Communications in Combinatorics and Optimization
Vol. 8, No. 4 (2023), pp. 737-749
DOI: 10.22049/CCO.2022.27794.1355

CCO
Commun. Comb. Optim.

Research Article

On several new closed-form evaluations for the generalized hypergeometric functions

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Received: 4 May 2022; Accepted: 1 September 2022
Published Online: 6 September 2022

Abstract: The main objective of this paper is to establish as many as thirty new closed-form evaluations of the generalized hypergeometric function ${}_{q+1}F_q(z)$ for $q = 2, 3$. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ for $q = 1, 2, 3$ into even and odd components together with the use of several known infinite series involving reciprocal of the non-central binomial coefficients obtained earlier by L. Zhang and W. Ji.

Keywords: Generalized hypergeometric function, central and non-central binomial coefficients, combinatorial sum, reciprocals

AMS Subject classification: Primary: 05A15, 11B68, 33C20, Secondary: 30C05, 33C05, 33C15

1. Introduction

A natural generalization of the well-known Gauss's hypergeometric function ${}_2F_1$ is the generalized hypergeometric function ${}_pF_q$ is defined by [12]

$$ {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \\ \\ b_1, b_2, \ldots, b_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}, \tag{1} $$

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where $(a)_n$ is the well-known Pochhammer's symbol (or the shifted or raised factorial, since $(1)_n = n!$) defined for the complex number $a(\neq 0)$ by

$$(a)_n = \begin{cases} a(a+1)\dots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases}$$

In terms of well-known gamma function, $(a)_n$ is defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The series (1) is convergent for all values of $|z| < \infty$ if $p \leq q$ and for all values of $|z| < 1$ if $p = q+1$ while it is divergent for all values of z, $z \neq 0$ if $p \geq q+1$. Also, when $|z| = 1$ with $p = q+1$, the series (1) converges absolutely if $Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$ conditionally convergent if $-1 < Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) \leq 0, \quad z \neq 1$ and divergent if $Re\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) < -1$.
It is not out of place to mention here that the generalized hypergeometric function occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering and statistics. For more details about this function, we refer [1–3, 9, 12, 15].
On the other hand, the binomial coefficients are defined by

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & ;\ n \geq k \\ 0 & ;\ n < k. \end{cases} \tag{2}$$

for nonnegative integers n and k. The central binomial coefficients are defined by

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \qquad (n = 0, 1, 2, \dots). \tag{3}$$

It is well known that the binomial and reciprocal of binomial coefficients play an important role in many areas of mathematics (including number theory, probability and statistics). Actually the sums containing the central binomial coefficients and reciprocals of the central binomial coefficients have been studied for a long time. A large number of very interesting results can be seen in the research papers by Lehmer [7], Mansour [8], Pla [10], Sherman [14], Sprugnoli [16, 17], Sury [18], Sury et al. [19], Trif [20], Wheelon [21] and Zhao and Wang [23]. Many facts about the central binomial coefficients and the reciprocals of the central binomial coefficients can be found in the book of Koshy [5]. Gould [4] has collected numerous identities involving central binomial coefficients. Riordan [13] is also a good reference. By employing several interesting and useful results containing infinite series involving central binomial coefficients and reciprocal of the central binomial coefficients obtained earlier by Lehmer [7], very recently, Kumar et al. [6] obtained several interesting closed-form

evaluations of the generalized hypergeometric functions.
Moreover, in 2013, by utilizing Gamma-Beta function, Zhang and Ji [22] obtained the following result containing infinite series involving reciprocal of non-central binomial coefficients viz.

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}} = \frac{2}{3} + \frac{4\sqrt{3}\pi}{27}$$

and by item splitting established the following fifteen results viz.

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)} = \frac{4\sqrt{3}\pi}{9} - 2, \tag{4}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+5)} = \frac{28\sqrt{3}\pi}{9} - \frac{50}{3}, \tag{5}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+7)} = \frac{148\sqrt{3}\pi}{9} - \frac{4018}{45} \tag{6}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+9)} = \frac{3548\sqrt{3}\pi}{45} - \frac{225158}{525}, \tag{7}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+5)} = -\frac{4\sqrt{3}\pi}{3} + \frac{22}{3}, \tag{8}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+7)} = -4\sqrt{3}\pi + \frac{982}{45} \tag{9}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+9)} = -\frac{196\sqrt{3}\pi}{15} + \frac{112054}{1575}, \tag{10}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+5)(2m+7)} = -\frac{20\sqrt{3}\pi}{3} + \frac{1634}{45}, \tag{11}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+5)(2m+9)} = -\frac{284\sqrt{3}\pi}{15} + \frac{18034}{175}, \tag{12}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+7)(2m+9)} = -\frac{156\sqrt{3}\pi}{5} + \frac{267422}{1575}, \tag{13}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+5)(2m+7)} = \frac{4\sqrt{3}\pi}{3} - \frac{326}{45}, \tag{14}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+5)(2m+9)} = \frac{44\sqrt{3}\pi}{15} - \frac{25126}{1575}, \tag{15}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+7)(2m+9)} = \frac{68\sqrt{3}\pi}{15} - \frac{38842}{1575}, \tag{16}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+5)(2m+7)(2m+9)} = \frac{92\sqrt{3}\pi}{15} - \frac{52558}{1575}, \tag{17}$$

$$\sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+5)(2m+7)(2m+9)} = -\frac{4\sqrt{3}\pi}{5} + \frac{762}{175}. \tag{18}$$

Also, in the same paper , they have obtained the following result of infinite series (containing positive and negative terms) involving reciprocal of non-central binomial coefficients viz.

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}} = \frac{2}{5} + \frac{8\sqrt{5}}{25} \ln \varphi$$

with the golden ration $\varphi = \frac{\sqrt{5}+1}{2}$ and by item splitting established the following results viz.

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)} = -\frac{8\sqrt{5}}{5} \ln \varphi + 2, \tag{19}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+5)} = \frac{72\sqrt{5}}{5} \ln \varphi - \frac{46}{3}, \tag{20}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+7}{m}(2m+3)} = -\frac{1144\sqrt{5}}{15} \ln \varphi + \frac{3698}{45}, \tag{21}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+9)} = \frac{1832\sqrt{5}}{5} \ln \varphi - \frac{206939}{525}, \tag{22}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+5)} = -8\sqrt{5} \ln \varphi + \frac{26}{3}, \tag{23}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+7)} = \frac{56\sqrt{5}}{3} \ln \varphi - \frac{902}{45}, \tag{24}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+9)} = -\frac{184\sqrt{5}}{3} \ln \varphi + \frac{103994}{1575}, \tag{25}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+5)(2m+7)} = \frac{136\sqrt{5}}{3} \ln \varphi - \frac{2194}{45}, \tag{26}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+5)(2m+9)} = -88\sqrt{5} \ln \varphi + \frac{16574}{175}, \tag{27}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+7)(2m+9)} = -\frac{664\sqrt{5}}{3} \ln \varphi + \frac{375122}{1575}. \tag{28}$$

Remark 1. The results (6), (7), (9), (12), (15), (18) and (25) are given here in corrected form.

The rest of the paper is organized as follows. In section 2, we shall establish the results (8) to (18) and (23) to (28) in a very elementary way using a partial fraction method. In addition to this, in this section, we shall add five new results that will be required in our present investigation. In section 3, we shall express the results (4) to (28) and (29) to (33) in terms of the generalized hypergeometric functions. Finally, in section

4, we shall establish as many as thirty new and interesting closed-form evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for $q = 2$ and 3 with argument $\frac{1}{16}$. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ for $q = 1, 2$ and 3 into even and odd components together with the results given in section 3.

2. Alternative derivations of the results (8) to (18) and (23) to (28) together with five new results

In this section, first we shall provide an alternative derivation of the results (8) to (18) and (23) to (28) and thereafter provide five new results by the same technique.

(a) Derivation of the results (8) to (18) and (23) to (28)
In order to derive the result (8), let us denote the left-hand side of (8) by S, we have

$$S = \sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)(2m+5)}.$$

Now, using partial fractions, it is easy to see that

$$\begin{aligned} S =& \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}} \left[\frac{1}{2m+3} - \frac{1}{2m+5} \right] \\ =& \frac{1}{2} \left\{ \sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+3)} - \sum_{m=0}^{\infty} \frac{1}{\binom{2m+1}{m}(2m+5)} \right\}. \end{aligned}$$

Finally, using the results (4) and (5), we at once arrive at the right-hand side of (8). This completes the proof of the result (8). In exactly the same manner, the results (9) to (18) and (23) to (28) can be derived. So we prefer to omit the details.

(b) Five new results
In this sub-section, we shall provide the following five new results without proof as these can be derived in a similar manner mentioned in the sub-section (a). These are

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+5)(2m+7)} = -\frac{40\sqrt{5}}{3} \ln \varphi + \frac{646}{45}, \tag{29}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+5)(2m+9)} = \frac{40\sqrt{5}}{3} \ln \varphi - \frac{180689}{12600}, \tag{30}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+7)(2m+9)} = 40\sqrt{5} \ln \varphi - \frac{271129}{6300}, \tag{31}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+5)(2m+7)(2m+9)} = \frac{200\sqrt{5}}{3} \ln\varphi - \frac{903827}{12600}, \tag{32}$$

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{\binom{2m+1}{m}(2m+3)(2m+5)(2m+7)(2m+9)} = -\frac{40\sqrt{5}}{3} \ln\varphi + \frac{120523}{8400}. \tag{33}$$

3. Results (4) to (28) and (29) to (33) in terms of hypergeometric functions

In terms of the generalized hypergeometric function, the results (4) to (28) and (29) to (33) can be written in the following manner.

$${}_2F_1\left[\begin{matrix} 1, 2 \\ \frac{5}{2} \end{matrix} ; \frac{1}{4}\right] = \frac{4\sqrt{3}\pi}{3} - 6, \tag{34}$$

$${}_3F_2\left[\begin{matrix} 1, 2, \frac{5}{2} \\ \frac{3}{2}, \frac{7}{2} \end{matrix} ; \frac{1}{4}\right] = 5\left(\frac{28\sqrt{3}\pi}{9} - \frac{50}{3}\right), \tag{35}$$

$${}_3F_2\left[\begin{matrix} 1, 2, \frac{7}{2} \\ \frac{3}{2}, \frac{9}{2} \end{matrix} ; \frac{1}{4}\right] = 7\left(\frac{148\sqrt{3}\pi}{9} - \frac{4018}{45}\right), \tag{36}$$

$${}_3F_2\left[\begin{matrix} 1, 2, \frac{9}{2} \\ \frac{3}{2}, \frac{11}{2} \end{matrix} ; \frac{1}{4}\right] = 9\left(\frac{3548\sqrt{3}\pi}{45} - \frac{225158}{525}\right), \tag{37}$$

$${}_2F_1\left[\begin{matrix} 1, 2 \\ \frac{7}{2} \end{matrix} ; \frac{1}{4}\right] = 5(22 - 4\sqrt{3}\pi), \tag{38}$$

$${}_3F_2\left[\begin{matrix} 1, 2, \frac{7}{2} \\ \frac{5}{2}, \frac{9}{2} \end{matrix} ; \frac{1}{4}\right] = 21\left(\frac{982}{45} - 4\sqrt{3}\pi\right), \tag{39}$$

$${}_3F_2\left[\begin{matrix} 1, 2, \frac{5}{2} \\ \frac{3}{2}, \frac{9}{2} \end{matrix} ; \frac{1}{4}\right] = 35\left(\frac{1634}{45} - \frac{20\sqrt{3}\pi}{3}\right), \tag{40}$$

$${}_3F_2\left[\begin{matrix} 1, 2, \frac{9}{2} \\ \frac{5}{2}, \frac{11}{2} \end{matrix} ; \frac{1}{4}\right] = 27\left(\frac{112054}{1575} - \frac{196\sqrt{3}\pi}{15}\right), \tag{41}$$

$${}_4F_3\left[\begin{matrix} 1, 2, \frac{5}{2}, \frac{9}{2} \\ \frac{3}{2}, \frac{7}{2}, \frac{11}{2} \end{matrix} ; \frac{1}{4}\right] = 45\left(\frac{18034}{175} - \frac{284\sqrt{3}\pi}{15}\right), \tag{42}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{7}{2}\\ \frac{3}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=63\left(\frac{267422}{1575}-\frac{156\sqrt{3}\pi}{5}\right), \tag{43}$$

$$ {}_2F_1\left[\begin{matrix}1,2\\ \frac{9}{2}\end{matrix};\frac{1}{4}\right]=105\left(\frac{4\sqrt{3}\pi}{3}-\frac{326}{45}\right), \tag{44}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{9}{2}\\ \frac{7}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=135\left(\frac{44\sqrt{3}\pi}{15}-\frac{25126}{1575}\right), \tag{45}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{7}{2}\\ \frac{5}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=189\left(\frac{68\sqrt{3}\pi}{15}-\frac{52558}{1575}\right), \tag{46}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{5}{2}\\ \frac{3}{2},\frac{11}{2}\end{matrix};\frac{1}{4}\right]=315\left(\frac{92\sqrt{3}\pi}{15}-\frac{52558}{1575}\right), \tag{47}$$

$$ {}_2F_1\left[\begin{matrix}1,2\\ \frac{11}{2}\end{matrix};\frac{1}{4}\right]=945\left(\frac{762}{175}-\frac{4\sqrt{3}\pi}{5}\right), \tag{48}$$

$$ {}_2F_1\left[\begin{matrix}1,2\\ \frac{5}{2}\end{matrix};-\frac{1}{4}\right]=6-\frac{24\sqrt{5}}{5}\ln\varphi, \tag{49}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{5}{2}\\ \frac{3}{2},\frac{7}{2}\end{matrix};-\frac{1}{4}\right]=5\left(\frac{72\sqrt{5}}{5}\ln\varphi-\frac{46}{3}\right), \tag{50}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{7}{2}\\ \frac{3}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=7\left(\frac{3698}{45}-\frac{1144\sqrt{5}}{15}\ln\varphi\right), \tag{51}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{9}{2}\\ \frac{3}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=9\left(\frac{1832\sqrt{5}}{5}\ln\varphi-\frac{206939}{525}\right), \tag{52}$$

$$ {}_2F_1\left[\begin{matrix}1,2\\ \frac{7}{2}\end{matrix};-\frac{1}{4}\right]=15\left(\frac{26}{3}-8\sqrt{5}\ln\varphi\right), \tag{53}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{7}{2}\\ \frac{5}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=21\left(\frac{56\sqrt{5}}{3}\ln\varphi-\frac{902}{45}\right), \tag{54}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{5}{2}\\ \frac{3}{2},\frac{9}{2}\end{matrix};-\frac{1}{4}\right]=35\left(\frac{136\sqrt{5}}{3}\ln\varphi-\frac{2194}{45}\right), \tag{55}$$

$$ {}_3F_2\left[\begin{matrix}1,2,\frac{9}{2}\\ \frac{5}{2},\frac{11}{2}\end{matrix};-\frac{1}{4}\right]=27\left(\frac{103994}{1575}-\frac{184\sqrt{5}}{3}\ln\varphi\right), \tag{56}$$

$$ {}_4F_3\left[\begin{matrix} 1, 2, \frac{5}{2}, \frac{9}{2} \\ \frac{3}{2}, \frac{7}{2}, \frac{11}{2} \end{matrix} ; -\frac{1}{4}\right] = 45\left(\frac{16574}{175} - 88\sqrt{5}\ln\varphi\right), \tag{57} $$

$$ {}_3F_2\left[\begin{matrix} 1, 2, \frac{7}{2} \\ \frac{3}{2}, \frac{11}{2} \end{matrix} ; -\frac{1}{4}\right] = 63\left(\frac{375122}{1575} - \frac{664\sqrt{5}}{3}\ln\varphi\right), \tag{58} $$

$$ {}_2F_1\left[\begin{matrix} 1, 2 \\ \frac{9}{2} \end{matrix} ; -\frac{1}{4}\right] = 105\left(\frac{646}{45} - \frac{40\sqrt{5}}{3}\ln\varphi\right), \tag{59} $$

$$ {}_3F_2\left[\begin{matrix} 1, 2, \frac{9}{2} \\ \frac{7}{2}, \frac{11}{2} \end{matrix} ; -\frac{1}{4}\right] = 135\left(\frac{40\sqrt{5}}{3}\ln\varphi - \frac{180689}{12600}\right), \tag{60} $$

$$ {}_3F_2\left[\begin{matrix} 1, 2, \frac{7}{2} \\ \frac{5}{2}, \frac{11}{2} \end{matrix} ; -\frac{1}{4}\right] = 189\left(40\sqrt{5}\ln\varphi - \frac{271129}{6300}\right), \tag{61} $$

$$ {}_3F_2\left[\begin{matrix} 1, 2, \frac{5}{2} \\ \frac{3}{2}, \frac{11}{2} \end{matrix} ; -\frac{1}{4}\right] = 315\left(\frac{200\sqrt{5}}{3}\ln\varphi - \frac{903827}{12600}\right), \tag{62} $$

$$ {}_2F_1\left[\begin{matrix} 1, 2 \\ \frac{11}{2} \end{matrix} ; -\frac{1}{4}\right] = 945\left(\frac{120523}{8400} - \frac{40\sqrt{5}}{3}\right). \tag{63} $$

4. Closed form evaluations

In this section, we shall establish the following thirty new closed-form evaluations for the generalized hypergeometric function ${}_3F_2(1/16)$, ${}_4F_3(1/16)$ and ${}_5F_4(1/16)$.

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ \frac{5}{4}, \frac{7}{4} \end{matrix} ; \frac{1}{16}\right] = 2\left(\frac{\sqrt{3}\pi}{3} - \frac{6\sqrt{5}}{5}\ln\varphi\right), \tag{64} $$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{3}{2}, 2 \\ \frac{7}{4}, \frac{9}{4} \end{matrix} ; \frac{1}{16}\right] = 10\left(\frac{\sqrt{3}\pi}{3} - 3 + \frac{6\sqrt{5}}{5}\ln\varphi\right), \tag{65} $$

$$ {}_3F_2\left[\begin{matrix} 1, 1, \frac{3}{2} \\ \frac{3}{4}, \frac{9}{4} \end{matrix} ; \frac{1}{16}\right] = 36\sqrt{5}\ln\varphi - 80 + \frac{70\sqrt{3}\pi}{9}, \tag{66} $$

$$ {}_3F_2\left[\begin{matrix} 1, \frac{3}{2}, 2 \\ \frac{5}{4}, \frac{11}{4} \end{matrix} ; \frac{1}{16}\right] = \frac{9}{4}\left(\frac{140\sqrt{3}\pi}{9} - \frac{20}{3} - 72\sqrt{5}\ln\varphi\right), \tag{67} $$

$$ {}_4F_3\left[\begin{matrix} 1, 1, \frac{3}{2}, \frac{7}{4} \\ \frac{3}{4}, \frac{5}{4}, \frac{11}{4} \end{matrix} ; \frac{1}{16}\right] = \frac{1}{3}\left(\frac{518\sqrt{3}\pi}{3} - \frac{224}{3} - \frac{4004\sqrt{5}}{5}\ln\varphi\right), \tag{68} $$

$$ {}_4F_3\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{9}{4} \\ \frac{5}{4}, \frac{7}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = \frac{9}{2}\left(\frac{148\sqrt{3}\pi}{3} - \frac{2572}{5} + \frac{1144\sqrt{5}}{5}\ln\varphi\right), \tag{69} $$

$$ {}_4F_3\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{9}{4} \\ \frac{3}{4}, \frac{11}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = \left(\frac{1774\sqrt{3}\pi}{5} - \frac{1296291}{350} + \frac{8244\sqrt{5}}{5}\ln\varphi\right), \tag{70} $$

$$ {}_4F_3\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{11}{4} \\ \frac{5}{4}, \frac{7}{4}, \frac{15}{4}\end{matrix}; \frac{1}{16}\right] = \frac{33}{2}\left(\frac{3548\sqrt{3}\pi}{45} - \frac{6073}{175} - \frac{1832\sqrt{5}}{5}\ln\varphi\right), \tag{71} $$

$$ {}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2} \\ \frac{7}{4}, \frac{9}{4}\end{matrix}; \frac{1}{16}\right] = 10(-\sqrt{3}\pi + 12 - 60\sqrt{5}\ln\varphi), \tag{72} $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{3}{2}, 2 \\ \frac{9}{4}, \frac{11}{4}\end{matrix}; \frac{1}{16}\right] = \frac{7}{2}\left(120\sqrt{5}\ln\varphi - 20\sqrt{3}\pi - 20\right), \tag{73} $$

$$ {}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2} \\ \frac{5}{4}, \frac{11}{4}\end{matrix}; \frac{1}{16}\right] = 196\sqrt{5}\ln\varphi - 42\sqrt{3}\pi + \frac{56}{3}, \tag{74} $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{3}{2}, 2 \\ \frac{7}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = \frac{135}{2}\left(\frac{628}{15} - \frac{56\sqrt{5}}{3}\ln\varphi - 4\sqrt{3}\pi\right), \tag{75} $$

$$ {}_4F_3\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{7}{4} \\ \frac{3}{4}, \frac{9}{4}, \frac{11}{4}\end{matrix}; \frac{1}{16}\right] = \frac{1}{3}\left(-350\sqrt{3}\pi - \frac{1960}{3} + 2380\sqrt{5}\ln\varphi\right), \tag{76} $$

$$ {}_4F_3\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{9}{4} \\ \frac{5}{4}, \frac{11}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = \frac{63}{2}\left(-20\sqrt{3}\pi + \frac{1276}{5} - 136\sqrt{5}\ln\varphi\right), \tag{77} $$

$$ {}_4F_3\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{9}{4} \\ \frac{5}{4}, \frac{7}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = \left(\frac{46296}{25} - 828\sqrt{5}\ln\varphi - \frac{882\sqrt{3}\pi}{5}\right), \tag{78} $$

$$ {}_4F_3\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{11}{4} \\ \frac{7}{4}, \frac{9}{4}, \frac{15}{4}\end{matrix}; \frac{1}{16}\right] = \frac{165}{2}\left(\frac{1612}{315} - \frac{196\sqrt{3}\pi}{15} + \frac{184\sqrt{5}}{3}\ln\varphi\right), \tag{79} $$

$$ {}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2} \\ \frac{3}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = -426\sqrt{3}\pi + \frac{22248}{5} - 1980\sqrt{5}\ln\varphi, \tag{80} $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{3}{2}, 2 \\ \frac{5}{4}, \frac{1}{4}\end{matrix}; \frac{1}{16}\right] = \frac{231}{2}\left(\frac{292}{35} + 88\sqrt{5}\ln\varphi - \frac{284\sqrt{3}\pi}{15}\right), \tag{81} $$

$$ {}_5F_4\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{7}{4}, \frac{9}{4} \\ \frac{3}{4}, \frac{5}{4}, \frac{11}{4}, \frac{13}{4}\end{matrix}; \frac{1}{16}\right] = \left(\frac{321272}{25} - 6972\sqrt{5}\ln\varphi - \frac{4914\sqrt{3}\pi}{5}\right), \tag{82} $$

$$ {}_5F_4\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{9}{4}, \frac{11}{4} \\ \frac{5}{4}, \frac{7}{4}, \frac{13}{4}, \frac{15}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{297}{2}\left(\frac{664\sqrt{5}}{3}\ln\varphi - \frac{1436}{21} - \frac{156\sqrt{3}\pi}{5}\right), \tag{83} $$

$$ {}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2} \\ \frac{9}{4}, \frac{11}{4}\end{matrix} ; \frac{1}{16}\right] = \left(70\sqrt{3}\pi + \frac{1120}{3} - 700\sqrt{5}\ln\varphi\right), \tag{84} $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{3}{2}, 2 \\ \frac{11}{4}, \frac{13}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{945}{2}\left(\frac{4\sqrt{3}\pi}{3} - \frac{108}{5} + \frac{40\sqrt{5}}{3}\ln\varphi\right), \tag{85} $$

$$ {}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2} \\ \frac{7}{4}, \frac{13}{4}\end{matrix} ; \frac{1}{16}\right] = 198\sqrt{3}\pi - \frac{1145091}{560} + 900\sqrt{5}\ln\varphi, \tag{86} $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{3}{2}, 2 \\ \frac{9}{4}, \frac{15}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{1155}{2}\left(\frac{44\sqrt{3}\pi}{15} - \frac{6773}{4200} - \frac{40\sqrt{5}}{3}\ln\varphi\right), \tag{87} $$

$$ {}_4F_3\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{9}{4} \\ \frac{5}{4}, \frac{11}{4}, \frac{13}{4}\end{matrix} ; \frac{1}{16}\right] = 3780\sqrt{5}\ln\varphi - \frac{1279491}{200} + \frac{2142\sqrt{3}\pi}{5}, \tag{88} $$

$$ {}_4F_3\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{11}{4} \\ \frac{7}{4}, \frac{13}{4}, \frac{15}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{1485}{2}\left(\frac{68\sqrt{3}\pi}{15} + \frac{38587}{2100} - 40\sqrt{5}\ln\varphi\right), \tag{89} $$

$$ {}_4F_3\left[\begin{matrix}1, 1, \frac{3}{2}, \frac{7}{4} \\ \frac{3}{4}, \frac{11}{4}, \frac{13}{4}\end{matrix} ; \frac{1}{16}\right] = 966\sqrt{3}\pi - \frac{1324291}{80} + 10500\sqrt{5}\ln\varphi, \tag{90} $$

$$ {}_4F_3\left[\begin{matrix}1, \frac{3}{2}, 2, \frac{9}{4} \\ \frac{5}{4}, \frac{13}{4}, \frac{15}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{2079}{2}\left(\frac{92\sqrt{3}\pi}{15} + \frac{53707}{1400} - \frac{200\sqrt{5}}{3}\ln\varphi\right), \tag{91} $$

$$ {}_3F_2\left[\begin{matrix}1, 1, \frac{3}{2} \\ \frac{11}{4}, \frac{13}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{1413891}{160} - 378\sqrt{3}\pi - 6300\sqrt{5}\ln\varphi, \tag{92} $$

$$ {}_3F_2\left[\begin{matrix}1, \frac{3}{2}, 2 \\ \frac{13}{4}, \frac{15}{4}\end{matrix} ; \frac{1}{16}\right] = \frac{10395}{2}\left(\frac{40\sqrt{5}}{3}\ln\varphi - \frac{4\sqrt{3}\pi}{5} - \frac{83947}{8400}\right). \tag{93} $$

Proof. In order to establish the results (64)-(93), we shall use the following general results recorded in [11].

$$ {}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q\end{matrix} ; z\right] + {}_{q+1}F_q\left[\begin{matrix}a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q\end{matrix} ; -z\right] $$
$$ = 2\ {}_{2q+2}F_{2q+1}\left[\begin{matrix}\frac{a_1}{2}, \frac{a_1}{2}+\frac{1}{2}, \ldots, \frac{a_{q+1}}{2}, \frac{a_{q+1}}{2}+\frac{1}{2} \\ \frac{1}{2}, \frac{b_1}{2}, \frac{b_1}{2}+\frac{1}{2}, \ldots, \frac{b_q}{2}, \frac{b_q}{2}+\frac{1}{2}\end{matrix} ; z^2\right] \tag{94} $$

and

$$
{}_{q+1}F_q\left[\begin{array}{c} a_1, a_2, \ldots, a_{q+1} \\ \\ b_1, b_2, \ldots, b_q \end{array} ; z\right] - {}_{q+1}F_q\left[\begin{array}{c} a_1, a_2, \ldots, a_{q+1} \\ b_1, b_2, \ldots, b_q \end{array} ; -z\right]
$$

$$
= \frac{2a_1 a_2 \ldots a_{q+1}}{b_1 b_2 \ldots b_q}\, {}_{2q+2}F_{2q+1}\left[\begin{array}{c} \frac{a_1}{2}+\frac{1}{2}, \frac{a_1}{2}+1, \ldots, \frac{a_{q+1}}{2}+\frac{1}{2}, \frac{a_{q+1}}{2}+1 \\ \\ \frac{3}{2}, \frac{b_1}{2}+\frac{1}{2}, \frac{b_1}{2}+1, \ldots, \frac{b_{q+1}}{2}+\frac{1}{2}, \frac{b_q}{2}+1 \end{array} ; z^2\right]. \tag{95}
$$

It is not out of place to mention here that the results, (94) and (95) can be established by resolving a generalized hypergeometric function

$$
{}_{q+1}F_q\left[\begin{array}{c} a_1, a_2, \ldots, a_{q+1} \\ \\ b_1, b_2, \ldots, b_q \end{array} ; \pm z\right]
$$

into even and odd components and making use of the following identities:

$$
(a)_{2n} = 2^{2n}\left(\frac{a}{2}\right)_n \left(\frac{a}{2}+\frac{1}{2}\right)_n
$$

and

$$
(a)_{2n+1} = a2^{2n}\left(\frac{a}{2}+\frac{1}{2}\right)_n \left(\frac{a}{2}+1\right)_n.
$$

Therefore, for the derivation of the results (64) and (65), we substitute the results (34) and (49) by letting $q = 1$ and substituting $a_1 = 1, a_2 = 2$, $b_1 = 5/2$ and $z = 1/4$ in (94) and (95) respectively, after some simplification, we obtain the results (64) and (65) respectively. Similarly, the other results (66) to (93) can be established by choosing the appropriate parameters and making use of the results (35) to (63) in (94) and (95) respectively. We however omit the details.

We conclude this section by remarking that the results (64) to (93) have been verified by using MAPLE software. □

5. Concluding remark

In this paper, as many as thirty new-closed form evaluations of the generalized hypergeometric functions ${}_{q+1}F_q(z)$ for $q = 2, 3$ with arguments $\frac{1}{16}$ have been established. This is achieved by means of separating the generalized hypergeometric function ${}_{q+1}F_q(z)$ into even and odd components together with the use of the several known results of interesting series involving reciprocals of the non-central binomial coefficients obtained earlier by Zhang and Ji [22]. We believe that the results established in this paper have not appeared in the literature before and represent a definite contribution to the theory of the generalized hypergeometric function. It is hoped that the results could be of potential use in the area of mathematics, statistics and mathematical physics.

Acknowledgements. The work of D. Lim was partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) NRF-2021R1C1C1010902.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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