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Research Article

Leech graphs

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Abstract: Let $t_p(G)$ denote the number of paths in a graph G and let $f : E \to \mathbb{Z}^+$ be an edge labeling of G. The weight of a path P is the sum of the labels assigned to the edges of P. If the set of weights of the paths in G is $\{1, 2, 3, \ldots, t_p(G)\}$, then f is called a Leech labeling of G and a graph which admits a Leech labeling is called a Leech graph. In this paper, we prove that the complete bipartite graphs $K_{2,n}$ and $K_{3,n}$ are not Leech graphs and determine the maximum possible value that can be given to an edge in the Leech labeling of a cycle.

Keywords: Leech labeling, Leech tree, Leech graph

AMS Subject classification: 05C78

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].
Let $f : E \to \mathbb{Z}^+$ be an edge labeling of G. The weight of a path P in G is the sum of the labels of the edges of P and is denoted by $w(P)$. Leech [5] introduced the concept of a Leech tree, while considering a problem in electrical engineering, where edge labels represent electrical resistance. Let T be a tree of order n. An edge labeling

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$f : E \to \mathbb{Z}^+$ is called a Leech labeling if the weights of the $\binom{n}{2}$ paths in T are exactly $1, 2, \ldots, \binom{n}{2}$. A tree which admits a Leech labeling is called a Leech tree. Since each edge label is the weight of a path of length one, it follows that f is an injection and 1,2 are edge labels for all $n \geq 3$. Leech found five Leech trees which are given in Figure 1 and these are the only known Leech trees.

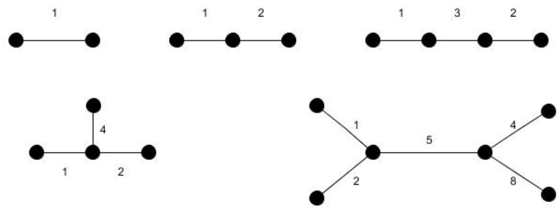

Figure 1. **Leech trees**

Taylor [8] proved that if T is a Leech tree of order n, then $n = k^2$ or $k^2 + 2$ for some integer k. Since then it has been proved by several authors ([1],[7],[9]) that no Leech trees of order 9, 11 or 16 exist, leaving $n = 18$ as the smallest open case. In [13] and [10], it is shown that bistars, tristars and a subclass of trees of diameter $n - 2$ are non-Leech trees. Some variations of Leech trees such as modular Leech trees ([3],[4]), minimal distinct distance trees [1] and leaf-Leech trees [6] have been investigated by several authors. A parameter called Leech index was introduced in [11], which measures how close a tree is towards being a Leech tree.

The total number of paths in a graph G is called the path number of G and is denoted by $t_p(G)$. Let $f : E \to \mathbb{Z}^+$ be an edge labeling of G. The weight of a path P in G is the sum of the labels of the edges of P and is denoted by $w(P)$. If the set of weights of the paths in G is $\{1, 2, 3, \ldots, t_p(G)\}$, then f is called a Leech labeling of G and a graph that admits a Leech labeling is called a Leech graph [12].

Let f be an edge labeling of a graph G such that both f and the weight function w on the set of all paths of G are both injective. Let S be the set of all path weights. Let k_f be the positive integer such that $\{1, 2, 3, \ldots, k_f\} \subseteq S$ and $k_f + 1 \notin S$. Let $k(G) = \max k_f$, where the maximum is taken over all such edge labelings f. Then $k(G)$ is called the Leech index of the graph G.

In [12], it has been proved that cycles of order at most 6 are Leech graphs, whereas complete graphs of order 4, 5 and 6 are non-Leech graphs. The case $n \geq 7$ is left as an open problem for both cycles and complete graphs. It is a simple observation that $K_4 - \{e\}$ and P_5 are non-Leech graphs of smallest order and smallest size. Since C_6 is a Leech graph and P_5 is not a Leech graph, it follows that the property of being a Leech graph is not hereditary and hence does not admit a forbidden subgraph characterization.

In this paper, we prove that $K_{2,n}$ and $K_{3,n}$, for $n > 2$ are non-Leech graphs and determine the Leech index of these graphs. We also prove some properties of Leech cycles.

2. Complete Bipartite Graphs

Throughout this section, let $X = \{u_1, u_2, \ldots, u_m\}$ and $Y = \{v_1, v_2, \ldots, v_n\}$ be the bipartition of $K_{m,n}$. If P_1 and P_2 are two paths having a common end vertex v and $V(P_1) \cap V(P_2) = \{v\}$, then the path obtained by concatenation of P_1 and P_2 is denoted by $P_1 \circ P_2$.

Lemma 1. *Let f be a Leech labeling of $K_{m,n}, m, n > 2$. Let P_1 be a $x - y$ path, P_2 be a $r - s$ path, $V(P_1) \cap V(P_2) = \phi, w(P_1) = k, w(P_2) = l$ and $xr \in E(K_{m,n})$. Then $w(P) \neq k + l$ for any path P with x as origin and $r \notin V(P)$.*

Proof. Suppose there exists a path P with x as origin, $r \notin V(P)$ and $w(P) = k + l$. Then the two paths $P \circ (x, r)$ and $P_1 \circ (x, r) \circ P_2$ have the weight $k + l + f(xr)$, which is a contradiction. □

Corollary 1. *Let f be a Leech labeling of $K_{m,n}, m, n > 2$. Let e_1 and e_2 be two nonadjacent edges, $f(e_1) = k$ and $f(e_2) = l$. Then $f(e) \neq k + l$ for any edge e adjacent to e_1 or e_2.*

Corollary 2. *Let f be a Leech labeling of $K_{3,n}$ and let $n > 3$. Let P_1 and P_2 be two vertex disjoint paths such that the end vertices of P_1 and P_2 cover all the vertices of $X = \{u_1, u_2, u_3\}$. Let $w(P_1) + w(P_2) = c$. Then $f(e) \neq c$ for any edge e.*

Proof. Suppose there exists an edge e with $f(e) = c$. Since the end vertices of P_1 and P_2 cover X, we may assume without loss of generality that u_1 is an end vertex of P_1 and e. We claim that no vertex of the other partite set $Y = \{v_1, v_2, \ldots, v_n\}$ is an end vertex of P_2. Suppose v_1 is an end vertex of P_2. If $e = u_1v_i$ where $i \neq 1$, then $w(P_1 \circ (u_1, v_1) \circ P_2) = w((v_i, u_1, v_1)) = c + f(u_1v_1)$, which is a contradiction. Hence $e = u_1v_1$ and the other end vertex of P_2 is u_2 or u_3. Let u_2 be the other end vertex of P_2. Since $n > 3$, there exists v_i in Y such that $v_i \notin (V(P_1) \cup V(P_2))$. Now, $w(P_1 \circ (u_1, v_i, u_2) \circ P_2) = w((v_1, u_1, v_i, u_2)) = c + f(u_1v_i) + f(v_iu_2)$, which is a contradiction. Thus no vertex of Y is an end vertex of P_2 and hence P_2 is a u_2-u_3 path. Clearly, P_1 has length 1 and P_2 has length 2. Let $P_1 = (u_1, v_1), P_2 = (u_2, v_2, u_3)$ and $e = u_1v_i$ where $i \neq 1$. Since $n > 3$, there exists $v_j \in Y$ such that $j \notin \{1, 2, i\}$. Then $w(P_1 \circ (u_1, v_j, u_2) \circ P_2) = w((v_i, u_1, v_j, u_2)) = c + f(u_1v_j) + f(v_ju_2)$, which is a contradiction. Hence $f(e) \neq c$ for any edge e. □

Corollary 3. *Let f be a Leech labeling of $K_{3,n}, n > 3$. Let P_1 and P_2 be two vertex disjoint paths and let P_3 and P_4 be another pair of vertex disjoint paths such that $w(P_1) + w(P_2) = w(P_3) + w(P_4) = c$ and the end vertices of P_1, P_2, P_3, P_4 covers all the vertices of $X = \{u_1, u_2, u_3\}$. Then $f(e) \neq c$ for any edge.*

Proof. Suppose there exists an edge e with $f(e) = c$. Since the end vertices of P_1, P_2, P_3 and P_4 cover X, we may assume without loss of generality that u_1 is an end vertex of P_1 and e. Proceeding as in Corollary 2, the proof follows. □

Lemma 2. *Let f be a Leech labeling of $K_{2,n}, n > 2$. Let $f(u_1v_i) + f(u_2v_j) = c$ where $i \neq j$. Then $f(e) \neq c$ for any edge e.*

Proof. Suppose $f(e) = c$ for some edge e. Since e is incident with u_1 or u_2, we may assume that $e = u_1v_k$ where $k \neq i$. If $k \neq j$, then $w((v_k, u_1, v_j)) = w((v_i, u_1, v_j, u_2)) = c + f(u_1v_j)$. If $k = j$, then for any $r \neq i, j$, $w((v_i, u_1, v_r, u_2, v_j)) = w((v_j, u_1, v_r, u_2)) = c + f(u_1v_r) + f(u_2v_r)$. Hence the result follows. □

We now proceed to prove that $K_{2,n}$ and $K_{3,n}$ are not Leech graphs for all $n \geq 3$. Throughout the proof S denotes the set of all path weights at each stage. For any positive integer r, the set $\{1, 2, \ldots, r\}$ is denoted by $[r]$.

Theorem 1. *The complete bipartite graph $K_{2,n}$, $n > 2$ is not a Leech graph.*

Proof. Suppose $K_{2,n}$ is a Leech graph with Leech labeling f. It follows from Lemma 2 that the edges with labels 1 and 2 are adjacent. Suppose $f(u_1v_1) = 1$ and $f(u_2v_1) = 2$. Then either $f(u_1v_2) = 4$ or $f(u_2v_2) = 4$. If $f(u_1v_2) = 4$, then $[5] \subseteq S$ and there cannot be a path of weight 6. Similarly, if $f(u_2v_2) = 4$, then there cannot be a path of weight 5, which is a contradiction. Hence $f(u_1v_1) = 1, f(u_1v_2) = 2$ and $[3] \subseteq S$. If 4 is assigned to an edge not adjacent to u_1v_1, u_1v_2, then it follows from Lemma 2 that the path weights 5 or 6 cannot be obtained. Hence let $f(u_1v_3) = 4$, so that $[6] \subseteq S$. Again it follows from Lemma 2 that if 7 is assigned to an edge not adjacent to u_1v_1, u_1v_2, then the path weight 8 or 9 cannot be obtained. Hence $f(u_1v_4) = 7$, so that $[9] \cup \{11\} \subseteq S$. Now let $f(e) = 10$. Since $11 \in S$, e is not adjacent to u_1v_1. If e is not adjacent to u_1v_2, then by Lemma 2, the path weight 12 cannot be obtained. Hence $e = u_2v_2$. Now u_2v_2 and u_1v_3 are nonadjacent and by Lemma 2 the path weight 14 cannot be obtained. Hence $K_{2,n}$ is not a Leech graph. □

Corollary 4. *The Leech index of $K_{2,n}$ is $k(K_{2,n}) = 13$, for $n \geq 4$. When $n = 3$, $k(K_{2,3}) = 8$ and when $n = 2$, $K_{2,2} = C_4$ which is a Leech graph.*

Theorem 2. *The complete bipartite graph $K_{3,n}, n \geq 3$ is not a Leech graph.*

Proof. Suppose $K_{3,n}$ is a Leech graph with Leech labeling f. Let $f(e_1) = 1$ and $f(e_2) = 2$. Suppose e_1 and e_2 are nonadjacent. Then by Corollary 1, the edge e_3 with $f(e_3) = 3$ is nonadjacent to e_1 and e_2 and the edge e_4 with $f(e_4) = 4$ is nonadjacent to e_1 and e_3. Thus 1 and 4 are assigned to a pair of nonadjacent edges and 2 and 3 are assigned to a pair of nonadjacent edges. Hence path weight 5 cannot be obtained. Thus e_1 and e_2 are adjacent. We consider two cases.

Case 1. $e_1 = u_1v_1$ and $e_2 = u_1v_2$.
Hence $[3] \subseteq S$. Let $f(e_3) = 4$. If e_3 is nonadjacent to e_1 and e_2, then by Corollary 1, the label 5 must be assigned to an edge not adjacent to e_1 and e_2. Thus 1 and 5 are assigned to two nonadjacent edges and 2 and 4 are assigned to two nonadjacent edges. Hence the path weight 6 cannot be obtained.

If e_3 is nonadjacent to e_1 and adjacent to e_2, then 5 must be assigned to an edge independent to e_1 and e_3, say, $f(u_3v_3) = 5$. Then path weight $6 + f(u_1v_3)$ repeats.

Now suppose e_3 is adjacent to e_1 and not adjacent to e_2. Let $e_3 = u_2v_1$. Then by Corollary 1, the label 6 must be assigned to an edge not adjacent to both $e_2 = u_1v_2$ and $e_3 = u_2v_1$. Hence $[7] \subseteq S$. By Corollary 1 the label 8 must be assigned to an edge not adjacent to the edges with labels 2 and 6. Thus 1 and 8 are assigned to two nonadjacent edges and 3 and 6 are path weights of two vertex disjoint paths. Hence, by Corollary 3, path weight 9 cannot be obtained.

Therefore e_3 is adjacent to both e_1 and e_2. Let $e_3 = u_1v_3$. Hence $[6] \subseteq S$. Now let $f(e_4) = 7$. If e_4 is not adjacent to e_1, then by Corollary 1, the label 8 must be assigned to an edge e not adjacent to e_1 and e_4. Now if e_4 is not adjacent to e_2, then $f(e_2) + f(e_4) = f(e_1) + f(e) = 9$ and hence by Corollary 3, the path weight 9 cannot be obtained. If e_4 is adjacent to e_2, then 8 must be assigned to an edge independent to e_1 and e_4, say, $f(u_3v_j) = 8, j \neq 1, 2$ and then $9 + f(v_ju_1)$ repeats. Hence e_4 is adjacent to e_1. Now if e_4 is not adjacent e_2, then by Corollary 1, the label 9 must be assigned to an edge e not adjacent to e_2 and e_4. Hence $f(e_3) + f(e_4) = f(e) + f(e_2) = 11$ and by Corollary 3, path weight 11 cannot be obtained. Thus e_4 is adjacent to e_2. Therefore $e_4 = u_1v_4$ and so $[9] \cup \{11\} \subseteq S$. Now let $f(e_5) = 10$.

Since $11 \in S$, e_5 is not adjacent to e_1. If e_5 is adjacent to e_2, then $[13] \cup \{16, 19\} \subseteq S$. By Corollary 1, the label 14 must be assigned to an edge e not adjacent to e_3 and e_5. If e is adjacent to e_1, then path weight 19 repeats. If e is non adjacent to e_1, then $f(e_1) + f(e) = f(e_5) + w(v_1, u_1, u_3) = 15$, and by Corollary 3, the path weight 15 cannot be obtained. Therefore e_5 is not adjacent to e_2. Hence let $e_5 = u_2v_j$ where $j \neq 1, 2$. Then the label 12 must be assigned to an edge not adjacent to e_2 and e_5. Let $f(u_3v_k) = 12$ where $k \neq 2, j$.

If $k = 1$, since e_2 and u_3v_k are nonadjacent, the edge $e_5 = u_2v_j$ with label

10 and the edge $e_3 = u_1v_3$ with label 4 are adjacent. Thus $j = 3$. Now $w((u_2, v_3, u_1, v_1)) = w((u_3, v_1, u_1, v_2)) = 15$, a contradiction. If $k \neq 1$, then $w((v_1, u_1, v_2)) + f(e_5) = f(u_1v_1) + f(u_3v_k) = 13$ and hence path weight 13 cannot be obtained. Hence the labels 1 and 2 cannot be assigned to the adjacent edges $e_1 = u_1v_1$ and $e_2 = u_1v_2$.

Case 2. $e_1 = u_1v_1$ and $e_2 = u_2v_1$.
Let $f(e_3) = 4$. If e_3 is nonadjacent to both e_1 and e_2, then by Corollary 1, the label 5 must be assigned to an edge nonadjacent to e_1 and e_3. Thus 1 and 5 are assigned to two nonadjacent edges and 2 and 4 are assigned to two nonadjacent edges. Hence path weight 6 cannot be obtained.

If e_3 is adjacent to e_2 and nonadjacent to e_1, then 5 must be assigned to an edge nonadjacent to both e_1 and e_3. Let $f(u_3v_3) = 5$. Then the paths (u_1, v_1, u_3, v_3) and (v_2, u_2, v_1, v_3) have weight $6 + f(u_3v_1)$ which is a contradiction.

Now suppose e_3 is adjacent to e_1 and nonadjacent to e_2. Let $e_3 = u_1v_2$. By Corollary 1, the label 6 must be assigned to an edge not adjacent to e_2 and e_3. Hence $[7] \subseteq S$. Again by Corollary 1, the label 8 must be assigned to an edge not adjacent to the edges with labels 2 and 6. But now 2 and 8 are assigned to two nonadjacent edges and 4 and 6 are assigned to nonadjacent edges. Hence by Corollary 3, path weight 10 cannot be obtained.

Therefore e_3 is adjacent to both e_1 and e_2. Let $e_3 = u_3v_1$. Hence $[6] \subseteq S$. Let $f(e_4) = 7$. If e_4 is nonadjacent to e_1, then by Corollary 1, the label 8 must be assigned to an edge e nonadjacent to e_1 and e_4. Now, if e_4 is nonadjacent to e_2, then $f(e_2) + f(e_4) = f(e_1) + f(e) = 9$ and by Corollary 3, the path weight 9 cannot be obtained. If e_4 is adjacent to e_2, then the label 8 must be assigned to an edge nonadjacent to both e_1 and e_4. Let $f(u_3v_3) = 8$. In this case we get two paths with weight 13. Hence e_4 is adjacent to e_1. Now if e_4 is nonadjacent to e_2, then by Corollary 1, the label 9 must be assigned to an edge e nonadjacent to both e_2 and e_4. Hence $f(e_3) + f(e_4) = f(e) + f(e_2) = 11$ and by Corollary 3, the path weight 11 cannot be obtained, which is a contradiction. Hence $K_{3,n}$ is not a Leech graph. □

Corollary 5. *The Leech index of $K_{3,n}$ is $k(K_{3,n}) = 14$, for $n \geq 4$. When $n = 3$, $k(K_{3,3}) = 10$.*

3. Leech Cycles

In [12], it has been proved that the cycle C_n, with $3 \leq n \leq 6$ is a Leech graph. The Leech labelings of these cycles are given in Figure 2, in which two Leech labelings are given for C_4. Thus for a Leech graph, the Leech labeling is not in general unique. In a cycle, since there exist exactly two paths between every pair of vertices, $t_p(C_n) =$

$n(n-1)$. In this section, we present results on the maximum label that can assigned to an edge in a Leech cycle.

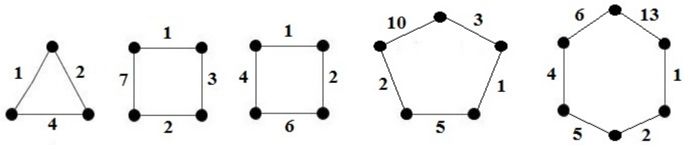

Figure 2. Leech labeling of cycles of order ≤ 6

Theorem 3. *Let f be the Leech labeling of a cycle C_n. Then, $w(f) = 2\binom{n}{2} + 1$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $f(v_1v_2) = 1$. Then $P = (v_2, v_3, \ldots, v_n, v_1)$ is the path of maximum weight. Hence $w(P) = t_p(G) = n(n-1)$. Therefore, $w(f) = w(P) + 1 = n^2 - n + 1 = 2\binom{n}{2} + 1$. □

Theorem 4. *The maximum value that can be assigned to an edge in a Leech labeling of a cycle C_n is $\binom{n}{2} + 1$. Also, this maximum value is attained only by C_3 and C_4.*

Proof. Let $f(e) = M$, where M is the maximum value assigned to an edge by f. Now, $P_n = C_n - e$ is a path of order n and hence $w(C_n - e) \geq 1 + 2 + \cdots + (n-1) = \binom{n}{2}$. Hence, $w(f) = 2\binom{n}{2} + 1 = w(C_n - e) + M \geq \binom{n}{2} + M$. Therefore, $M \leq \binom{n}{2} + 1$. Also, equality holds if and only if $w(P_n) = \binom{n}{2}$ and the path weights of all subpaths of P_n are exactly $\{1, 2, \ldots, \binom{n}{2} - 1\}$. Hence P_n is a Leech path. Since P_2, P_3 and P_4 are the only Leech paths, the maximum edge label M is attained only for C_3 and C_4. □

Theorem 5. *The only Leech cycles which admits a Leech labeling in which the maximum label is $\binom{n}{2}$ are C_4 and C_5.*

Proof. Let $f(e_1) = M = \binom{n}{2}$, where M is the maximum value assigned to an edge by f and $e_1 = v_1v_2$. Let $P_n = C_n - e_1$. Then, $w(P_n) = \binom{n}{2} + 1$ and all path weights $1, 2, \ldots, \binom{n}{2} - 1$ must be obtained from P_n. Hence, the set of edge labels of P_n is $\{1, 2, \ldots, n-2, n\}$. Let $f(e_2) = 1$. If $e_2 = v_2v_3$, then $w(v_3, v_4, \ldots, v_n, v_1) = \binom{n}{2} = w(v_1v_2)$. Hence, $e_2 \neq v_2v_3$. Similarly, $e_2 \neq v_nv_1$. If $e_2 = v_iv_{i+1}$ and $f(v_{i-1}v_i) = k$ then $w(v_{i-1}, v_i, v_{i+1}) = k + 1$. Since $1, 2, \ldots, n-2$ and n are already edge weights, $k \notin \{2, 3, \ldots, n-3\}$. A similar argument holds for $f(v_{i+1}v_{i+2}) = l$ also. Therefore, k and l are $n-2$ and n in some order. If $n = 4$, this gives a Leech labeling of C_4 with maximum label 6 as given in Figure 2.

Now, let $n \geq 5$. Let $f(e_3) = 2$. If e_3 is adjacent to an edge labeled k then these two edges together gives a path of weight $k + 2$. Since, $1, 2, \ldots, n-2$ and

n are already edge weights and $\{w(v_{i-1}v_iv_{i+1}), w(v_iv_{i+1}v_{i+2})\} = \{n-1, n+1\}$, $k \notin \{1, 3, \ldots, n-2\}$. Hence, e_3 is adjacent only to the edge with label n.

Now, since the edges with labels 1 and 2 are non-adjacent, there exists an edge e_4 with $f(e_4) = 3$. If e_4 is adjacent to an edge labeled $k \in \{1, 2, 4, \ldots, n-2\}$ then the path weight of e_4 together with this edge will be in $\{4, 5, 7, \ldots, n+1\}$ which is not possible. Hence, the only possibility is $n - 2 = 3$ and this gives a Leech labeling of C_5 with the maximum edge label is 10 as given in Figure 2. □

Theorem 6. *The only Leech cycle which admits a Leech labeling in which the maximum label is* $\binom{n}{2} - 1$ *is* C_6.

Proof. Let $f(e) = \binom{n}{2} - 1$, where $e = v_1v_2$. Let $P_n = C_n - e = (v_2, v_3, \ldots, v_n, v_1)$. Then $w(P_n) = \binom{n}{2} + 2$ and all path weights less than $\binom{n}{2} - 1$ must be obtained from P_n. Hence the set of all path weights of the subpaths of P_n is $\{1, 2, \ldots, \binom{n}{2} - 2, \binom{n}{2} + 2, k\}$, where k is $\binom{n}{2}$ or $\binom{n}{2} + 1$. Also, the sum of edge weights of P_n is $\binom{n}{2} + 2$ and hence the set of all edge labels of P_n is $\{1, 2, \ldots, n-2, n+1\}$ or $\{1, 2, \ldots, n-3, n-1, n\}$. We consider four cases.

Case 1. $k = \binom{n}{2}$ and the set of edge labels of P_n is $\{1, 2, \ldots, n-2, n+1\}$.
Since $w(P_n) = \binom{n}{2} + 2$ and $k = \binom{n}{2}$ is a path weight of a subpath of P_n, the label 2 must be assigned to a pendant edge of P_n. Let $f(v_2v_3) = 2$. Now since $f(e) = \binom{n}{2} - 1$ and $\binom{n}{2}$ is a path weight of a subpath of P_n, the label 1 cannot be assigned to a pendant edge of P_n. Let $f(e_1) = 1$, where e_1 is an internal edge of P_n. Now, if a and b are edge labels of two adjacent edges of P_n, then $a + b$ is not an edge label. Hence, $a + b = n - 1$ or n or $a + b \geq n + 2$. Hence the two edges adjacent to e_1 have labels $n - 2$ and $n + 1$ and we get path weights $n - 1$ and $n + 2$. Now, if $f(v_3v_4) = x$ then, $w(v_2, v_3, v_4) = x + 2 = n$ or $x + 2 \geq n + 3$. If $x + 2 = n$, then $x = n - 2$ and hence $e_1 = v_4v_5$. But, then $w(v_2, v_3, v_4, v_5) = 2 + n - 2 + 1 = n + 1$ which is already an edge weight, a contradiction. Therefore, the only possibility is $f(v_3v_4) = n + 1$ and eventually $f(v_4v_5) = 1$ and $f(v_5v_6) = n - 2$ and we get path weights $n + 3$ and $n + 4$. Now, there is an edge with label 3 and if the edge adjacent to it is labeled y then $y + 3 = n$ or $y + 3 \geq n + 5$. But, $y \geq n + 2$ is not possible and hence 3 is assigned to a pendant edge of P_n. But, then the path weight $3 + \binom{n}{2} - 1 = \binom{n}{2} + 2$ repeats. Hence, this case is not possible.

Case 2. $k = \binom{n}{2}$ and the set of edge labels of P_n is $\{1, 2, \ldots, n-3, n-1, n\}$.
As in Case 1, $f(v_2v_3) = 2$ and $f(e_1) = 1$, where e_1 is an internal edge of P_n. Also, if a and b are edge labels of two adjacent edges of P_n, then $a + b = n - 2$ or $a + b \geq n + 1$. Hence the two edges adjacent to e_1 have labels $n - 3$ and n and we get path weights $n - 2$ and $n + 1$. Now if $f(v_3v_4) = x$ then, $w(v_2, v_3, v_4) = x + 2 \geq n + 2$. Therefore, $f(v_3v_4) = n$ and eventually, $f(v_4v_5) = 1$ and $f(v_5v_6) = n - 3$. Also, we have obtained all path weights up to $n + 3$. Now, if the label y is assigned to an edge adjacent to the edge labeled 3, then $y + 3 \geq n + 4$ which is not possible.

Case 3. $k = \binom{n}{2} + 1$ and the set of edge labels of P_n is $\{1, 2, \ldots, n-2, n+1\}$.
Since $w(P_n) = \binom{n}{2} + 2$ and $k = \binom{n}{2} + 1$ is a path weight of a subpath of P_n, the label 1 must be assigned to a pendant edge of P_n. Let $f(v_2v_3) = 1$. Also, since $\binom{n}{2}$ is not a path weight of a subpath of P_n, 2 cannot be assigned to a pendent edge of P_n. Let $f(e_1) = 2$, where e_1 is an internal edge of P_n. Now, if a and b are edge labels of two adjacent edges of P_n, then $a + b = n - 1$ or n or $a + b \geq n + 2$. Therefore, $f(v_3v_4) = n-2$ or $n+1$. If $f(v_3v_4) = n-2$ then we get path weight $n-1$ also, so that if an edge adjacent to e_1 is given label x, then $x + 2 = n$ or $x + 2 \geq n + 2$. Therefore, the edges adjacent to e_1 are labeled $n - 2$ and $n + 1$. Therefore, $f(v_4v_5) = 2$ and $f(v_5v_6) = n + 1$. But, then $w(v_2, v_3, v_4, v_5) = 1 + n - 2 + 2 = n + 1$, which is already an edge weight. Therefore, let $f(v_3v_4) = n + 1$ and then we get the path weight $n + 2$ also. Again, if an edge adjacent to e_1 is given label x, then $x + 2 = n - 1$ or n or $x + 2 \geq n + 3$. Therefore, the edges adjacent to e_1 are labeled $n - 3$ or $n - 2$ or $n + 1$. If e_1 is adjacent to an edge labeled $n + 1$ then, $e_1 = v_4v_5$ and we get paths of weight $n + 3$ and $n + 4$. Now, the edge labeled 3 can be adjacent only to either $n - 3$ or the edge labeled $n - 4$, which implies 3 is assigned to a pendant edge of P_n. But, then $f(e) + 3$ gives another path weight of weight $\binom{n}{2} + 2$, a contradiction. Therefore, the labels of edges adjacent to 2 are $n - 3$ and $n - 2$, so that the path weights $n - 1$ and n are obtained. Again, if the label y is assigned to an edge adjacent to the edge labeled 3, then $y + 3 \geq n + 3$ in which case also 3 is assigned to a pendant edge and hence is not possible.

Case 4. $k = \binom{n}{2} + 1$ and the set of edge labels of P_n is $\{1, 2, \ldots, n-3, n-1, n\}$.
As in Case 3, $f(v_2v_3) = 1$ and $f(e_1) = 2$, where e_1 is an internal edge of P_n. Again, if a and b are edge labels of two adjacent edges of P_n, then $a+b = n-2$ or $a+b \geq n+1$. Therefore, $f(v_3v_4) = n - 3$ or n.

If $f(v_3v_4) = n$, then we get path weight $n + 1$ also. Now, if an edge adjacent to e_1 is given label x, then $x + 2 = n - 2$ or $x + 2 \geq n + 2$. Therefore, the edges adjacent to 2 are labeled $n - 4$ and n, so that we get all path weights upto $n + 3$. Now, if label y is assigned to an edge adjacent to the edge labeled 3, then $y + 3 \geq n + 4$ which is not possible. Therefore, let $f(v_3v_4) = n - 3$ so that we get path weight $n - 2$ also, so that if an edge adjacent to e_1 is given label x, then $x + 2 \geq n + 1$. Therefore, the edges adjacent to e_1 are labeled $n - 1$ and n, so that we get all path weights up to $n + 2$. In this case also, if the label y is assigned to an edge adjacent to the edge labeled 3, then $y + 3 \geq n + 3$ in which case 3 is assigned to a pendant edge and hence is not possible. So, the only possibility is $n - 3 = 3$ and this gives the Leech labeling of C_6. □

Corollary 6. *The maximum value that can be assigned to an edge in a Leech labeling of a cycle C_n, where $n > 7$, is less than $\binom{n}{2} - 1$.*

The following figure gives three different Leech labelings of C_6 in which the maximum labels are $\binom{n}{2} - 1$, $\binom{n}{2} - 2$ and $\binom{n}{2} - 3$. We strongly believe that cycles of length greater than 6 are not Leech graphs.

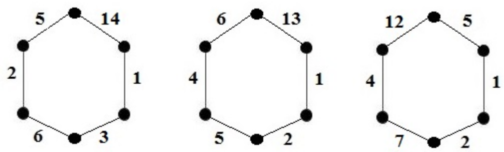

Figure 3. **Leech labelings of C_6**

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