Communications in Combinatorics and Optimization
Vol. 8, No. 2 (2023), pp. 423-430
DOI: 10.22049/CCO.2022.27916.1394

Research Article

Coalition graphs

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Received: 9 July 2022; Accepted: 29 August 2022
Published Online: 3 September 2022

Abstract: A coalition in a graph $G = (V, E)$ consists of two disjoint sets V_1 and V_2 of vertices, such that neither V_1 nor V_2 is a dominating set, but the union $V_1 \cup V_2$ is a dominating set of G. A coalition partition in a graph G of order $n = |V|$ is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i either is a dominating set consisting of a single vertex of degree $n - 1$, or is not a dominating set but forms a coalition with another set V_j. Associated with every coalition partition π of a graph G is a graph called the coalition graph of G with respect to π, denoted $CG(G, \pi)$, the vertices of which correspond one-to-one with the sets $V_1, V_2, \ldots, V_k$ of π and two vertices are adjacent in $CG(G, \pi)$ if and only if their corresponding sets in π form a coalition. In this paper, we initiate the study of coalition graphs and we show that every graph is a coalition graph.

Keywords: Dominating set; Coalition; Independent dominating set

AMS Subject classification: 05C69

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1. Introduction

A coalition is generally defined as a temporary alliance of two or more parties to work together to achieve some common objective. Coalitions are often formed in governments when no political party achieves an absolute majority after an election, but a cooperative agreement between two or more political parties is adequate to established a majority and form a government, with members of the coalition serving on a cabinet. But coalitions can just as well be formed among various trade unions in order to improve employee wages, benefits, and safer working conditions.

In mathematics, however, the term *coalition* has many different definitions, such as the following sample of three.

In [6], Monsuur and Janssen define a *coalition* in a graph $G = (V, E)$ to consist of a pair of vertex-disjoint sets $S = M \cup P$, the vertices in M are called *members* of the coalition, and the vertices in P are called *participants*. The coalition S must satisfy the following three conditions:
(i) every vertex $w \in P$ is adjacent to at least one vertex in M,
(ii) no vertex in $V - S$ is adjacent to a vertex in M, and
(iii) the induced subgraph $G[M]$ is connected.

In [7] Peleg defines a *coalition* in a graph $G = (V, E)$ to be a set $M \subset V$, which is said to have a *local majority* over a vertex $w \in V - M$ if more than half of the vertices in the set consisting of w and all of the vertices adjacent to w are in M. If a set M has a local majority over every vertex in $V - M$, then M is called a *monopoly* (see also Bermond, Bond, Peleg, and Perennes in [1]).

In [8] Voice, Polukarov, and Jennings consider *coalition structure generation*, which is defined as follows. You are given a set N of order n and a valuation function $v : \mathcal{P}(N) \to \mathcal{R}$, where $\mathcal{P}(N)$ denotes the power set of N, and the problem is to create a partition $\pi = \{N_1, N_2, \ldots, N_m\}$ of N into *coalitions* N_i, which maximizes the sum $\Sigma_{i=1}^{m} v(N_i)$.

Motivated by the idea that the union of two sets can have a property that neither set has, the authors introduced a different kind of coalition in 2020 [3] and studied it further in [2, 4, 5].

In order to understand these types of coalitions, we will need a few definitions. Let $G = (V, E)$ be a graph and $\overline{G}$ be the complement of G. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \,|\, uv \in E\}$ and its closed neighborhood is $N[v] = N(v) \cup \{v\}$. Each vertex $u \in N(v)$ is called a *neighbor* of v, and $|N(v)|$ is the *degree* of v, denoted $\deg(v)$. In a graph G of order $n = |V|$, a vertex of degree $n - 1$ is called a *dominating* vertex, while a vertex of degree 0 is an *isolate*. For a set S of vertices, we denote the *subgraph induced by* S by $G[S]$. We denote the family of paths, cycles, and complete graphs of order n by P_n, C_n, and K_n, respectively, and the complete bipartite graph having r vertices in one partite set and s vertices in the other by $K_{r,s}$. The *corona* $G \circ K_1$ of a graph G is the graph obtained from G by adding for each vertex $v \in V$ a new vertex v' and the edge vv'.

A set $S \subseteq V$ is a *dominating set* of a graph G if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ equals the minimum cardinality

of any dominating set of G. A dominating set of G with cardinality $\gamma(G)$ is called a *γ-set* of G.
A non-empty subset $X \subseteq V$ is called a *singleton set* if $|X| = 1$ and a *non-singleton set* if $|X| \geq 2$. For an integer k, we use the standard notation $i \in [k]$ to mean that i is an integer and $1 \leq i \leq k$.

2. $\mathcal{P}$-Coalitions in Graphs

Let $\mathcal{P}$ be any property of a set $S \subset V$ of vertices in a graph G, such as (i) $\mathcal{P}_1$: $G[S]$ is a connected graph, (ii) $\mathcal{P}_2$: $G[S]$ contains a cycle, (iii) $\mathcal{P}_3$: $G[S]$ contains a vertex of degree at least k, (iv) $\mathcal{P}_4$: S is a dominating set of G.
A set with property $\mathcal{P}$ is called a $\mathcal{P}$-set. A *$\mathcal{P}$-coalition* consists of two disjoint sets V_i and V_j, neither of which is a $\mathcal{P}$-set but whose union $V_i \cup V_j$ is a $\mathcal{P}$-set. Thus, in a $\mathcal{P}_1$-coalition, or a *connected coalition*, the subgraph $G[V_i \cup V_j]$ induces a connected subgraph, in a $\mathcal{P}_2$-coalition, or a *cyclic coalition*, the subgraph $G[V_i \cup V_j]$ contains a cycle, and in a $\mathcal{P}_3$-coalition, or a *degree-k coalition*, the subgraph $G[V_i \cup V_j]$ contains a vertex of degree at least k.
A *$\mathcal{P}$-coalition partition* is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that no set V_i is a $\mathcal{P}$-set, but for every V_i there exists a V_j such that $V_i \cup V_j$ is a $\mathcal{P}$-set.
For the $\mathcal{P}$-coalitions in this paper, we consider the property $\mathcal{P}_4$ of being a dominating set.

Definition 1. A *(dominating) coalition* in a graph G consists of two disjoint sets of vertices V_1 and V_2, neither of which is a dominating set but the union $V_1 \cup V_2$ is a dominating set. We say that the sets V_1 and V_2 *form a coalition*, and are *coalition partners*.

Since we will focus only on dominating coalitions, in the remainder of this paper whenever we say *coalition* we will mean a *dominating coalition*.
In order to study the $\mathcal{P}_4$-coalition partitions of interest in this paper, we modify the definition of a $\mathcal{P}_4$-coalition partition slightly to allow for singleton dominating sets.

Definition 2. A *coalition partition*, henceforth called a *c-partition*, in a graph G is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i of π is either a singleton dominating set, or is not a dominating set but forms a coalition with another non-dominating set in π.

It is proven in [3] that every graph has a c-partition. Note that if G has no dominating vertex, then no set V_i in a c-partition is a dominating set, and hence must form a coalition with another set V_j in the partition. By the *singleton partition* of a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$, we mean the partition $\pi_1 = \{V_1, V_2, \ldots, V_n\}$, where for $i \in [n]$, $V_i = \{v_i\}$. We shall use the notation π_1 throughout to denote the singleton partition of G.
For example, consider the cycle $C_5 = (u_1, u_2, u_3, u_4, u_5, u_1)$. The singleton partition π_1 of C_5 is a c-partition, since every pair of non-adjacent vertices in C_5 is a dominating

set of C_5, implying that every singleton set of π_1 forms a coalition with some other singleton set. Another c-partition of C_5 is $\pi_2 = \{\{u_1, u_2\}, \{u_3, u_4\}, \{u_5\}\}$. None of the three sets in π_2 is a dominating set, but every set forms a dominating set with each of the other two sets. Also, note that for the complete graph K_n, the singleton partition is its only c-partition. Its complement, the empty graph $\overline{K}_n$ for $n \geq 2$, may have more than one c-partition, but it has only one type of c-partition, namely a partition of the vertices into two non-empty sets.

3. Coalition Graphs

In the paper [3] introducing coalitions, the authors suggest several related areas for future study, one of which is a graph naturally associated with a c-partition π of a graph G defined as follows.

Definition 3. Let G be a graph with a c-partition $\pi = \{V_1, V_2, \dots, V_k\}$. The *coalition graph* $CG(G, \pi)$ of G is the graph with vertex set $V_1, V_2, \dots, V_k$, corresponding one-to-one with the sets of π, and two vertices V_i and V_j are adjacent in $CG(G, \pi)$ if and only if the sets V_i and V_j are coalition partners in π, that is, neither V_i nor V_j is a dominating set of G, but $V_i \cup V_j$ is a dominating set of G.

For ease of discussion, we use V_i to simultaneously denote a set in G and its corresponding vertex in $CG(G, \pi)$, depending on the context to make it clear. Clearly, if V_iV_j is an edge in $CG(G, \pi)$, then $|V_i| + |V_j| \geq \gamma(G)$.

The complete graph K_n has exactly one c-partition, namely, its singleton partition π_1, for which $CG(K_n, \pi_1) \simeq \overline{K}_n$. That is, the coalition graph of K_n with its singleton partition is isomorphic to the complement $\overline{K}_n$ of K_n. The singleton c-partition π_1 of the cycle C_5 gives $CG(C_5, \pi_1) \simeq C_5$, while the c-partition $\pi_2 = \{\{u_1, u_2\}, \{u_3, u_4\}, \{u_5\}\}$ of C_5 results in $CG(C_5, \pi_2) \simeq K_3$.

In Section 4, we show that every graph G is a coalition graph of some graph H and some c-partition of H. In Section 5, we suggest several avenues for future research.

4. Every Graph is a Coalition Graph

Since every graph G has at least one c-partition π, every graph has at least one associated coalition graph $H = CG(G, \pi)$ and can have many associated coalition graphs, depending on the number of c-partitions that it has. In this section, we show that, conversely, every graph G is a coalition graph.

Theorem 1. *For every graph G, there is a graph H and some c-partition π of H, such that $CG(H, \pi) \simeq G$, that is, every graph G is a coalition graph.*

Proof. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\} \cup \{w_1, w_2, \dots, w_t\}$, where each vertex v_i has degree at least 1, and each vertex w_i is an isolate. Let $G' =$

$G - \{w_1, w_2, \ldots, w_t\}$, $E(G') = E(G) = \{e_1, e_2, \ldots, e_m\}$, and $E(\overline{G}) = \{c_1, c_2, \ldots, c_p\}$, where $m + p = \binom{n+t}{2}$. We will construct a graph H and a c-partition π of H, such that the coalition graph of H and π is isomorphic to G, that is, $CG(H, \pi) \simeq G$.
We begin the construction of H with a complete graph K_n, whose vertices are labeled $\{v_1, v_2, \ldots, v_n\}$ corresponding to the n non-isolated vertices of G. We will refer to these as the n *base vertices* of H, since we will add additional vertices to H. Our partition π of H begins as the singleton partition, and we denote the sets of π as $V_i = \{v_i\}$, for each $v_i \in V(G')$. We will add to the sets in the partition as we build H.

For each edge $e_i \in E(G)$, we proceed as follows. Let $e_i = v_j v_k$.

- Add two new vertices $u_{i,j}$ and $u_{i,k}$ to H.
- Add edges between $u_{i,j}$ and all of the base vertices except for v_j. Similarly, add edges between $u_{i,k}$ and all of the base vertices except for v_k. Thus, each of the vertices $u_{i,j}$ and $u_{i,k}$ has degree $n-1$ in H.
- Add vertex $u_{i,j}$ to the set V_k and add vertex $u_{i,k}$ to the set V_j.

For each edge $c_i = v_j v_k \in E(\overline{G})$, we proceed as follows.

- Add one new vertex $x_{j,k}$ to H.
- Add edges between $x_{j,k}$ and all of the base vertices except for v_j and v_k. Thus, vertex $x_{j,k}$ has degree $n-2$ in H.
- Add the vertex $x_{j,k}$ to any set of the partition π except for V_j and V_k.

Finally, if w_i is an isolate in G, then it necessarily corresponds to a dominating vertex in H. Thus, we conclude our construction of H by adding t vertices to H labeled $w_1, w_2, \ldots, w_t$ corresponding to the isolates in G. Then add edges such that each of these vertices is a dominating vertex in H. We extend π so that each w_i is in a singleton set $W_i = \{w_i\}$ of π for $i \in [t]$.
At this point in the construction, notice several things:

- Every edge in H is incident either with at least one base vertex or with a dominating vertex.
- No set V_j in the partition is a dominating set of H. This follows since the degree of v_j in G is at least one, in which case there exists at least one edge $e_i = v_j v_k$ incident to v_j, and therefore no vertex in V_j can dominate $u_{i,j}$ in H.
- If $v_j v_k$ is an edge in G, then V_j and V_k form a coalition in H because the base vertices v_j and v_k collectively dominate every vertex in H.
- If $v_j v_k$ is not an edge in G, then V_j and V_k do not form a coalition in H because of the vertex $x_{j,k}$ which is not adjacent to either v_j or v_k, and is thus not dominated by any vertex in $V_j \cup V_k$.

- Each w_i is a dominating vertex in H, so w_i is an isolate in G.

It follows that $\pi = \{V_1, V_2, \ldots, V_n, W_1, W_2, \ldots, W_t\}$ is a c-partition of H and that $CG(H, \pi) \simeq G$. □

Notice that for a given graph $G = (V, E)$, having n non-isolates and t isolates, and size $m = |E|$, where $m + \overline{m} = \binom{n}{2}$, the graph H constructed in the proof of Theorem 1 has:
(i) order $n(H) = n + m + \binom{n}{2} + t$, and
(ii) size $m(H) = \binom{n}{2} + 2m(n-1) + \overline{m}(n-2) + t(n(H)-1)$.
This raises the question: does there exist a graph H^* of smaller order n^* and size m^* with a c-partition π^* such that $CG(H^*, \pi^*) \simeq G$?
A partial answer to this question is given in the following corollary to Theorem 1.

Corollary 1. *For every graph G, there is a graph H and a c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of H, such that $CG(H, \pi) \simeq G$ and every set $V_i \in \pi$ is an independent set.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_n, W_1, W_2, \ldots, W_t\}$ be the c-partition of the graph H constructed in the proof of Theorem 1, such that $CG(H, \pi) \simeq G$.
Let H' be the graph constructed from the graph H by deleting all edges between any two vertices in the same set V_i in π, for all $1 \leq i \leq k$. Let $\pi' = \{V_1', V_2', \ldots, V_k'\}$ be the vertex partition of H', where for all $1 \leq i \leq k$, $V_i' = V_i$. It is easy to see that $V_i \cup V_j$ forms a coalition in H if and only if $V_i' \cup V_j'$ forms a coalition in H'. Therefore, $CG(H', \pi') \simeq G$. □

A *proper coloring* of a graph $G = (V, E)$ is a vertex partition $\pi = \{V_1, V_2, \ldots, V_k\}$ in which every set V_i is an independent set. Corollary 1 can be restated in terms of proper colorings of graphs.

Corollary 2. *Every graph G is isomorphic to a coalition graph that can be obtained from a c-partition which is a proper coloring of some graph H.*

5. Open Problems and Future Research

Several interesting classes of graphs emerge from the study of coalition graphs. Here we explore some directions for research in this area.

5.1. Coalition Graphs of Singleton Partitions

A coalition graph $H = CG(G, \pi_1)$ obtained from the singleton c-partition π_1 of a graph G is called the *singleton coalition graph* or *SC-graph* of G.

Problem 1. Characterize the class of SC-graphs.

As we have seen, for some graphs the singleton partition π_1 is a c-partition, while for others it is not. A graph whose singleton partition is a c-partition is called a *singleton-partition graph*, or simply an *SP-graph*. Graphs whose singleton partition is not a c-partition are called *XSP-graphs*. If a graph G is an XSP-graph, then G has a non-dominating vertex x which is not in a dominating set of cardinality 2 with any other vertex. Conversely, if a graph G is an SP-graph, then every vertex $v \in V$ is either a dominating vertex or is in a dominating set of cardinality two with at least one other vertex w. Thus, for every SP-graph G, $1 \leq \gamma(G) \leq 2$. We note, for example, that 14 of the 34 graphs of order $n = 5$ are SP-graphs.

Problem 2. Characterize the class of SP-graphs.

Consider a (singleton) coalition graph chain to be a sequence of graphs $G_1, G_2, G_3, \ldots$ such that the singleton coalition graph of G_i is G_{i+1}. Do arbitrarily long singleton coalition graph chains exist? Here, for example, is a chain of singleton coalition graphs for graphs of order $n = 4$, where the chain terminates at $\overline{K}_4$ since $\overline{K}_4$ is not an SP-graph:

$$P_4 \rightarrow C_4 \rightarrow K_4 \rightarrow \overline{K}_4.$$

Problem 3. Investigate singleton coalition graph chains.

A graph G is called a *self-coalition graph* if $CG(G, \pi_1) \simeq G$. We noted earlier that $CG(C_5, \pi_1) \simeq C_5$. Thus, C_5 is a self-coalition graph, and so is the complete bipartite graph $K_{r,s}$, since $CG(K_{r,s}, \pi_1) \simeq K_{r,s}$. A characterization of self-coalition graphs is given by the authors in [5].

A graph G is called a *complementary coalition graph* if $CG(G, \pi_1) \simeq \overline{G}$. We note that there are five graphs G of order $n = 4$ and nine graphs G of order $n = 5$ (including the cycle C_5) for which $CG(G, \pi_1) \simeq \overline{G}$.

Problem 4. Characterize complementary coalition graphs.

5.2. Partitions into Independent Sets

An *independent c-partition* is a c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ in which every set V_i is an independent set. A graph is an *independent coalition graph* if it is the coalition graph of an independent c-partition. As we have shown in Corollary 1, every graph is an independent coalition graph of some graph. Since any dominating set formed by a coalition in an independent c-partition is either independent or induces a bipartite subgraph, we define two more types of coalitions.

An independent c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ is called an *independent dominating c-partition* if every set $V_i \in \pi$ is either a singleton set containing a dominating vertex or there exists a set $V_j \in \pi$ such that $V_i \cup V_j$ is an independent dominating set. We note that not every graph has an independent dominating c-partition. For example, the corona $K_3 \circ K_1$ does not.

Problem 5. Characterize the graphs having an independent dominating c-partition.

An independent c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ is called a *bipartite (independent dominating) c-partition* if every set $V_i \in \pi$ is either a singleton set containing a dominating vertex or there exists a set $V_j \in \pi$ such that $V_i \cup V_j$ is a dominating set that induces a bipartite subgraph. Again not every graph has a bipartite c-partition. For example, the corona $K_5 \circ K_1$ does not.

Problem 6. Characterize the graphs having a bipartite c-partition.

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