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Research Article

Total restrained Roman domination

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Abstract: Let G be a graph with vertex set $V(G)$. A Roman dominating function (RDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$. If f is an RDF on G, then let $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. An RDF f is called a restrained (total) Roman dominating function if the subgraph induced by V_0 (induced by $V_1 \cup V_2$) has no isolated vertex. A total and restrained Roman dominating function is a total restrained Roman dominating function. The total restrained Roman domination number $\gamma_{trR}(G)$ on a graph G is the minimum weight of a total restrained Roman dominating function on the graph G.

We initiate the study of total restrained Roman domination number and present several sharp bounds on $\gamma_{trR}(G)$. In addition, we determine this parameter for some classes of graphs.

Keywords: Total restrained domination, total restrained Roman domination, total restrained Roman domination number

AMS Subject classification: 05C69

1. Introduction

For definitions and notations not given here we refer to [11]. We consider simple and finite graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$. The *degree* of vertex $v \in V$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. For a subset D of vertices in a graph G, we denote by $G[D]$ the subgraph of G induced by D. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*. We denoted the sets of all leaves and all support vertices of G by $L(G)$ and $S(G)$, respectively. Let $K_{n_1,n_2,\ldots,n_p}$ denote the *complete p-partite graph* with vertex set $S_1 \cup S_2 \cup \ldots \cup S_p$ where $|S_i| = n_i$ for $1 \leq i \leq p$.

A set $S \subseteq V(G)$ is called a *dominating set* if every vertex is either an element of S or is adjacent to an element of S. The *domination number* $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set of G. A *total restrained dominating set* of a graph G without isolated vertices is defined in [14] as a dominating set D with the property that the subgraphs induced by D and $V(G) \setminus D$ do not contain isolated vertices. The cardinality of a minimum total restrained dominating set in G is the *total restrained domination number*, denoted by $\gamma_{tr}(G)$. A total restrained dominating set of G of cardinality $\gamma_{tr}(G)$ is called a $\gamma_{tr}(G)$-set.

In this paper we continue the study of Roman dominating functions in graphs (see, for example, the survey articles [7–9]). A *Roman dominating function* (RDF) on a graph G is defined in [10] as a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$. The weight of an RDF f is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. Moreover, if f is an RDF on G, we let $V_i^f = \{v \in V \mid f(v) = i\}$ for every $i \in \{0, 1, 2\}$. Consequently, any RDF f can be represented by $f = (V_0^f, V_1^f, V_2^f)$, where the superscript f can be deleted in V_i^f when no confusion arises.

A *total Roman dominating function* (TRDF) on a graph G without isolated vertices is defined in [13] as a Roman dominating function f with the property that the subgraph induced by $V_1 \cup V_2$ has no isolated vertex. The *total Roman domination number* $\gamma_{tR}(G)$ is the minimum weight of a TRDF on G. A TRDF on G with weight $\gamma_{tR}(G)$ is called a $\gamma_{tR}(G)$-function. Total Roman domination has been studied by several authors [1–3, 5, 6].

A *restrained Roman dominating function* (RRDF) on a graph G is defined in [15] as a Roman dominating function f with the property that the subgraph induced by V_0 has no isolated vertex. The *restrained Roman domination number* $\gamma_{rR}(G)$ is the minimum weight of an RRDF on G. An RRDF on G with weight $\gamma_{rR}(G)$ is called a $\gamma_{rR}(G)$-function. The restrained Roman domination number has also been studied in [4, 16].

A total and restrained Roman dominating function on a graph without isolated vertices is a *total restrained Roman dominating function* (TRRDF). The *total restrained Roman domination number* $\gamma_{trR}(G)$ on a graph G is the minimum weight of a TRRDF on G. A TRRDF on G with weight $\gamma_{trR}(G)$ is called a $\gamma_{trR}(G)$-function.

If G is a graph without isolated vertices, then the definitions lead to

$$\gamma_R(G) \leq \gamma_{tR}(G) \leq \gamma_{trR}(G) \tag{1}$$

and

$$\gamma_R(G) \leq \gamma_{rR}(G) \leq \gamma_{trR}(G). \tag{2}$$

We initiate the study of total restrained Roman domination and present several sharp bounds on $\gamma_{trR}(G)$. In addition, we determine this parameter for some classes of graphs. Furthermore, if T is a tree of order $n \geq 6$, then we prove the sharp lower bound $\gamma_{trR}(T) \geq \frac{3(n+2)}{4}$.

We make use of the following results.

Proposition 1. *[3] If G is a nontrivial path or a cycle of order n, then $\gamma_{tR}(G) = n$.*

Proposition 2. *[15] If $p, q \geq 2$ are integers, then $\gamma_{rR}(K_{p,q}) = 4$.*

Proposition 3. *[14] If C_n is a cycle of order n, then $\gamma_{tr}(C_n) = n - 2\lfloor \frac{n}{4} \rfloor$.*

Proposition 4. *[3] If G is a graph with no isolated vertex, then $2\gamma(G) \leq \gamma_{tR}(G)$.*

Proposition 5. *[1] If G is a connected graph of order $n \geq 3$, then $\gamma_{tR}(G) \geq \lceil \frac{2n}{\Delta(G)} \rceil$.*

2. Complexity of total restrained Roman domination number

Our aim in this section is to show that the decision problem associated with the total restrained Roman domination is NP-hard even when restricted to bipartite graphs.

Total restrained Roman domination number problem (TRRDN)
Instance: *A bipartite graph G with no isolated vertices and a positive integer k.*
Question: *Is $\gamma_{trR}(G) \leq k$?*

We show the NP-hardness of TRRDN problem by transforming the 3-SAT problem to it in polynomial time. Recall that the 3-SAT problem specified below was proven to be NP-complete in [12].

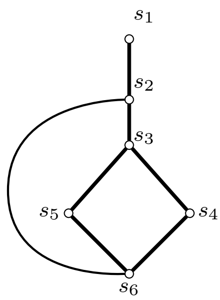
3-SAT problem
Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.
Question: Is there a truth assignment for U that satisfies all the clauses in $\mathcal{C}$?

Now, we show that the problem above is NP-hard, even when restricted to bipartite graphs.

Theorem 1. *Problem TRRDN is NP-complete for bipartite graphs.*

Proof. The problem clearly belongs to NP since checking that a given function is indeed a TRRDF, on a bipartite graph, of weight at most k can be done in polynomial time. Now let us show how to transform any instance of 3-SAT into an instance G of TRRD so that one of them has a solution if and only if the other one has a solution.

Let $U = \{u_1, u_2, \ldots, u_n\}$ and $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary instance of 3-SAT. We will construct a bipartite graph G and a positive integer k such that $\mathscr{C}$ is satisfiable if and only if $\gamma_{trR}(G) \leq k$. We construct such a graph G as follows. For each $i = 1, 2, \ldots, n$, corresponding to the variable $u_i \in U$, associate a complete



F

Figure 1. **The graph F**

bipartite graph $H_i = K_{3,4}$ with bipartite sets $X = \{x_i, y_i, z_i, w_i\}$ and $Y = \{u_i, t_i, \overline{u_i}\}$. For each $j = 1, 2, \ldots, m$, corresponding to the clause $C_j = \{p_j, q_j, r_j\} \in \mathscr{C}$, associate a single vertex c_j and add edge-set $E_j = \{c_j p_j, c_j q_j, c_j r_j\}$. Finally, add the graph F depicted in Figure 1 and connect s_1 to every vertex c_j with $1 \leq j \leq m$. Set $k = 4n + 4$. Clearly, G is a bipartite graph of order $7n + m + 6$. The graph obtained when $U = \{u_1, u_2, u_3, u_4\}$ and $\mathscr{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{u_1, u_2, \overline{u_3}\}, C_2 = \{\overline{u_1}, u_2, u_4\}, C_3 = \{\overline{u_2}, u_3, u_4\}$ is illustrated in Figure 2. Now, we only need to prove that $\gamma_{trR}(G) = 4n + 4$ if and only if there is a truth assignment for U satisfying each clause in $\mathscr{C}$. This goal can be established by proving the next two claims.

Claim 1. $\gamma_{trR}(G) \geq 4n + 4$. Moreover, if $\gamma_{trR}(G) = 4n + 4$, then for any $\gamma_{trR}(G)$-function $f = (V_0, V_1, V_2)$, $f(V(H_i)) = 4$, at most one of $f(u_i)$ and $f(\overline{u_i})$ is 2 for each i, $f(s_2) = f(s_6) = 2$ or $f(s_2) = f(s_3) = 2$, $f(s) = 0$ for the remaining vertices of F and $f(c_j) = 0$ for each j.

Proof of Claim 1. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{trR}(G)$-function. It is easily verified that $f(V(H_i)) \geq 3$ for each $i \in \{1, 2, \ldots, n\}$. Define $Q = \{i \mid f(V(H_i)) = 3 \text{ and } 1 \leq i \leq n\}$ and let $i \in Q$. It is easy to see that $f(t_i) = 2$, $f(p_i) = 1$ for only one vertex $p_i \in \{x_i, y_i, z_i, w_i\}$, and $f(v) = 0$ for the other vertices of H_i. Since f is a TRRDF of G, there exist some vertices c_j, with $f(c_j) = 2$, adjacent to the vertices u_i and $\overline{u_i}$. Let Q' be the set of such vertices c_j. We moreover observe that $2|Q| \leq |[Q, Q']| \leq 3|Q'|$. On the other hand, it is a routine matter to see that $f(V(F)) \geq 4$. Therefore,

$$\gamma_{trR}(G) = f(V(G)) \geq 3|Q| + 2|Q'| + 4(n - |Q|) + 4 = 4n - |Q| + 2|Q'| + 4 \geq 4n + 4.$$

Note that if $|Q| > 0$, then we have $\gamma_{trR}(G) > 4n + 4$.

Suppose that $\gamma_{trR}(G) = 4n + 4$. Then $f(V(H_i)) = 4$ for each $i = 1, 2, \ldots, n$. If $f(s_1) \neq 0$, then for totally restrained Roman dominating the vertices s_3, s_4, s_5 and s_6

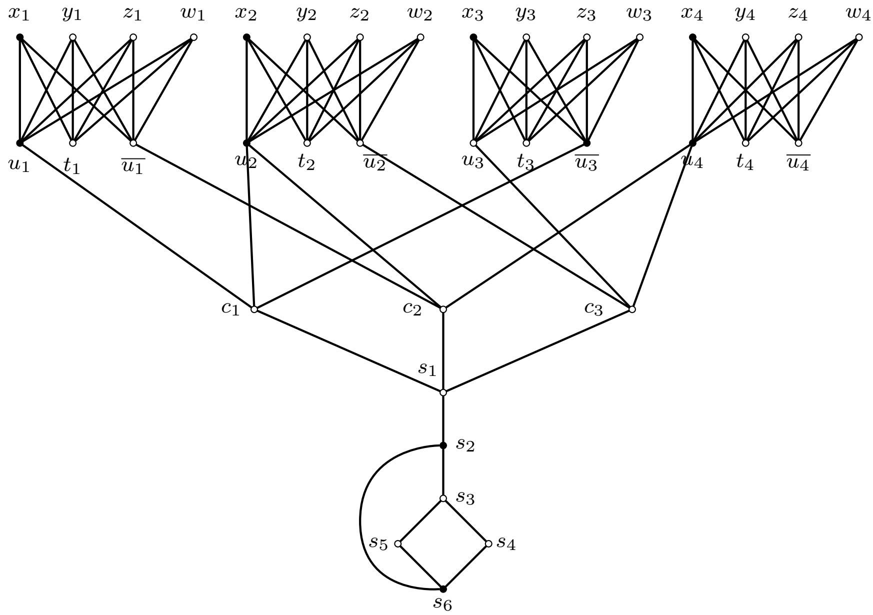


Figure 2. **The graph** G

we must have $f(N[s_3]) \cup \{s_6\}) \geq 4$ which leads to a contradiction. Hence $f(s_1) = 0$ and similar as above, it is clear to see that $f(s_2) = f(s_3) = 2$ or $f(s_2) = f(s_6) = 2$. Therefore $f(s) = 0$ for the remaining vertices of F, and $\sum_{j=1}^{m} f(c_j) = 0$. Now we show that at most one of $f(u_i)$ and $f(\overline{u_i})$ is 2. Let $f(u_i) = f(\overline{u_i}) = 2$ for some $1 \leq i \leq n$. Since $f(V(H_i)) = 4$, it follows that $f(x) = 0$ for each $x \in V(H_i) \setminus \{u_i, \overline{u_i}\}$. This implies that $f(t_i) = f(N[t_i]) = 0$, a contradiction. Therefore, at most of one $f(u_i)$ and $f(\overline{u_i})$ equals two. ♦

Claim 2. $\gamma_{trR}(G) = 4n + 4$ if and only if $\mathscr{C}$ is satisfiable.

Proof of Claim 2. Suppose that $\gamma_{trR}(G) = 4n + 4$ and let f be a $\gamma_{trR}(G)$-function. By Claim 2, at most one of $f(u_i)$ and $f(\overline{u_i})$ is 2 for each $i = 1, 2, \ldots, n$. Define a mapping $t : U \longrightarrow \{T, F\}$ by

$$t(u_i) = \begin{cases} T & \text{if} \quad f(u_i) = 2, \\ F & \text{otherwise.} \end{cases} \quad (i = 1, \ldots, n) \tag{3}$$

We now show that t is a satisfying truth assignment for $\mathscr{C}$. It is sufficient to show that every clause in $\mathscr{C}$ is satisfied by t. To this end, we arbitrarily choose a clause $C_j \in \mathscr{C}$ with $1 \leq j \leq m$. By Claim 2, $f(s_1) = f(c_j) = 0$. Hence, there exists some i with $1 \leq i \leq n$ such that $f(u_i) = 2$ or $f(\overline{u_i}) = 2$ where c_j is adjacent to u_i or $\overline{u_i}$. Suppose that c_j is adjacent to u_i where $f(u_i) = 2$. Since u_i is adjacent to c_j in G, the literal u_i is in the clause C_j by the construction of G. Since $f(u_i) = 2$, it follows that $t(u_i) = T$ by (3), which implies that the clause C_j is satisfied by t. Suppose that c_j is adjacent to $\overline{u_i}$ where $f(\overline{u_i}) = 2$. Since $\overline{u_i}$ is adjacent to c_j in G, the literal $\overline{u_i}$ is in the clause C_j. Since $f(\overline{u_i}) = 2$, it follows that $t(u_i) = F$ by (3). Thus, t assigns $\overline{u_i}$ the truth value T, that is, t satisfies the clause C_j . By the arbitrariness of j with $1 \leq j \leq m$, we have shown that t satisfies all the clauses in $\mathscr{C}$, that is, $\mathscr{C}$ is satisfiable. Conversely, suppose that $\mathscr{C}$ is satisfiable, and let $t : U \to \{T, F\}$ be a satisfying truth assignment for $\mathscr{C}$. We construct a subset D of vertices of G as follows. If $t(u_i) = T$, then put the vertices u_i and x_i in D; if $t(u_i) = F$, then put the vertices $\overline{u_i}$ and x_i in D. Hence $|D| = 2n$. Define the function $g : V(G) \longrightarrow \{0, 1, 2\}$ by $g(x) = 2$ for every $x \in D, g(s_2) = g(s_3) = 2$ and $g(y) = 0$ for the remaining vertices. Since t is a satisfying truth assignment for $\mathscr{C}$, the corresponding vertex c_j in G is adjacent to at least one vertex in D. One can easily check that g is a TRRDF on G of weight $4n+4$ and so $\gamma_{trR}(G) \leq 4n+4$. By Claim 2, $\gamma_{trR}(G) \geq 4n+4$. Therefore, $\gamma_{trR}(G) = 4n+4$. ♦

This completes the proof. □

3. Properties and bounds

In this section we present basic properties and bounds on the total restrained Roman domination number. Since the function f with $f(x) = 1$ for each vertex $x \in V(G)$ is a TRRDF on a graph G without isolated vertices, we obtain our first bound.

Observation 2. If G is a graph of order n without isolated vertices, then $\gamma_{trR}(G) \leq n$.

The next result follows from the inequality $\gamma_{tR}(G) \leq \gamma_{trR}(G)$, Proposition 1 and Observation 2.

Observation 3. If G is a nontrivial path or a cycle of order n, then $\gamma_{trR}(G) = n$.

Proposition 6. *For any connected graph G of order n with minimum degree at least three, $\gamma_{trR}(G) \leq n - 1$.*

Proof. Let G be a graph with $\delta(G) \geq 3$. If G has a triangle $uvwu$, then the function f defined by $f(v) = 2$, $f(u) = f(w) = 0$ and $f(x) = 1$ otherwise, is a TRRDF on G of weight $n - 1$ and we are done. Hence, we assume that G is triangle-free. Let v be a vertex of G with minimum degree and let u_1, u_2 be two neighbors of v. Assume that $w_i \in N(u_i) \setminus \{v\}$ for $i = 1, 2$. If $w_1 = w_2$, then let $w \in N(w_1) \setminus \{u_1, u_2\}$ and define the function g on G by $g(v) = g(w) = 2$, $g(u_1) = g(u_2) = g(w_1) = 0$ and $g(x) = 1$ for the remaining vertices. Clearly, g is a TRRDF on G of weight $n - 1$ and hence $\gamma_{trR}(G) \leq n - 1$. Therefore we assume that $w_1 \neq w_2$. Since G is triangle-free, w_1 has a neighbor z_1 not in $\{u_1, u_2, w_2\}$ and w_2 has a neighbors z_2 not in $\{u_2, u_1, w_1\}$. If $z_1 = z_2$, then the function g defined on G by $g(v) = g(z_1) = 2$,

$g(u_1) = g(u_2) = g(w_1) = g(w_2) = 0$ and $g(x) = 1$ for the remaining vertices, is a TRRDF on G of weight $n-2$ as desired. Let $z_1 \neq z_2$ and define the function g on G by $g(v) = g(z_1) = g(z_2) = 2$, $g(u_1) = g(u_2) = g(w_1) = g(w_2) = 0$ and $g(x) = 1$ for the remaining vertices. Clearly, g is a TRRDF of G of weight $n-1$ and hence $\gamma_{trR}(G) \leq n-1$. This completes the proof. □

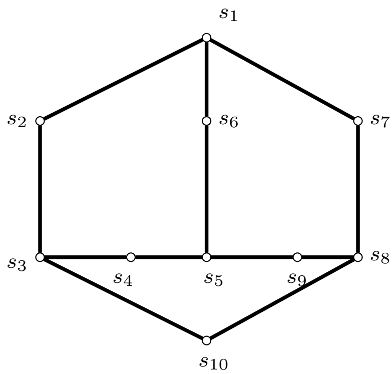


Figure 3. **A graph G of order 10 with girth 6 and $\gamma_{trR}(G) = 10$**

Proposition 7. *For any connected graph G of order n with minimum degree at least two and girth at least seven different from cycles, $\gamma_{trR}(G) \leq n-1$.*

Proof. Let G be a connected graph of order n with $\delta(G) \geq 2$ and $g = g(G) \geq 7$. Let $C = x_1x_2 \dots x_gx_1$ be a cycle of G on $g(G)$ vertices. Since G is a connected graph different from a cycle and because C has length $g(G)$, we may assume that x_1 has a neighbor $w_1 \in V(G) \setminus V(C)$. Let $w_2 \in N(w_1) \setminus \{x_1\}$. Notice that since $g(G) \geq 7$, each vertex in $V(G) \setminus V(C)$ has at most one neighbor in $V(C)$. Define the function g on G by $g(x_1) = g(x_4) = g(x_{g-2}) = 2$, $g(x_2) = g(x_3) = g(x_g) = g(x_{g-1}) = 0$ and $g(x) = 1$ for the remaining vertices. It is easy to verify that g is a TRRDF on G and consequently $\gamma_{trR}(G) \leq n-1$. □

The graph illustrated in Figure 3 shows that the assumption of $g(G) \geq 7$ in Proposition 7 is necessary.

Observation 4. If $p, q \geq 2$ are integers, then $\gamma_{trR}(K_{p,q}) = 4$.

Proof. Proposition 2 leads to $\gamma_{trR}(K_{p,q}) \geq \gamma_{rR}(K_{p,q}) = 4$. Now let X, Y be a bipartition of $K_{p.q}$. If $x \in X$ and $y \in Y$, then define f by $f(x) = f(y) = 2$ and $f(u) = 0$ for $u \in V(K_{p,q}) \setminus \{x, y\}$. Then f is a TRRDF on $K_{p,q}$ and thus $\gamma_{trR}(K_{p,q}) \leq 4$ and so $\gamma_{trR}(K_{p,q}) = 4$. □

The inequality $\gamma_{tR}(G) \leq \gamma_{trR}(G)$ and Proposition 5 yield the following lower bound.

Corollary 1. *If G is a connected graph of order $n \geq 3$, then $\gamma_{trR}(G) \geq \lceil \frac{2n}{\Delta(G)} \rceil$.*

Using Observations 3 and 4, we observe that we have equality for paths, cycles and the complete bipartite graphs $K_{p,p}$ in the inequality of Corollary 1.

Observation 5. Let G be a graph of order $n \geq 2$ without isolated vertices, and let f be a TRRDF of G. Then $f(x) \geq 1$ for every leaf and every support vertex and thus $\gamma_{trR}(G) \geq |L(G)| + |S(G)|$.

Let H be the graph consisting of a path $x_1 x_2 \ldots x_{3t}$ for an integer $t \geq 2$, further vertices $v_1, v_2, \ldots, v_t$ such that v_i is adjacent with x_{3i-2}, x_{3i-1} and x_{3i} for $1 \leq i \leq t$. Then $\gamma_{rR}(H) = 2t$ and $\gamma_{trR}(H) = 3t$. Hence we have

Proposition 8. *There exists a graph H for which $\gamma_{trR}(H) - \gamma_{rR}(H)$ can be made arbitrarily large.*

If $S_{p,q}$ is a double star of order n with $p, q \geq 2$, then $\gamma_{trR}(S_{p,q}) = n$ by Observation 5 and $\gamma_{tR}(S_{p,q}) = 4$ and thus

Proposition 9. *There exists a graph H for which $\gamma_{trR}(H) - \gamma_{tR}(H)$ can be made arbitrarily large.*

Using the inequality $\gamma_{tR}(G) \leq \gamma_{trR}(G)$ and Proposition 4, we obtain the next lower bound.

Corollary 2. *If G is a graph with no isolated vertex, then $2\gamma(G) \leq \gamma_{tR}(G) \leq \gamma_{trR}(G)$.*

The *corona* $H \circ K_1$ of a graph H is the graph obtained from H by adding a pendant edge to each vertex of H. If $G = H \circ K_1$ with a connected graph H, then $\gamma(G) = \frac{n}{2}$, and $\gamma_{trR}(G) = n$ by Observations 2 and 5. Hence $\gamma_{trR}(H \circ K_1) = 2\gamma(H \circ K_1)$ and thus Corollary 2 is sharp.

Theorem 6. *If G is a graph of order $n \geq 4$ without isolated vertices, then $\gamma_{trR}(G) \geq 3$, with equality if and only if $\Delta(G) = n - 1$, and G contains a vertex w of maximum degree such that $G[N_G(w)]$ has exactly one isolated vertex or no isolated vertex and at least one component of order at least three.*

Proof. Let f be a $\gamma_{trR}(G)$-function. If $f(x) \geq 1$ for all $x \in V(G)$, then $\gamma_{trR}(G) \geq n > 3$. If there exists a vertex u with $f(u) = 0$, then u has a neighbor v with $f(v) = 2$. Since f is a TRRDF, the vertex v is adjacent to a vertex z such that $f(z) \geq 1$. Therefore, $\gamma_{trR}(G) \geq 3$.

If $\Delta(G) = n-1$ and G contains a vertex w of maximum degree such that $G[N_G(w)]$ has exactly one isolated vertex u, then the function f with $f(w) = 2$, $f(u) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{u, w\}$ is a TRRDF on G and thus $\gamma_{trR}(G) = 3$. Assume next that $\Delta(G) = n-1$ and G contains a vertex w of maximum degree such that $G[N_G(w)]$ has no isolated vertex and at least one component H of order at least three. Let T be a spanning tree of H and let v be a leaf of T. Then $G[V(T) \setminus \{v\}]$ is connected, and hence the function f with $f(w) = 2$, $f(v) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{v, w\}$ is a TRRDF on G and thus $\gamma_{trR}(G) = 3$.
Conversely, assume that $\gamma_{trR}(G) = 3$, and let f be a $\gamma_{trR}(G)$-function. Since $n \geq 4$, we note that there exist two vertices w and u with $f(w) = 2$, $f(u) = 1$ and $f(x) = 0$ for $x \in V(G) \setminus \{u, w\}$. Therefore all vertices $x \neq w$ are adjacent to w and $G[N_G(w) \setminus \{u\}]$ does not contain an isolated vertex. So $\Delta(G) = n-1$ and $G[N_G(w)]$ has exatly one isolated vertex u or u is adjacent to a vertex of $N_G(w) \setminus \{u\}$, and then $G[N_G(w)]$ has no isolated vertex and at least one component of order at least three. □

Since $\gamma_{trR}(K_3) = 3$, Theorem 6 leads to the next special case immediately.

Corollary 3. *If $n \geq 3$, then $\gamma_{trR}(K_n) = 3$.*

Corollary 4. *If G and $\overline{G}$ are graphs of order $n \geq 4$ without isolated vertices, then $\gamma_{trR}(G) + \gamma_{trR}(\overline{G}) \geq 8$.*

Proof. Since G and $\overline{G}$ are without isolated vertices, we observe that $\Delta(G) \leq n-2$ and $\Delta(\overline{G}) \leq n-2$. Hence Theorem 6 implies $\gamma_{trR}(G), \gamma_{trR}(\overline{G}) \geq 4$ and so $\gamma_{trR}(G) + \gamma_{trR}(\overline{G}) \geq 8$. □

Example 1. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y = \{y_1, y_2, \ldots, y_q\}$ be a bipartition of the complete bipartite graph $K_{p.q}$ for $p, q \geq 3$, and let $B = K_{p,q} - e$ with $e = x_1y_1$. We note that $\Delta(B) \leq n(B) - 2$ and $\Delta(\overline{B}) \leq n(B) - 2$. Define f by $f(x_p) = f(y_q) = 2$ and $f(x) = 0$ otherwise. Then f is a TRRDF on B and therefore $\gamma_{trR}(B) = 4$ according to Theorem 6. Next define g by $g(x_1) = g(y_1) = 2$ and $g(x) = 0$ otherwise. Then g is a TRRDF on $\overline{B}$ and thus $\gamma_{trR}(\overline{B}) = 4$. Consequently, $\gamma_{trR}(B) + \gamma_{trR}(\overline{B}) = 8$.

Example 1 demonstrates that Corollary 4 is sharp.

Observation 7. Let $G = K_{n_1, n_2, \ldots, n_p}$ be a complete p-partite graph with $p \geq 3$ and $n_1 \leq n_2 \leq \ldots \leq n_p$. If $n = n_1 + n_2 + \ldots + n_p \geq 4$, then $\gamma_{trR}(G) = 3$ when $n_1 = 1$ and $\gamma_{trR}(G) = 4$ when $n_1 \geq 2$.

Proof. If $n_1 = 1$, then Theorem 6 leads to $\gamma_{trR}(G) = 3$. If $n_1 \geq 2$, then $\Delta(G) \leq n-2$ and therefore $\gamma_{trR}(G) \geq 4$ by Theorem 6. Now let $u \in S_1$ and $v \in S_2$, where S_1 and S_2 are two different partite sets of G. Then the function f defined by $f(u) = f(v) = 2$ and $f(x) = 0$ for $x \in V(G) \setminus \{u, v\}$ is a TRRDF on G of weight 4 and so $\gamma_{trR}(G) \leq 4$. Consequently, $\gamma_{trR}(G) = 4$ when $n_1 \geq 2$. □

Theorem 8. *If G is a graph of order $n \geq 4$ without isolated vertices, then*

$$\gamma_{tr}(G) \leq \gamma_{trR}(G) \leq 2\gamma_{tr}(G).$$

In addition, $\gamma_{tr}(G) = \gamma_{trR}(G)$ if and only if $\gamma_{tr}(G) = \gamma_{trR}(G) = n$.

Proof. Let $f = (V_0, V_1, V_2)$ be an arbitrary $\gamma_{trR}(G)$-function. Then $V_1 \cup V_2$ is a total restrained dominating set of G, and hence it follows that

$$\gamma_{tr}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{trR}(G).$$

This establishes the lower bound in the statement of the theorem. In particular, if $\gamma_{tr}(G) = \gamma_{trR}(G)$, then $V_2 = \emptyset$, implying $V(G) = V_1$ and so $\gamma_{tr}(G) = \gamma_{trR}(G) = n$. Clearly, if $\gamma_{tr}(G) = \gamma_{trR}(G) = n$, then $\gamma_{tr}(G) = \gamma_{trR}(G)$.
For the upper bound, let D be a $\gamma_{tr}(G)$ set. Define f by $f(x) = 2$ for $x \in D$ and $f(x) = 0$ for $x \in V(G) \setminus D$. Then f is a TRRDF on G of weight $2|D| = 2\gamma_{tr}(G)$ and thus $\gamma_{trR}(G) \leq 2\gamma_{tr}(G)$. □

If C_n is a cycle of order $n = 4t$, then we deduce from Proposition 3 that $\gamma_{tr}(C_{4t}) = 2t$, and Observation 3 implies $\gamma_{trR}(C_{4t}) = 4t$. This example shows that the upper bound in Theorem 8 is sharp.

4. Trees

In this section we first characterize all trees T with $\gamma_{trR}(T) = n(T)$, and then we present a lower bound for the total restrained Roman domination number of trees.

Theorem 9. *Let T be a tree of order n. Then $\gamma_{trR}(T) = n$ if and only if there is no path $v_1v_2v_3v_4v_5$ in T such that $d(v_3) \geq 3$ and v_i is not a leaf or a support vertex for each $i \in \{1, 2, 4, 5\}$.*

Proof. Let $\gamma_{trR}(T) = n$. We assume, by contradiction, that there is a path $v_1v_2v_3v_4v_5$ in T such that $d(v_3) \geq 3$ and v_i is not a leaf or support vertex for each $i \in \{1, 2, 4, 5\}$. Root T at v_3, and let u_i be a child of v_i for $i \in \{1, 5\}$. Define the function f by $f(v_3) = f(u_1) = f(u_5) = 2$, $f(v_i) = 0$ for $i \in \{1, 2, 4, 5\}$ and $f(x) = 1$ for the remaining vertices. One can easily see that f is a TRRDF on T of weight $n - 1$ which leads to the contradiction $\gamma_{trR}(T) < n$.
Conversely, suppose there is no path $v_1v_2v_3v_4v_5$ in T such that $d(v_3) \geq 3$ and v_i is not a leaf or a support vertex for each $i \in \{1, 2, 4, 5\}$. We assume, by contradiction, that $\gamma_{trR}(T) < n$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{trR}(T)$-function. We deduce from $\gamma_{trR}(T) < n$ that $|V_0| > |V_2|$. Therefore there is a vertex $v \in V_2$ such that $|N(v) \cap V_0| \geq 2$. Let $u_1, u_2 \in N(v) \cap V_0$. Since f is a TRRDF, we must have $d(v) \geq 3$ and that u_1, u_2 have neighbors in V_0. Assume that $w_1 \in N(u_1) \cap V_0$ and $w_2 \in N(u_2) \cap V_0$. It follows from Observation 5 that no vertex in $\{u_1, u_2, w_1, w_2\}$ is a leaf or support vertex, which is a contradiction. □

Theorem 10. *If T is a tree of order $n \geq 6$, then $\gamma_{trR}(T) \geq \frac{3(n+2)}{4}$.*

Proof. We proceed by induction on $n \geq 6$. If $\mathrm{diam}(T) \leq 4$, then Observations 2 and 5 easily show that $\gamma_{trR}(T) = n \geq \frac{3(n+2)}{4}$. Let now $\mathrm{diam}(T) \geq 5$. If $n = 6$, then $T \cong P_6$ and we have the equality in the lower bound. Assume that $n \geq 7$ and let the lower bound hold for all trees T' of order $6 \leq n' < n$. Let T be a tree of order n and let f be a $\gamma_{trR}(T)$-function. By the above, it suffices to assume that $\mathrm{diam}(T) \geq 5$. If T has a strong support vertex v with leaf neighbors u_1, u_2, then by Observation 5 we have $f(v), f(u_1), f(u_2) \geq 1$ and hence $f|_{V(T-u_1)}$ is a TRRDF on the tree $T - u_1$. The induction hypothesis implies

$$\gamma_{trR}(T) \geq 1 + \sum_{x \in V(T-u_1)} f(x) \geq 1 + \frac{3((n-1)+2)}{4} > \frac{3(n+2)}{4}.$$

Thus we assume that T has no strong support vertex. Let $v_1 v_2 \ldots v_p$ be a diametral path and root T at v_p. By our earlier assumption we have $d(v_2) = 2$, and according to Observation 5, we have $f(v_1), f(v_2) \geq 1$.
If $f(v_3) \geq 1$, then $f|_{V(T-v_1)}$ is a TRRDF on $T - v_1$, and the induction hypothesis yields $\gamma_{trR}(T) > \frac{3(n+2)}{4}$ as above. Let now $f(v_3) = 0$.
Assume that $d(v_3) \geq 3$. Since $f(v_3) = 0$, we deduce from Observation 5 that v_3 is not adjacent to a leaf. Let $u_2 \neq v_2$ be a support vertex adjacent to v_3, and let u_1 be a leaf adjacent to u_2. Since T has no strong support vertex, we have $d(u_2) = 2$. Since f is a TRRDF on T, we observe from Observation 5 that $f(v_4) = 0$ and $f(u_1) + f(u_2) \geq 2$. To Roman dominate v_3, we assume without loss of generality that $f(v_2) = 2$. Therefore we note that $f|_{V(T-\{u_1,u_2\})}$ is a TRRDF on $T - \{u_1, u_2\}$, and the induction hypothesis implies

$$\gamma_{trR}(T) \geq 2 + \sum_{x \in V(T-\{u_1,u_2\})} f(x) \geq 2 + \frac{3((n-2)+2)}{4} > \frac{3(n+2)}{4}.$$

Next we assume that $d(v_3) = 2$. Since $f(v_3) = 0$, it follows from Observation 5 and the fact f is a TRRDF on T that $f(v_2) = 2$ and $f(v_4) = 0$.
Assume now that $d(v_4) \geq 3$. Since $f(v_4) = 0$, the vertex v_4 is not adjacent to a leaf. Considering above arguments we may assume that for each path $v_4 z_3 z_2 z_1$ in T where $z_3 \notin \{v_3, v_5\}$, we have $d(z_3) = d(z_2) = 2$. Next we distinguish three cases.
Case 1. Let $f(v_5) = 2$.
Assume that $u_3 \notin \{v_3, v_5\}$ is a support vertex adjacent to v_4 and u_2 a leaf adjacent to u_3. Since T has no strong support vertex, we have $d(u_3) = 2$. Since $f(v_5) = 2$, we observe that $f(u_3) + f(u_2) = 2$. Then $f|_{V(T-\{u_3,u_2\})}$ is a TRRDF on $T - \{u_3, u_2\}$, and we obtain the desired bound as above. Now let $u_3 \notin \{v_3, v_5\}$ be adjacent to v_4, u_2 a support vertex adjacent to u_3 and u_1 a leaf adjacent to u_2. Without loss of generality, we can assume that $d(u_3) = d(u_2) = 2$. We note that $f(u_3) + f(u_2) + f(u_1) = 3$, and

the function f restricted to $T-\{u_1,u_2,u_3\}$ is a TRRDF on $T-\{u_1,u_2,u_3\}$. The induction hypothesis leads to the desired bound as above.

Case 2. Let $f(v_5)=1$.
Then v_4 has a neighbor $u_3 \notin \{v_3,v_5\}$ with $f(u_3)=2$, and u_3 has a neighbor $u_2 \neq v_4$ with $f(u_2)=1$. Now define the function g by $g(v_5)=2$, $g(u_3)=1$ and $g(x)=f(x)$ otherwise. Then g is a TRRDF on T of the same weight as f, and we are in the position of Case 1.

Case 3. Let $f(v_5)=0$. Then v_4 has a neighbor $u_3 \notin \{v_3,v_5\}$ with $f(u_3)=2$, and u_3 has a neighbor $u_2 \neq v_4$ with $f(u_2)=1$. Similarly, v_5 has a neighbor $w \neq v_4$ with $f(w)=2$, and w has a neighbor $w' \neq v_5$ with $f(w') \geq 1$. It follows that $p \geq 7$. In this case we observe that the function f restricted to $T-\{v_1,v_2,v_3\}$, is a TRRDF on $T-\{v_1,v_2,v_3\}$, and since $p \geq 7$, the induction hypothesis implies

$$\gamma_{trR}(T) = 3 + \sum_{x\in V(T-\{v_1,v_2,v_3\})} f(x) \geq 3 + \frac{3((n-3)+2)}{4} > \frac{3(n+2)}{4}.$$

Finally, we assume that $d(v_4)=2$. Since $f(v_4)=0$, we conclude that $f(v_5)=2$. If $7 \leq n \leq 9$, then it is straightforward to verify that $\gamma_{trR}(T) \geq \frac{3(n+2)}{4}$. If $n \geq 10$, then the function f restricted to $T-\{v_1,v_2,v_3,v_4\}$ is a TRRDF on $T-\{v_1,v_2,v_3,v_4\}$, and the induction hypothesis leads to

$$\gamma_{trR}(T) = 3 + \sum_{x\in V(T-\{v_1,v_2,v_3,v_4\})} f(x) \geq 3 + \frac{3((n-4)+2)}{4} = \frac{3(n+2)}{4}.$$

□

Example 2. Let H be the tree consisting of the vertices w and z and the paths $v_i^1v_i^2v_i^3v_i^4$ for $1 \leq 1 \leq p$ such that w is adjacent to z and v_i^1 for $1 \leq i \leq p$. Then $n(H)=4p+2$, and the function f with $f(z)=f(v_i^4)=1$, $f(w)=f(v_i^3)=2$ and $f(v_i^1)=f(v_i^2)=0$ for $1 \leq i \leq p$ is a TRRDF on H of weight $3p+3$. Therefore $\gamma_{trR}(H) \leq 3p+3 = \frac{3(n(H)+2)}{4}$. Using Theorem 10, we note that $\gamma_{trR}(H) = 3p+3 = \frac{3(n(H)+2)}{4}$. This example demonstrates that Theorem 10 is sharp.

We conclude this section with an open problem.

Problem. Characterize all connected graphs G with $\gamma_{trR}(G) = n(G)$.

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