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Research Article

On Sombor coindex of graphs

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Abstract: In this paper, we explore several properties of Sombor coindex of a finite simple graph and we derive a bound for the total Sombor index. We also explore its relations to the Sombor index, the Zagreb coindices, forgotten coindex and other important graph parameters. We further compute the bounds of the Somber coindex of some graph operations and derived explicit formulae of Sombor coindex for some well-known graphs as application.

Keywords: Sombor index, Sombor coindex, total Sombor index, graph operations

AMS Subject classification: 05C07, 05C35

1. Introduction

We consider only finite simple graph in this paper. Let G be a finite simple graph on n vertices and m edges. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. The complement of G, denoted by $\overline{G}$, is a simple graph on $V(G)$ in which two vertices u and v are adjacent, i.e., joined by an edge uv, if and only if they are not adjacent in G. Hence, $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. Clearly, $E(G) \cup E(\overline{G}) = E(K_n)$, where K_n is the complete graph on n vertices and $\overline{m} = |E(\overline{G})| = \binom{n}{2} - m$. The degree of a vertex u in G is denoted by $d_G(u)$. Then $d_{\overline{G}}(u) = n - 1 - d_G(u)$. Let Δ and δ denote the maximum vertex degree and the minimum vertex degree of the graph G, respectively.

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In chemical graph theory, one generally considers various graph-theoretical invariants of molecular graphs (also known as topological indices or molecular descriptors), and study how strongly are they correlated with various properties of the corresponding molecules. The first such topological index was introduced in 1947 by Harry Wiener and used for correlation with boiling points of alkanes [20]. Wiener's index is related to the distances in molecular graphs. Historically, the first vertex-degree-based topological indices were the graph invariants that nowadays are called Zagreb indices. Along this line of approach, numerous graph invariants have been employed with varying degree of success in QSAR (quantitative structure-activity relationship) and QSPR (quantitative structure-property relationship) studies. The Zagreb indices are among the most studied invariants and they are defined as sums of contributions dependent on the degrees of adjacent vertices over all edges of a graph. The Zagreb indices of a graph G, i.e., the *first Zagreb index* $M_1(G)$ and the *second Zagreb index* $M_2(G)$, are defined [11] as follows.

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]; \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The Zagreb indices can be viewed as the contributions of pairs of adjacent vertices to additively and multiplicatively weighted versions of Wiener numbers and polynomials [11]. When computing the weighted Wiener polynomials of certain composite graphs, similar contributions of non-adjacent pairs of vertices had to be taken into account [6]. These quantities were called *Zagreb coindices* since the defining sums run over the edges of the complement of G although the degrees of the vertices are with respect to the graph itself. The *first Zagreb coindex* $\overline{M}_1(G)$ and the *second Zagreb coindex* $\overline{M}_2(G)$ are defined [6] as follows.

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)]; \quad \overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u) d_G(v).$$

Generalised version of the first Zagreb index has also been introduced [12] and is known as the *general first Zagreb index* which is defined as

$$M_1^p(G) = \sum_{u \in V(G)} d_G(u)^p.$$

When $p = 3$, $M_1^3(G) = \sum_{u \in V(G)} d_G(u)^3$ is known as the *forgotten index* $F(G)$ and is also equal to

$$F(G) = \sum_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2].$$

The forgotten coindex of a graph G is defined [5] as

$$\overline{F}(G) = \sum_{uv \notin E(G)} [d_G(u)^2 + d_G(v)^2].$$

Several properties of the Zagreb coindices have been explored and the Zagreb coindices for some derived graphs and several graph operations have also been studied recently in [1], [13] and [18]. In [5], forgotten coindex of some graph operations are obtained and in [2] relations between this invariant and some well-known graph invariants are explored. Motivated by such results, we consider in this paper the recently introduced but much studied topological index of a graph, the *Sombor index* [9]. The chemical applicability of Sombor index is explored and has shown good predictive potential [16]. Computations of Sombor index for chemical graphs are done in [3], basic properties of Sombor index are obtained in [4, 10, 15] and its relations to some well known topological indices are explored in [8], [14] and [19]. In [17], a structural result for graphs with integer values of Sombor index and some bounds on the Sombor index are derived among other results and it also indicates many possible future directions, for instance the global or total Sombor index which is the sums of contributions $\sqrt{d_G(u)^2 + d_G(v)^2}$ for all pairs of vertices u, v, whether adjacent or not. It further indicates another potentially interesting future work which is the difference between the total and the classical Sombor index that corresponds to the *Sombor coindex*. Following this direction, we explore the Sombor coindex of graphs as indicated in [17]. We give several properties of the Sombor coindex and its relations to Sombor index, Zagreb coindices, forgotten coindex and other important graph parameters. Since several complicated (and important) graphs often arise from simpler graphs via some graph operations, we also present the Sombor coindex of some graph operations. The paper is arranged as follows: In section 2, we recall the definitions of the Sombor index and the Sombor coindex along with some examples. In section 3, we give several properties of the Sombor coindex and its bounds in terms of important graph parameters. In section 4, we explore the relations between Sombor coindex and other topological coindices: Zagreb coindices and forgotten coindex. In section 5, we present the Sombor coindex of some graph operations and we also compute the Sombor coindex of some (chemical) graphs as application. We end the paper with some possible future work in the concluding section.

2. Sombor coindex of graphs

Motivated by the geometric interpretation of the degree radius of an edge uv, which is the distance from the origin to the ordered pair $(d_G(u), d_G(v))$, where $d_G(u) \leq d_G(v)$, Gutman recently introduced a new vertex-degree-based molecular structure descriptor [9], the *Sombor index*, which is defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

We present some examples of the Somber index. For more details we refer the reader to [9].

Example 1. (i) $SO(K_n) = n(n-1)^2/\sqrt{2}$ and $SO(\overline{K_n}) = 0$.

(ii) For the cycle C_n $(n \geq 3)$, $SO(C_n) = 2\sqrt{2}n$.

(iii) Notice that $SO(P_2) = SO(K_2) = \sqrt{2}$. For $n \geq 3$, $SO(P_n) = 2(n-3)\sqrt{2} + 2\sqrt{5}$.

We now recall the Sombor coindex. It is defined by considering analogous contributions from the pairs of non-adjacent vertices to the formula of Sombor index which was indicated in [17]. The *Sombor coindex* of G is defined as

$$\overline{SO}(G) = \sum_{uv \notin E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Remark 1. Notice that the total Sombor index, denoted by $SO_t(G)$, is expressed as

$$SO_t(G) = SO(G) + \overline{SO}(G).$$

Remark 2. The Sombor coindex of G is also expressed as

$$\overline{SO}(G) = \sum_{uv \in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Remark 3. Notice that in the definition of Sombor coindex of G, the defining sums run over $E(\overline{G})$ but the degrees of the vertices are with respect to G. It is, therefore, not equal to the Sombor index of $\overline{G}$.

Remark 4. Deleting an edge in a graph increases the Sombor coindex of the graph whereas adding an edge will decrease the Sombor coindex.

Example 2. (i) In the case of complete graphs K_n, the defining sums in the Sombor coindex are taken over the empty set of edges and hence it has zero contribution to the Sombor coindex. And in the case of empty graphs, all degrees are zero. Thus

$$\overline{SO}(K_n) = \overline{SO}(\overline{K_n}) = 0.$$

(ii) Notice that the complement of the cycle C_n $(n \geq 3)$ has $n(n-3)/2$ edges and since it is a 2-regular graph, the Sombor coindex of the cycle C_n is given by

$$\overline{SO}(C_n) = n(n-3)\sqrt{2}.$$

(iii) By an (x, y)-edge of a complement graph $\overline{G}$, we mean an edge $e = uv$ where $d_G(u) = x$ and $d_G(v) = y$. Let $n \geq 3$. The complement of the path P_n has only one $(1, 1)$-edge, $2(n-3)$ number of $(1, 2)$-edges and $\binom{n-4}{2}$ number of $(2, 2)$-edges. Thus

$$\overline{SO}(P_n) = [(n-4)(n-3) + 1]\sqrt{2} + 2(n-3)\sqrt{5}.$$

3. Properties of Sombor coindex of graphs

In this section, we give some basic properties of $\overline{SO}(G)$ and we also give several bounds on $\overline{SO}(G)$ in terms of some useful graph parameters. First we give an upper bound on $\overline{SO}(G)$ for a triangle-free graph G.

Theorem 1. *Let G be a traingle-free graph on n vertices and m edges. Then*

$$\overline{SO}(G) \leq \begin{cases} \overline{m}\sqrt{\delta^2 + (n-\delta)^2} & \text{if } \Delta + \delta \leq n, \\ \overline{m}\sqrt{\Delta^2 + (n-\Delta)^2} & \text{if } \Delta + \delta \geq n. \end{cases}$$

Proof. Since G is triangle-free, we have $d(u) + d(v) \leq n$ for any edge uv in G. From the definition of the Sombor coindex, we have

$$\begin{aligned} \overline{SO}(G) &= \sum_{uv \notin E(G)} \sqrt{d(u)^2 + d(v)^2} \\ &= \sum_{uv \in E(\overline{G})} \sqrt{d(u)^2 + d(v)^2} \\ &\leq \sum_{uv \in E(\overline{G})} \sqrt{d(u)^2 + (n - d(u))^2}. \end{aligned}$$

Let us consider a function $f(x) = x^2 + (n-x)^2$ for $x \in [\delta, \Delta]$. Since $f'(x) = 2(2x - n)$, we see that $f(x)$ is decreasing on $[\delta, n/2]$ and increasing on $[n/2, \Delta]$. Thus

$$\overline{SO}(G) \leq \begin{cases} \overline{m}\sqrt{\delta^2 + (n-\delta)^2} & \text{if } \Delta + \delta \leq n, \\ \overline{m}\sqrt{\Delta^2 + (n-\Delta)^2} & \text{if } \Delta + \delta \geq n. \end{cases}$$

□

Next, we give the upper and lower bounds on $\overline{SO}(G)$ in terms of n, Δ and δ.

Theorem 2. *Let G be a graph on n vertices and m edges. Then*

$$\frac{\delta n}{\sqrt{2}}\left(n - 1 - \Delta\right) \leq \overline{SO}(G) \leq \frac{\Delta n}{\sqrt{2}}\left(n - 1 - \delta\right).$$

Equality holds if G is a regular graph.

Proof. We just prove the upper bound as the lower bound can be similarly obtained. From the definition of the Sombor coindex, we have

$$\overline{SO}(G) = \sum_{uv \notin E(G)} \sqrt{d_G(u)^2 + d_G(v)^2} = \sum_{uv \in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2} \leq \sqrt{2}\Delta\overline{m}.$$

Notice that $2\overline{m} = n(n-1) - 2m$. And by the Handshaking lemma, we have

$$2m = \sum_{u \in V(G)} d_G(u) \geq n\delta.$$

Thus, $2\overline{m} \leq n(n-1-\delta)$. It follows that $\overline{SO}(G) \leq \frac{\Delta n}{\sqrt{2}} (n-1-\delta)$. Moreover it is easy to see that the equality holds for a regular graph. $\square$

As corollary we can compute the Somber coindex of r-regular graphs.

Corollary 1. *Let G be an r-regular graph on n vertices. Then*

$$\overline{SO}(G) = \frac{nr(n-1-r)}{\sqrt{2}}.$$

Remark 5. Let G be a graph on n vertices. Since $\Delta \leq n-1$, we have

$$\overline{SO}(G) \leq \frac{n(n-1)(n-1-\delta)}{\sqrt{2}}.$$

Next, we have another upper bound for the Sombor coindex in terms of $\overline{m}$, δ and $\overline{M}_1(G)$.

Theorem 3. *Let G be a graph with m edges. Then*

$$\overline{SO}(G) \leq \overline{M}_1(G) - (2-\sqrt{2})\delta\overline{m}.$$

Equality holds if G is a regular graph.

Proof. By the definition of the Sombor coindex, we have

$$\overline{SO}(G) = \sum_{uv \notin E(G)} \sqrt{d_G(u)^2 + d_G(v)^2} = \sum_{uv \in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

Notice that for any u and v with $d_G(u) \geq d_G(v)$, we have

$$\sqrt{d_G(u)^2 + d_G(v)^2} \leq d_G(u) + (\sqrt{2}-1)d_G(v).$$

Thus,

$$\begin{aligned}
\overline{SO}(G) &\leq \sum_{\substack{uv \in E(\overline{G}) \\ d_G(u) \geq d_G(v)}} [d_G(u) + (\sqrt{2}-1)d_G(v)] \\
&= \sum_{uv \in E(\overline{G})} [d_G(u) + d_G(v)] - \sum_{\substack{uv \in E(\overline{G}) \\ d_G(u) \geq d_G(v)}} (2-\sqrt{2})d_G(v) \\
&\leq \overline{M}_1(G) - (2-\sqrt{2})\delta\overline{m}.
\end{aligned}$$

Further, the above inequalities become equalities if G is a regular graph. $\square$

In [1], it is proven that $\overline{M}_1(G) = 2m(n-1) - M_1(G)$. Thus we have the following corollary.

Corollary 2. *Let G be a graph on n vertices and m edges. Then*

$$\overline{SO}(G) \leq 2m(n-1) - M_1(G) - \left(1 - \frac{1}{\sqrt{2}}\right)[n(n-1) - 2m]\delta.$$

Equality holds if G is a regular graph.

We remarked that the Sombor coindex of a graph G is not the same as the Sombor index of $\overline{G}$. However, they are related closely which is reflected in the bounds of the total Sombor index as follows.

Theorem 4. *Let G be a graph on n vertices and m edges. Then*

(i) $SO_t(G) = SO(G) + \overline{SO}(G) \leq \dfrac{n(n-1)\Delta}{\sqrt{2}}$. *Equality holds if G is a regular graph.*

(ii) $SO(\overline{G}) + \overline{SO}(G) \leq \overline{m}(n-1+\Delta-\delta)\sqrt{2}$. *Equality holds if G is a regular graph.*

Proof. By the definition of the Sombor index and the Sombor coindex, we have

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2} \leq m\sqrt{2}\Delta \tag{1}$$

$$\overline{SO}(G) = \sum_{uv \in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2} \leq \overline{m}\sqrt{2}\Delta \tag{2}$$

From (1) and (2), we have

$$SO_t(G) = SO(G) + \overline{SO}(G) \leq (m+\overline{m})\sqrt{2}\Delta \leq \binom{n}{2}\sqrt{2}\Delta = \frac{n(n-1)\Delta}{\sqrt{2}}.$$

Further, it is easy to see that the equality holds if G is a regular graph. This proves the first part. For the second part, notice that $d_{\overline{G}}(u) = n-1-d_G(u)$. Hence

$$\begin{aligned} SO(\overline{G}) &= \sum_{uv \in E(\overline{G})} \sqrt{d_{\overline{G}}(u)^2 + d_{\overline{G}}(v)^2} \\ &= \sum_{uv \in E(\overline{G})} \sqrt{(n-1-d_G(u))^2 + (n-1-d_G(v))^2} \\ &\leq \overline{m}(n-1-\delta)\sqrt{2}. \end{aligned} \tag{3}$$

From (2) and (3), we have

$$SO(\overline{G}) + \overline{SO}(G) \leq \overline{m}(n-1+\Delta-\delta)\sqrt{2}.$$

Moreover, the equality holds if G is a regular graph. This completes the proof. □

Theorem 5. *Let G be a graph with m edges. Then*

$$\overline{SO}(G) + \overline{SO}(\overline{G}) \leq 2\overline{M}_1(G) - \left(1 - \frac{1}{\sqrt{2}}\right) n(n-1)\delta.$$

Equality holds if G is a regular graph.

Proof. Applying Theorem 3 to $\overline{G}$, we have

$$\overline{SO}(\overline{G}) \leq \overline{M}_1(\overline{G}) - (2-\sqrt{2})\delta m.$$

Equality holds if G is a regular graph. Now,

$$\overline{SO}(G) + \overline{SO}(\overline{G}) \leq \overline{M}_1(G) - (2-\sqrt{2})\delta\overline{m} + \overline{M}_1(\overline{G}) - (2-\sqrt{2})\delta m.$$

It is proven in [1] that $\overline{M}_1(G) = \overline{M}_1(\overline{G})$. It follows that

$$\begin{aligned}\overline{SO}(G) + \overline{SO}(\overline{G}) \leq & 2\overline{M}_1(G) - (2-\sqrt{2})(\overline{m}+m)\delta \\ = & 2\overline{M}_1(G) - \left(1 - \frac{1}{\sqrt{2}}\right) n(n-1)\delta.\end{aligned}$$

Moreover, the equality holds if G is a regular graph. □

4. Relations between Sombor coindex and some topological coindices

In this section we present relations between Sombor coindex and some well-studied coindices: Zagreb coindices and forgotten coindex. First we recall the following well-known inequalities which are needed for our results in this section.

Lemma 1 (Pólya-Szegö inequality [7]). *Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ be two sequences of positive real numbers. If there exists real numbers A, a, B and b such that $0 < a \leq a_k \leq A < \infty$ and $0 < b \leq b_k \leq B < \infty$ for $k = 1, 2, \ldots, m$ then*

$$\frac{\sum_{k=1}^{m} a_k^2 \sum_{k=1}^{m} b_k^2}{\left(\sum_{k=1}^{m} a_k b_k\right)^2} \leq \frac{(ab+AB)^2}{4abAB}$$

where the equality holds if and only if

$$p = m\frac{A}{a}\Big/\left(\frac{A}{a}+\frac{B}{b}\right),\ q = m\frac{B}{b}\Big/\left(\frac{A}{a}+\frac{B}{b}\right)$$

are integers and if p *of the numbers* $a_1, a_2, \ldots, a_m$ *are equal to* a *and* q *of these numbers are equal to* A*, and if the corresponding numbers* b_k *are equal to* B *and* b*, respectively.*

Lemma 2 (Radon's inequality [4]). *If* $a_k, b_k > 0$ *for* $k = 1, 2, \ldots, m$ *and* $p > 0$*, then*

$$\sum_{k=1}^{m} \frac{{a_k}^{p+1}}{{b_k}^{p}} \geq \frac{\left(\sum\limits_{k=1}^{m} a_k\right)^{p+1}}{\left(\sum\limits_{k=1}^{m} b_k\right)^{p}}.$$

Equality holds if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_m}{b_m}$.

Remark 6. The upper bound of Sombor coindex involving forgotten coindex is an easy consequence of Cauchy-Schwarz inequality. More precisely, for a graph with m edges $\overline{SO}(G) \leq \sqrt{\overline{m}\overline{F}(G)}$, where the equality holds if G is a regular graph. Here, we present a lower bound for $\overline{SO}(G)$ which is still sharp for a regular graph.

Theorem 6. *Let* G *be a graph on* n *vertices and* m *edges. Then*

$$\sqrt{\overline{m}\overline{F}(G)} \leq \frac{1}{2}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right)\overline{SO}(G).$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let d_i denotes the degree of the vertex v_i in G. Letting $a_k \to \sqrt{{d_i}^2 + {d_j}^2}$ and $b_k = \delta$ in Lemma 1 and choosing $a = \delta = b$ and $A = \Delta = B$, we have $0 < a \leq a_k \leq A < \infty$ and $0 < b \leq b_k \leq B < \infty$ for $k = 1, 2, \ldots, m$. Notice that $\frac{(ab + AB)^2}{4abAB} = \frac{1}{4}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right)^2$. Applying Lemma 1 with the sums running over the edges in $\overline{G}$, we have

$$\frac{\sum\limits_{v_iv_j\in E(\overline{G})} [{d_i}^2 + {d_j}^2] \sum\limits_{v_iv_j\in E(\overline{G})} \delta^2}{\left(\sum\limits_{v_iv_j\in E(\overline{G})} \delta\sqrt{{d_i}^2 + {d_j}^2}\right)^2} \leq \frac{1}{4}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right)^2.$$

So

$$\frac{\overline{F}(G)\overline{m}}{\overline{SO}(G)^2} \leq \frac{1}{4}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right)^2.$$

Hence

$$\sqrt{\overline{m}\overline{F}(G)} \leq \frac{1}{2}\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right)\overline{SO}(G).$$

$\square$

We now present the relation between Sombor coindex and the first Zagreb coindex.

Theorem 7. *Let G be a graph on n vertices and m edges. Then*

$$2\sqrt{\overline{m}\Delta\overline{M}_1(G)} \leq \left(1+\frac{\Delta}{\delta}\right)\overline{SO}(G)$$

and

$$\overline{SO}(G) \leq \sqrt{\overline{m}\Delta\overline{M}_1(G)}.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let d_i denotes the degree of the vertex v_i in G. Letting $a_k \to \sqrt{d_i + d_j}$ and $b_k \to \sqrt{\dfrac{{d_i}^2 + {d_j}^2}{d_i + d_j}}$ in Lemma 1 and choosing $a = \sqrt{2\delta}$, $A = \sqrt{2\Delta}$, $b = \sqrt{\delta}$ and $B = \sqrt{\Delta}$, we have $0 < a \leq a_k \leq A < \infty$ and $0 < b \leq b_k \leq B < \infty$ for $k = 1, 2, \ldots, m$. Notice that

$$\frac{(ab+AB)^2}{4abAB} = \frac{1}{4\delta\Delta}\left(\Delta+\delta\right)^2.$$

Applying Lemma 1 with the sums running over the edges in $\overline{G}$, we have

$$\frac{\displaystyle\sum_{v_iv_j\in E(\overline{G})}[d_i+d_j]\sum_{v_iv_j\in E(\overline{G})}\frac{{d_i}^2+{d_j}^2}{d_i+d_j}}{\left(\displaystyle\sum_{v_iv_j\in E(\overline{G})}\sqrt{{d_i}^2+{d_j}^2}\right)^2} \leq \frac{1}{4\delta\Delta}\left(\Delta+\delta\right)^2.$$

Notice that $\dfrac{{d_i}^2+{d_j}^2}{d_i+d_j} \geq \delta$. So, $\displaystyle\sum_{v_iv_j\in E(\overline{G})}\frac{{d_i}^2+{d_j}^2}{d_i+d_j} \geq \overline{m}\delta$. Thus

$$\frac{\overline{M}_1(G)\overline{m}\delta}{\overline{SO}(G)^2} \leq \frac{1}{4\delta\Delta}\left(\Delta+\delta\right)^2$$

and so

$$2\sqrt{\overline{m}\Delta\overline{M}_1(G)} \leq \left(1+\frac{\Delta}{\delta}\right)\overline{SO}(G).$$

For the upper bound, letting $a_k \to \sqrt{{d_i}^2 + {d_j}^2}$ and $b_k \to d_i + d_j$ in Lemma 2 with the sums running over the edges in $\overline{G}$, we have

$$\sum_{v_iv_j \in E(\overline{G})} \frac{{d_i}^2 + {d_j}^2}{d_i + d_j} \geq \frac{\left(\sum\limits_{v_iv_j \in E(\overline{G})} \sqrt{{d_i}^2 + {d_j}^2}\right)^2}{\sum\limits_{v_iv_j \in E(\overline{G})} [d_i + d_j]}. \tag{4}$$

Notice that $\frac{{d_i}^2 + {d_j}^2}{d_i + d_j} \leq \Delta$. So, $\sum\limits_{v_iv_j \in E(\overline{G})} \frac{{d_i}^2 + {d_j}^2}{d_i + d_j} \leq \overline{m}\Delta$. Thus (4) becomes $\frac{\overline{SO}(G)^2}{\overline{M}_1(G)} \leq \overline{m}\Delta$. Hence $\overline{SO}(G) \leq \sqrt{\overline{m}\Delta\overline{M}_1(G)}$. $\square$

Lastly, we present the relation of Sombor coindex and the second Zagreb coindex.

Theorem 8. *Let G be a graph on n vertices and m edges. Then*

$$\overline{SO}(G) \leq \sqrt{\left(\frac{\delta}{\Delta} + \frac{\Delta}{\delta}\right)\overline{m}\overline{M}_2(G)}.$$

Proof. We first recall the Cauchy-Schwarz inequality. Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ be two sequences of real numbers. Then

$$\left(\sum_{k=1}^{m} a_k b_k\right)^2 \leq \sum_{k=1}^{m} a_k^2 \sum_{k=1}^{m} b_k^2.$$

Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let d_i denotes the degree of the vertex v_i in G. Letting $a_k \to \sqrt{d_i d_j}$ and $b_k \to \sqrt{\frac{{d_i}^2 + {d_j}^2}{d_i d_j}}$ in the Cauchy-Schwarz inequality with the sums running over the edges in $\overline{G}$, we have

$$\left(\sum_{v_iv_j \in E(\overline{G})} \sqrt{{d_i}^2 + {d_j}^2}\right)^2 \leq \sum_{v_iv_j \in E(\overline{G})} d_i d_j \sum_{v_iv_j \in E(\overline{G})} \frac{{d_i}^2 + {d_j}^2}{d_i d_j}. \tag{5}$$

Since $0 < \delta \leq d_i \leq \Delta$ for any v_i, we have $\frac{\delta}{\Delta} \leq \frac{d_i}{d_j} \leq \frac{\Delta}{\delta}$. Now for any edge v_iv_j of G $(d_i \geq d_j)$, we have

$$\begin{aligned}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right)^2 &= \left(\frac{d_i}{d_j} - \frac{d_j}{d_i}\right)^2 + 4 \\ &\leq \left(\frac{\Delta}{\delta} - \frac{\delta}{\Delta}\right)^2 + 4 = \left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right)^2.\end{aligned}$$

Thus (5) becomes $\overline{SO}(G) \leq \sqrt{\left(\frac{\delta}{\Delta}+\frac{\Delta}{\delta}\right)\overline{m}\overline{M}_2(G)}$. □

Remark 7. We note that with some easy manipulations, lower bound of Sombor coindex in terms of second Zagreb coindex is obtained. More precisely, $\frac{\sqrt{2}}{\Delta}\overline{M}_2(G) \leq \overline{SO}(G)$, since

$$\begin{aligned}\overline{SO}(G) &= \sum_{uv\in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2} \\ &= \sum_{uv\in E(\overline{G})} d_u(G)d_v(G)\sqrt{\frac{1}{d_G(u)^2}+\frac{1}{d_G(v)^2}} \\ &\geq \sum_{uv\in E(\overline{G})} d_u(G)d_v(G)\sqrt{\frac{1}{\Delta^2}+\frac{1}{\Delta^2}} = \frac{\sqrt{2}\overline{M}_2(G)}{\Delta}.\end{aligned}$$

We give another upper bound following the same argument, $\overline{SO}(G) \leq \frac{\sqrt{2}}{\delta}\overline{M}_2(G)$.

5. Bounds on the Sombor coindex of graph operations

Since several complicated and important graphs often arise from simpler graphs via some graph operations, we also present the Sombor coindex of some graph operations in this section. We give the bounds on the Sombor coindex of some graph operations namely, union, sum, composition and Cartesian product. As an application, the Sombor coindex of some well-known (chemical) graphs are computed. First, we recall the definitions of the following operations which are found in standard references or [1].

A *union* $G_1 \cup G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$.

A *sum* $G_1 + G_2$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2) \cup \{u_1u_2 : u_1 \in V(G_1), u_2 \in V(G_2)\}$. Hence, we keep all edges of both the graphs and also join each vertex of one graph to each vertex of the other graph.

The *Cartesian product* $G_1 \square G_2$ of graphs G_1 and G_2 is the graph with the vertex set $V(G_1) \times V(G_2)$ in which $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever ($u_1 = v_1$ and $u_2v_2 \in E(G_2)$) or ($u_2 = v_2$ and $u_1v_1 \in E(G_1)$). Notice that the number of edges in $G_1 \square G_2$ is $n_1m_2 + m_1n_2$ and the degree of a vertex (u_1, u_2) of $G_1 \square G_2$ is $d_{G_1}(u_1) + d_{G_2}(u_2)$, where $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$ for $i = 1, 2$.

The *composition* $G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets and edge sets is the graph with the vertex set $V(G_1) \times V(G_2)$ in which $u = (u_1, u_2)$ is adjacent with $v = (v_1, v_2)$ whenever (u_1 is adjacent with v_1) or ($u_1 = v_1$ and u_2 is adjacent with v_2). Notice that the number of edges in $G_1[G_2]$ is $n_1m_2 + m_1n_2^2$ and the degree of a vertex (u_1, u_2) of $G_1[G_2]$ is $n_2d_{G_1}(u_1) + d_{G_2}(u_2)$, where $n_i = |V_i|$ and $m_i = |E_i|$ for $i = 1, 2$.

Theorem 9. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Then we have the following.*

(i) $\overline{SO}(G_1 \cup G_2) \leq \overline{SO}(G_1) + \overline{SO}(G_2) + n_1 n_2 \sqrt{{\Delta_1}^2 + {\Delta_2}^2}$.

(ii) $\overline{SO}(G_1 \cup G_2) \geq \overline{SO}(G_1) + \overline{SO}(G_2) + n_1 n_2 \sqrt{{\delta_1}^2 + {\delta_2}^2}$.

Here, Δ_i and δ_i denote the maximum degree vertex and the minimum degree vertex of G_i, respectively for $i = 1, 2$. Moreover, the equality holds if G_1 and G_2 are regular.

Proof. Let $G = G_1 \cup G_2$. By the definition of Sombor coindex, we have

$$\begin{aligned}
\overline{SO}(G) &= \sum_{uv \in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2} \\
&= \sum_{uv \in E(\overline{G}_1)} \sqrt{d_{G_1}(u)^2 + d_{G_1}(v)^2} + \sum_{uv \in E(\overline{G}_2)} \sqrt{d_{G_2}(u)^2 + d_{G_2}(v)^2} \\
&+ \sum_{u \in V(G_1)} \left[\sum_{v \in V(G_2)} \sqrt{d_{G_1}(u)^2 + d_{G_2}(v)^2} \right].
\end{aligned}$$

Notice that the last sum is the contribution to the Sombor coindex of the union from the missing edges between the components, which are the edges of the complete bipartite graph K_{n_1,n_2}. Thus,

$$\overline{SO}(G_1 \cup G_2) \leq \overline{SO}(G_1) + \overline{SO}(G_2) + n_1 n_2 \sqrt{{\Delta_1}^2 + {\Delta_2}^2}.$$

Moreover, it is easy to notice that the equality holds if G_1 and G_2 are regular. Similarly, the lower bound follows. $\square$

Theorem 10. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then we have the following.*

(i) $\overline{SO}(G_1 + G_2) \leq \sqrt{2}[\overline{m}_1(\Delta_1 + n_2) + \overline{m}_2(\Delta_2 + n_1)]$.

(ii) $\overline{SO}(G_1 + G_2) \geq \sqrt{2}[\overline{m}_1(\delta_1 + n_2) + \overline{m}_2(\delta_2 + n_1)]$.

Here, Δ_i and δ_i denote the maximum degree vertex and the minimum degree vertex of G_i, respectively for $i = 1, 2$. Moreover, the equality holds if G_1 and G_2 are regular.

Proof. Let $G = G_1 + G_2$. Notice that $d_G(u) = d_{G_1}(u) + n_2$ and $d_G(v) = d_{G_2}(v) + n_1$ for $u \in V(G_1), v \in V(G_2)$. Since all possible edges between G_1 and G_2 are present in

G, there are no missing edges, and hence their contribution is zero. Thus,

$$\begin{aligned}\overline{SO}(G) &= \sum_{uv\in E(\overline{G}_1)} \sqrt{d_G(u)^2 + d_G(v)^2} + \sum_{uv\in E(\overline{G}_2)} \sqrt{d_G(u)^2 + d_G(v)^2} \\ &= \sum_{uv\in E(\overline{G}_1)} \sqrt{(d_{G_1}(u)+n_2)^2 + (d_{G_1}(v)+n_2)^2} \\ &+ \sum_{uv\in E(\overline{G}_2)} \sqrt{(d_{G_2}(u)+n_1)^2 + (d_{G_2}(v)+n_1)^2} \\ &\leq \sqrt{2}[\overline{m}_1(\Delta_1+n_2) + \overline{m}_2(\Delta_2+n_1)].\end{aligned}$$

Moreover, the equality holds if G_1 and G_2 are regular. Similarly, the lower bound follows. □

Corollary 3. *The Sombor coindex of the complete bipartite graph $K_{p,q}$ is given by*

$$\overline{SO}(K_{p,q}) = \overline{SO}(\overline{K_p} + \overline{K_q}) = \frac{pq(p+q-2)}{\sqrt{2}}.$$

Remark 8. We thus obtain explicit formulae for the Sombor coindex of the n-vertex star graph $S_n = K_{1,n-1}$ for $n \geq 2$ via Corollary 3, i.e.,

$$\overline{SO}(S_n) = \frac{(n-1)(n-2)}{\sqrt{2}}.$$

Theorem 11. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

$$\overline{m}\sqrt{2}(\delta_1+\delta_2) \leq \overline{SO}(G_1 \square G_2) \leq \overline{m}\sqrt{2}(\Delta_1+\Delta_2).$$

Here, Δ_i and δ_i denote the maximum degree vertex and the minimum degree vertex of G_i, respectively for $i = 1,2$; and $\overline{m}$ is the number of edges in $\overline{G_1 \square G_2}$. Moreover, the equality holds if G_1 and G_2 are regular.

Proof. Let $G = G_1 \square G_2$. Let $n = |V(G)|$ and $m = |E(G)|$. Notice that $n = n_1 n_2$ and $m = n_1 m_2 + m_1 n_2$. So, the number of edges in $\overline{G}$, $\overline{m} = \binom{n_1 n_2}{2} - n_1 m_2 - m_1 n_2$. By the definition of the Sombor coindex, we have

$$\begin{aligned}\overline{SO}(G) &= \sum_{uv\in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2} \\ &= \sum_{uv\in E(\overline{G})} \sqrt{(d_{G_1}(u_1) + d_{G_2}(u_2))^2 + (d_{G_1}(v_1) + d_{G_2}(v_2))^2} \\ &\leq \sum_{uv\in E(\overline{G})} \sqrt{2}(\Delta_1+\Delta_2) = \overline{m}\sqrt{2}(\Delta_1+\Delta_2).\end{aligned}$$

Moreover, the equality holds if G_1 and G_2 are regular. Similarly, the lower bound follows. □

The following corollary is immediate for the graph $G = C_p \square C_q$. This graph is called C_4 nanotorus.

Corollary 4. *The Sombor coindex of the C_4 nanotorus is given by*

$$\overline{SO}(C_p \square C_q) = 2pq(pq-5)\sqrt{2}.$$

Theorem 12. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Then we have the following.*

$$\overline{m}\sqrt{2}(n_2\delta_1 + \delta_2) \leq \overline{SO}(G_1[G_2]) \leq \overline{m}\sqrt{2}(n_2\Delta_1 + \Delta_2).$$

Here, Δ_i and δ_i denote the maximum degree vertex and the minimum degree vertex of G_i, respectively for $i = 1, 2$; and $\overline{m}$ is the number of edges in $\overline{G_1[G_2]}$. Moreover, the equality holds if G_1 and G_2 are regular.

Proof. Let $G = G_1[G_2]$. Let $n = |V(G)|$ and $m = |E(G)|$. Notice that $n = n_1 n_2$ and $m = n_1 m_2 + m_1 n_2^2$. So, the number of edges in $\overline{G}$, $\overline{m} = \binom{n_1 n_2}{2} - n_1 m_2 - m_1 n_2^2$. By the definition of the Sombor coindex, we have

$$\begin{aligned}
\overline{SO}(G) &= \sum_{uv \in E(\overline{G})} \sqrt{d_G(u)^2 + d_G(v)^2} \\
&= \sum_{uv \in E(\overline{G})} \sqrt{(n_2 d_{G_1}(u_1) + d_{G_2}(u_2))^2 + (n_2 d_{G_1}(v_1) + d_{G_2}(v_2))^2} \\
&\leq \sum_{uv \in E(\overline{G})} \sqrt{2}(n_2\Delta_1 + \Delta_2) = \overline{m}\sqrt{2}(n_2\Delta_1 + \Delta_2).
\end{aligned}$$

Moreover, the equality holds if G_1 and G_2 are regular. Similarly, the lower bound follows. This completes the proof. □

As a corollary, the Sombor coindex of the closed fences $C_n[K_2]$ is immediate.

Corollary 5. *The Sombor coindex of the closed fences $C_n[K_2]$ is given by*

$$\overline{SO}(C_n[K_2]) = 5[2n(n-3)+4]\sqrt{2}.$$

6. Conclusion

Several vertex-degree-based graph invariants (topological indices) have been introduced and studied extensively in (chemical) graph theory and we continue further exploration in this direction based on the recently introduced Sombor (co)index. We give several properties of the Somber coindex and its relations to the Sombor index, the Zagreb coindices, forgotten coindex and other important graph parameters. We also compute the bounds of the Sombor coindex of some graph operations and compute the Sombor coindex of some graphs as application. One could explore further relations between Sombor coindex and other well-known (co)indices. One could also explore Sombor coindex of derived graphs and other graph operations which are of interest in chemical graph theory, such as splices and links of two or more graphs.

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Data Availibility Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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