four colors, say $1, 2, 3, 4$. In the following we extend this coloring to a facial total-coloring of G. We distinguish two cases according to the degree of v_2 in G.
First assume that v_2 has degree three in G. In this case $r = s$. W.l.o.g. we can assume that $\varphi(v_2) = 1$ and $\varphi(r) = 2$. We color the vertices and edges of C by the coloring defined in the proof of Lemma 1 with $c = 1, b = 3, a = 4$ if the length of C is a multiple of 3, and $c = 1, b = 3, d = 4, a = 2$ otherwise. Observe that this coloring does not change the color of v_2.

Now assume that the degree of v_2 is at least four. In this case the two edges r and s are facially adjacent in H. Therefore, we can assume that $\varphi(r) = 1, \varphi(v_2) = 2$, and $\varphi(s) = 3$. We color the vertices and edges of C by the coloring defined in the proof of Lemma 1 with $c = 2, b = 1, a = 3, d = 4$.
From Lemma 1 it follows that the bound is tight. □

Let $\chi''_o(G)$ denote the minimum number of colors used in an odd facial total-coloring of a plane graph G. The main result of this paper is Theorem 1.

Theorem 1. *If G is an outerplane graph with no inner edge (i.e., G is a cactus forest), then $\chi''_o(G) \leq 16$. Moreover, this bound is tight.*

Proof. Let φ be a facial total-coloring of G with colors A, B, C, D (such a coloring exists by Lemma 2). In the following we show that for every $i \in \{A, B, C, D\}$ the vertices and edges of color i can be recolored with at most four colors i_1, i_2, i_3, i_4 so that we obtain an odd facial total-coloring of G (which uses at most 16 colors).
First, for each component of G we find a facial total-coloring with (at most) eight colors such that for every inner face f and every color c, either no element or an odd number of elements incident with f is colored by c.
Let $i \in \{A, B, C, D\}$ be a fixed color. Let K be a component of G. We number the inner faces of K in the following way. Let f_1 be an arbitrary inner face of K. If there are inner faces of K which share a vertex with f_1, then we choose one of them and denote it as f_2. If there is no such face, then f_2 is an arbitrary inner face of K different from f_1. Now assume that $f_1, f_2, \ldots, f_m$ were already chosen. If there is a face which shares a vertex with at least one of the faces $f_1, f_2, \ldots, f_m$, then we denote it as f_{m+1}, otherwise we denote an arbitrary face different from $f_1, f_2, \ldots, f_m$ as f_{m+1}.
In the following, in each step we recolor all vertices and edges of color i incident with an inner face of K. In the m-th step we recolor the elements of color i incident with the inner face f_m. If the face f_1 is incident with an odd number of elements of color i, then we recolor all of them with color i_1. Otherwise, we recolor one element with color i_1 and the other elements with color i_2. In each next step, at most one element (with original color i) is precolored with i_1 or i_2. If no element is precolored, then we proceed as above. Now, without loss of generality, assume that one element is precolored with i_1. If there is an even number of elements of color i, then we recolor all of them with i_1, otherwise we recolor all of them with i_2. In such a way we obtain a facial total-coloring ψ of G which uses at most 8 colors.
Now, each inner face is well colored, i.e., if a color c appears on the boundary of an inner face f, then the total number of occurrences of c is odd on f.
Let c be a color which appears on an element of G under the coloring ψ. We say that c is good, if for every face f of G, either no element or an odd number of elements incident with f is colored by c, otherwise it is bad. Assume that there is a bad color, say j. This means that an even number of elements incident with the outer face of G is colored with j. If there are cut-edges in G of color j, then we choose one of

them and recolor it with a new color. In such a way j and the new color became good because we recolored one edge on the outer face, and the inner faces remain well colored, since the inner faces are incident with no cut-edges. Now assume that no cut-edge of G has color j. If there is an inner face, say f, such that all the elements of f of color j are edges, then we recolor all of these edges of f with a new color. Again, j and the new color became good, since we recolored an odd number of edges on f and also on the outer face of G. Now assume that every inner face which is incident with an element of color j is incident with a vertex of color j. We say that a vertex is a j-vertex if its color is j, and a j-face is an inner face which is incident with a j-vertex. Let H be an auxiliary graph whose vertices are j-vertices and j-faces of G. Two vertices u, v are adjacent in H if one of them, say u, is a j-vertex and v is a j-face which is incident with u. (Observe that H can be obtained so that we put one vertex v_f into each j-face f and then we join v_f with j-vertices incident with f.) Now we show that H has at least two components. Suppose that H is connected. This implies that the j-faces (more precisely, the vertices and edges incident with j-faces) induce a connected bridgeless subgraph $\bar{G}$ of G, moreover, if two j-faces share a vertex, then their common vertex is a j-vertex. The graph $\bar{G}$ is connected, therefore we can number its inner faces $f_1, f_2, f_3, \ldots$ so that for every $\ell \geq 2$ the face f_ℓ shares a vertex with a face f_k for some $k < \ell$. There is an odd number of elements of color j on f_1. (In fact, each of these inner faces is incident with an odd number of elements of color j.) Since f_1 and f_2 share a j-vertex, there is an even number of elements of color j which appear on f_2 and do not appear on f_1. By the same argument, there is an even number of elements of color j which appear on f_ℓ and do not appear on $f_1, f_2, \ldots, f_{\ell-1}$. Consequently, there is an odd number of elements of color j in G. This means that j is good, a contradiction. So H has at least two components. Let $\bar{H}$ be a subgraph of G which corresponds to a component of H. Using the above mentioned arguments for $\bar{G}$, we can show that $\bar{H}$ is incident with an odd number of elements of color j. Therefore, if we recolor the vertices and edges of color j incident with $\bar{H}$ with a new color, then j and the new color became good. This means that the facial total-coloring ψ of G (which uses at most 8 colors) can be modified to an odd facial total-coloring of G which uses at most 16 colors.
The bound 16 is tight since the graph consisting of two cycles on four vertices needs 16 colors in every odd facial total-coloring. □

We finish the paper with the following question.

Problem 1. Is there a connected outerplane graph G with no inner edges such that $\chi''_o(G) = 16$?

Note that there are connected outerplane graphs G with no inner edges such that $\chi''_o(G) = 15$. For instance, two tetragons with one vertex identified, or two tetragons with one extra edge (which join a vertex from the first tetragon with a vertex from the second one). Therefore, if the bound 16 from Theorem 1 can be improved for connected outerplane graphs, then only by one.

Finally, for results and challenging open problems on facially-constrained colorings of plane graphs, we refer the readers to [2, 3].

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