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Short Note

On signs of several Toeplitz–Hessenberg determinants whose elements contain central Delannoy numbers

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Dedicated to Dr. Professor Aliakbar Montazer Haghighi at Prairie View A&M University

Abstract: In the paper, by virtue of Wronski's formula and Kaluza's theorem for the power series and its reciprocal, and with the aid of the logarithmic convexity of a sequence constituted by central Delannoy numbers, the authors present negativity of several Toeplitz–Hessenberg determinants whose elements contain central Delannoy numbers and combinatorial numbers.

Keywords: sign; negativity; logarithmic convexity; Toeplitz–Hessenberg determinant; central Delannoy number; factorial; Wronski's formula; Kaluza's theorem

AMS Subject classification: 11B83, 11C20, 15A15, 15B05, 26A51

1. Introduction

In combinatorial analysis, the Delannoy numbers, denoted by $D(p,q)$, are the number of lattice paths from $(0,0)$ to (p,q) in which only east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ steps are allowed. They can be analytically generated by

$$\frac{1}{1-x-y-xy}=\sum_{p,q=0}^{\infty}D(p,q)x^py^q.$$

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When $p=q=r$, we call $D(r,r)$, shortly written as $D(r)$, central Delannoy numbers, which are the number of "king walks" from the $(0,0)$ corner of an $r\times r$ square to the upper right corner (r,r). Central Delannoy numbers $D(r)$ can be analytically generated by

$$\frac{1}{\sqrt{1-6x+x^2}}=\sum_{r=0}^{\infty}D(r)x^r=1+3x+13x^2+63x^3+\cdots,\quad |x|<3-2\sqrt{2}\,. \tag{1}$$

A lower (respectively upper) Hessenberg matrix is an $n\times n$ matrix $H_n=(h_{ij})_{1\le i,j\le n}$, where $h_{ij}=0$ for all pairs (i,j) such that $i+1<j$ (respectively $j+1<i$). See [20, Chapter 10]. A Toeplitz matrix is an $n\times n$ matrix $T_n=(t_{k,j})_{0\le k,j\le n-1}$, where $t_{k,j}=t_{k-j}$, that is, a matrix of the form

$$\begin{pmatrix}
t_0 & t_{-1} & t_{-2} & \cdots & t_{-n+3} & t_{-n+2} & t_{-n+1}\\
t_1 & t_0 & t_{-1} & \cdots & t_{-n+4} & t_{-n+3} & t_{-n+2}\\
t_2 & t_1 & t_0 & \cdots & t_{-n+5} & t_{-n+4} & t_{-n+3}\\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots\\
t_{n-3} & t_{n-4} & t_{n-5} & \cdots & t_0 & t_{-1} & t_{-2}\\
t_{n-2} & t_{n-3} & t_{n-4} & \cdots & t_1 & t_0 & t_{-1}\\
t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_2 & t_1 & t_0
\end{pmatrix}.$$

See [4]. For our convenience, we call the determinants $|H_n|$ and $|T_n|$ the Hessenberg determinant and the Toeplitz determinant, respectively. If an $n\times n$ matrix M_n is both a Hessenberg matrix and a Toeplitz matrix, we call its determinant $|M_n|$ the Toeplitz–Hessenberg determinant.

In [13, Theorem 1.1], among other things, central Delannoy numbers $D(r)$ for $r\in\mathbb{N}$ were proved to satisfy

$$\begin{vmatrix}
D(1) & 1 & 0 & \cdots & 0 & 0 & 0\\
D(2) & D(1) & 1 & \cdots & 0 & 0 & 0\\
D(3) & D(2) & D(1) & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
D(r-2) & D(r-3) & D(r-4) & \cdots & D(1) & 1 & 0\\
D(r-1) & D(r-2) & D(r-3) & \cdots & D(2) & D(1) & 1\\
D(r) & D(r-1) & D(r-2) & \cdots & D(3) & D(2) & D(1)
\end{vmatrix}=-\frac{1}{6^r}\sum_{\ell=1}^{r}(-1)^\ell 6^{2\ell}\frac{(2\ell-3)!!}{(2\ell)!!}\binom{\ell}{r-\ell}. \tag{2}$$

For more information and new results on central Delannoy numbers $D(n)$, please refer to [1, 2, 7–10, 15, 16, 18] and closely related references therein.

In this paper, we are interested in several Toeplitz–Hessenberg determinants similar to the one in the left hand side of (2). We will derive negativity of several Toeplitz–Hessenberg determinants whose elements contain the products of the (rising) factorials and central Delannoy numbers $D(r)$.

2. Lemmas

In this paper, we need the following lemmas.

Lemma 1 (Wronski's formula). *If $a_0 \neq 0$ and $P(x) = a_0 + a_1x + a_2x^2 + \cdots$ is a formal series, then the coefficients of the reciprocal series $\frac{1}{P(x)} = b_0 + b_1x + b_2x^2 + \cdots$ are given by*

$$b_r = \frac{(-1)^r}{a_0^{r+1}} \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{r-2} & a_{r-3} & a_{r-4} & a_{r-5} & \cdots & a_1 & a_0 & 0 \\ a_{r-1} & a_{r-2} & a_{r-3} & a_{r-4} & \cdots & a_2 & a_1 & a_0 \\ a_r & a_{r-1} & a_{r-2} & a_{r-3} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}, \quad r = 1, 2, \dots. \tag{3}$$

Wronski's formula (3) in Lemma 1 can be found in [3, p. 17, Theorem 1.3], [5, p. 347], [14, Lemma 2.4], [17, Lemma 2.3], [19, Section 2], and the paper [21].

Lemma 2 (Kaluza's theorem). *Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots$ be a formal power series over the field of real numbers such that the sequence a_r for $r = 0, 1, 2, \dots$ is positive and logarithmically convex. If $\frac{1}{P(x)} = b_0 + b_1x + b_2x^2 + \cdots$, then $b_r < 0$ for $r = 1, 2, \dots$.*

Kaluza's theorem stated in Lemma 2 can be found in [3, p. 13, Problem 6] and the paper [6].

Lemma 3 ([13, Corollary 1.2]). *The sequence $r!D(r)$ for $r \geq 0$ is logarithmically convex.*

3. Negativity of several Toeplitz–Hessenberg determinants

In this section, we present negativity of several Toeplitz–Hessenberg determinants whose elements contain the products of the (rising) factorials and central Delannoy numbers $D(r)$ generated in (1).

Theorem 1. *For $z \in \mathbb{C}$ and $r \in \{0\} \cup \mathbb{N}$, let $(z)_r$ stand for the rising factorial*

$$(z)_r = \prod_{\ell=0}^{r-1}(z+\ell) = \begin{cases} z(z+1)\cdots(z+r-1), & r \geq 1; \\ 1, & r = 0. \end{cases}$$

For $m \in \{0\} \cup \mathbb{N}$ and $r \in \mathbb{N}$, we have

$$(-1)^r\begin{vmatrix}
(m+1)_1D(m+1) & (m+1)_0D(m) & 0 & \cdots & 0 & 0\\
(m+1)_2D(m+2) & (m+1)_1D(m+1) & (m+1)_0D(m) & \cdots & 0 & 0\\
(m+1)_3D(m+3) & (m+1)_2D(m+2) & (m+1)_1D(m+1) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
(m+1)_{r-2}D(m+r-2) & (m+1)_{r-3}D(m+r-3) & (m+1)_{r-4}D(m+r-4) & \cdots & (m+1)_0D(m) & 0\\
(m+1)_{r-1}D(m+r-1) & (m+1)_{r-2}D(m+r-2) & (m+1)_{r-3}D(m+r-3) & \cdots & (m+1)_1D(m+1) & (m+1)_0D(m)\\
(m+1)_rD(m+r) & (m+1)_{r-1}D(m+r-1) & (m+1)_{r-2}D(m+r-2) & \cdots & (m+1)_2D(m+2) & (m+1)_1D(m+1)
\end{vmatrix} < 0. \quad (4)$$

In particular, for $r\in\mathbb{N}$, we have

$$(-1)^r\begin{vmatrix}
1!D(1) & 0!D(0) & 0 & \cdots & 0 & 0 & 0\\
2!D(2) & 1!D(1) & 0!D(0) & \cdots & 0 & 0 & 0\\
3!D(3) & 2!D(2) & 1!D(1) & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
(r-2)!D(r-2) & (r-3)!D(r-3) & (r-4)!D(r-4) & \cdots & 1!D(1) & 0!D(0) & 0\\
(r-1)!D(r-1) & (r-2)!D(r-2) & (r-3)!D(r-3) & \cdots & 2!D(2) & 1!D(1) & 0!D(0)\\
r!D(r) & (r-1)!D(r-1) & (r-2)!D(r-2) & \cdots & 3!D(3) & 2!D(2) & 1!D(1)
\end{vmatrix} < 0. \quad (5)$$

Proof. For fixed $m\in\{0\}\cup\mathbb{N}$, let

$$a_r=(r+m)!D(r+m),\quad r\ge 0. \tag{6}$$

By virtue of Wronski's formula (3) in Lemma 1, we obtain

$$b_r=\frac{(-1)^r}{[m!D(m)]^{r+1}}\begin{vmatrix}
(m+1)!D(m+1) & m!D(m) & 0 & \cdots & 0 & 0\\
(m+2)!D(m+2) & (m+1)!D(m+1) & m!D(m) & \cdots & 0 & 0\\
(m+3)!D(m+3) & (m+2)!D(m+2) & (m+1)!D(m+1) & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
(m+r-2)!D(m+r-2) & (m+r-3)!D(m+r-3) & (m+r-4)!D(m+r-4) & \cdots & m!D(m) & 0\\
(m+r-1)!D(m+r-1) & (m+r-2)!D(m+r-2) & (m+r-3)!D(m+r-3) & \cdots & (m+1)!D(m+1) & m!D(m)\\
(m+r)!D(m+r) & (m+r-1)!D(m+r-1) & (m+r-2)!D(m+r-2) & \cdots & (m+2)!D(m+2) & (m+1)!D(m+1)
\end{vmatrix}$$

$$
= \frac{(-1)^r}{(m!)^2 D^{r+1}(m)}
\begin{vmatrix}
(m+1)_1 D(m+1) & (m+1)_0 D(m) & 0 & \cdots & 0 & 0 \\
(m+1)_2 D(m+2) & (m+1)_1 D(m+1) & (m+1)_0 D(m) & \cdots & 0 & 0 \\
(m+1)_3 D(m+3) & (m+1)_2 D(m+2) & (m+1)_1 D(m+1) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(m+1)_{r-2} D(m+r-2) & (m+1)_{r-3} D(m+r-3) & (m+1)_{r-4} D(m+r-4) & \cdots & (m+1)_0 D(m) & 0 \\
(m+1)_{r-1} D(m+r-1) & (m+1)_{r-2} D(m+r-2) & (m+1)_{r-3} D(m+r-3) & \cdots & (m+1)_1 D(m+1) & (m+1)_0 D(m) \\
(m+1)_r D(m+r) & (m+1)_{r-1} D(m+r-1) & (m+1)_{r-2} D(m+r-2) & \cdots & (m+1)_2 D(m+2) & (m+1)_1 D(m+1)
\end{vmatrix}
$$

for $m \in \{0\} \cup \mathbb{N}$ and $r \in \mathbb{N}$.

From Lemma 3, it follows that the sequence a_r for $r \geq 0$ defined by (6) is logarithmically convex. Utilizing Lemma 2 and logarithmic convexity of the sequence (6) arrives at negativity of the sequence b_r, that is, negativity in (4) is valid for $m \in \{0\} \cup \mathbb{N}$ and $r \in \mathbb{N}$.

When taking $m = 0$ in the inequality (4), we derive (5) readily. The proof of Theorem 1 is complete. □

Theorem 2. *For $r \in \mathbb{N}$, when $a_0 > \frac{1}{3}$, we have*

$$
(-1)^r
\begin{vmatrix}
0!D(0) & a_0 & 0 & \cdots & 0 & 0 & 0 \\
1!D(1) & 0!D(0) & a_0 & \cdots & 0 & 0 & 0 \\
2!D(2) & 1!D(1) & 0!D(0) & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
(r-3)!D(r-3) & (r-4)!D(r-4) & (r-5)!D(r-5) & \cdots & 0!D(0) & a_0 & 0 \\
(r-2)!D(r-2) & (r-3)!D(r-3) & (r-4)!D(r-4) & \cdots & 1!D(1) & 0!D(0) & a_0 \\
(r-1)!D(r-1) & (r-2)!D(r-2) & (r-3)!D(r-3) & \cdots & 2!D(2) & 1!D(1) & 0!D(0)
\end{vmatrix} < 0. \quad (7)
$$

Proof. Let

$$a_r = (r-1)!D(r-1), \quad r \geq 1.$$

Since

$$a_2 a_0 - a_1^2 = 1!D(1)a_0 - [0!D(0)]^2 > 0$$

is equivalent to

$$a_0 > \frac{0!D(0)}{1!D(1)} = \frac{1}{3},$$

considering Lemma 3, the sequence a_r for $r \geq 0$ is logarithmically convex.

By virtue of Wronski's formula (3) in Lemma 1, we obtain

$$b_r=\frac{(-1)^r}{a_0^{r+1}}\begin{vmatrix}
0!D(0) & a_0 & 0 & 0 & \cdots & 0 & 0 & 0\\
1!D(1) & 0!D(0) & a_0 & 0 & \cdots & 0 & 0 & 0\\
2!D(2) & 1!D(1) & 0!D(0) & a_0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
(r-3)!D(r-3) & (r-4)!D(r-4) & (r-5)!D(r-5) & (r-6)!D(r-6) & \cdots & 0!D(0) & a_0 & 0\\
(r-2)!D(r-2) & (r-3)!D(r-3) & (r-4)!D(r-4) & (r-5)!D(r-5) & \cdots & 1!D(1) & 0!D(0) & a_0\\
(r-1)!D(r-1) & (r-2)!D(r-2) & (r-3)!D(r-3) & (r-4)!D(r-4) & \cdots & 2!D(2) & 1!D(1) & 0!D(0)
\end{vmatrix}.$$

Utilizing Lemma 2 and logarithmic convexity of the sequence a_r for $r\ge0$ reveals negativity of the sequence b_r, that is, the inequality (7) is valid. The proof of Theorem 2 is complete. □

Remark 1. This paper is a companion of the articles [11, 12].

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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