Communications in Combinatorics and Optimization
Vol. 8, No. 3 (2023), pp. 505-511
DOI: 10.22049/CCO.2022.27816.1359

CCO
Commun. Comb. Optim.

Research Article

An upper bound on triple Roman domination

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Received: 16 April 2022; Accepted: 15 July 2022
Published Online: 18 July 2022

Abstract: For a graph $G = (V, E)$, a triple Roman dominating function (3RD-function) is a function $f : V \to \{0, 1, 2, 3, 4\}$ having the property that (i) if $f(v) = 0$ then v must have either one neighbor u with $f(u) = 4$, or two neighbors u, w with $f(u) + f(w) \geq 5$ or three neighbors u, w, z with $f(u) = f(w) = f(z) = 2$, (ii) if $f(v) = 1$ then v must have one neighbor u with $f(u) \geq 3$ or two neighbors u, w with $f(u) = f(w) = 2$, and (iii) if $f(v) = 2$ then v must have one neighbor u with $f(u) \geq 2$. The weight of a 3RDF f is the sum $f(V) = \sum_{v \in V} f(v)$, and the minimum weight of a 3RD-function on G is the triple Roman domination number of G, denoted by $\gamma_{[3R]}(G)$. In this paper, we prove that for any connected graph G of order n with minimum degree at least two, $\gamma_{[3R]}(G) \leq \frac{3n}{2}$.

Keywords: triple Roman dominating function, triple Roman domination number

AMS Subject classification: 05C69

1. Introduction

Let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = |V(G)|$ and $m = |E(G)|$ are the *order* and the *size* of the graph G, respectively.

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The *open neighborhood* of vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. Let C_n the cycle of order n and P_n the path of order n. A set S of vertices of G is called a *dominating set* if $N[S] = \bigcup_{v\in S} N[v] = V(G)$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G.

Let k be a positive integer and let f be a function that assigns labels from the set $\{0, 1, \ldots, k+1\}$ to the vertices of the graph G. Given a vertex $v \in V(G)$, the *active neighborhood* of v, denoted by $AN(v)$, is the set of vertices $w \in N_G(v)$ such that $f(w) \geq 1$. Let $AN[v] = AN(v) \cup \{v\}$. A $[k]$-*Roman dominating function* is defined in [1] as a function $f : V \to \{0, 1, \ldots, k+1\}$ such that for every vertex $v \in V$ with $f(v) < k$,

$$f\left(AN[v]\right) \geq |AN(v)| + k.$$

The weight of a [k]-Roman dominating function is the value $f(V) = \sum_{v\in V} f(v)$, and the minimum weight of such a function is the [k]-*Roman domination number of* G, denoted by $\gamma_{[kR]}(G)$. Let us point out that for $k = 1$ the definition matches that of the Roman domination. Roman domination was introduced by Cockayne et al. in [8] and was inspired by the marvelous paper of Arquilla and Fredricksen [2] and the work of ReVelle and Rosing [9] and Stewart [10]. Furthermore, for $k = 2$ the above definition matches that of the double Roman domination which was introduced by Beeler et al. in [3]. Roman domination and its variants have many applications in the areas such as facility location problems, planning of defence strategies and surveillance related problems, etc. The literature on this topic has been detailed in two book chapters and two surveys [4–7].

For any function $f : V(G) \to \{0, 1, 2, \ldots, k\}$, define $V_i = \{u \in V(G) \mid f(u) = i\}$ for each $i \in \{0, 1, \ldots, k\}$. Since these $k+1$-sets determine f uniquely, we write $f = (V_0, V_1, \ldots, V_k)$. The weight of f is defined as $\omega(f) = \sum_{v\in V(G)} f(v)$. In this paper we focus on $k = 3$.

Not that, for a graph $G = (V, E)$, a triple Roman dominating function (3RD-function) is a function $f : V \to \{0, 1, 2, 3, 4\}$ having the property that

1. If $f(v) = 0$ then v must have either one neighbor in V_4, or either two neighbors u and w in $V_2 \cup V_3$ such that $f(u) + f(w) \geq 5$ or either three neighbors in V_2,

2. If $f(v) = 1$ then v must have either one neighbor in $V_3 \cup V_4$ or either two neighbors in V_2,

3. If $f(v) = 2$ then v must have one neighbor in $V_2 \cup V_3 \cup V_4$.

The weight of a 3RD-function f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of a 3RD-function on G is the triple Roman domination number of G, denoted by $\gamma_{[3R]}(G)$. The authors of [1] proved that for any connected graph G of order n, $\gamma_{[3R]}(G) \leq \frac{7n}{4}$. In this paper, we prove that for any connected graph G of order n with minimum degree at least two $\gamma_{[3R]}(G) \leq \frac{3n}{2}$.

We make use of the following results in this paper.

Proposition A. [1] *Let G be a graph of order n and maximum degree $\Delta(G)$. Then*

$$\gamma_{[3R]}(G) \leq 3n - 3\Delta(G) + 1.$$

Proposition B. [1] For any integer $n \geq 3$,

$$\gamma_{[3R]}(C_n) = \begin{cases} \lceil \frac{4n}{3} \rceil & \text{if either } n = 4, 5, 7, 10 \text{ or } n \equiv 0 \bmod 3, \\ \lceil \frac{4n}{3} \rceil + 1 & \text{if } n \neq 4, 5, 7, 10 \text{ and } n \equiv 1, 2 \bmod 3. \end{cases}$$

Proposition C. [1] Let n be a positive integer. Then $\gamma_{[3R]} P_n = M_n$, such that

$$M_n = \begin{cases} 4\left\lfloor \dfrac{n}{3} \right\rfloor, & \text{if } n \equiv 0 \bmod 3 \\ 4\left\lfloor \dfrac{n}{3} \right\rfloor + 3, & \text{if } n \equiv 1 \bmod 3 \\ 4\left\lfloor \dfrac{n}{3} \right\rfloor + 4, & \text{if } n \equiv 2 \bmod 3. \end{cases}$$

2. Main results

In this section we establish an upper bound on the triple Roman domination number of graphs with minimum degree 2. We first construct the graph $C_{m,k}$ as follows. For integers $m \geq 3$ and $k \geq 1$, let $C_{m,k}$ be the graph obtained from a cycle $C_m = (x_1 x_2 \dots x_m)$ by adding a pendant path $x_1 y_1 y_2 \dots y_k$, with $y_i \notin V(C_m)$ for all possible i.

Proposition 1. For integers $m \geq 3$ and $k \geq 1$ with $m + k \geq 4$, $\gamma_{[3R]}(C_{m,k}) \leq \frac{3(m+k)}{2}$.

Proof. If $m + k = 4$, then clearly $\gamma_{[3R]}(C_{m,k}) \leq \gamma_{[3R]}(C_{3,1}) = 4 < \frac{3(m+k)}{2}$. Let $m + k \geq 5$. Since $C_{m,k}$ has a Hamilton path and since adding an edge cannot increase the triple Roman domination number, we deduce from Proposition B that $\gamma_{[3R]}(C_{m,k}) \leq \left\lceil \frac{4(m+k)}{3} \right\rceil + 1 \leq \frac{3(m+k)}{2}$ for $m + k \neq 5, 7$. In the cases $m + k = 5$ or $m + k = 7$, it is straightforward to verify the desired bound. $\square$

Let $\mathcal{F}_1$ be the family of all loopless connected multigraphs G with $\delta(G) \geq 3$ and let $\mathcal{F}$ be the family of all graphs obtained from some graph in $\mathcal{F}_1$ by subdividing any edge at least once and at most seven times. Observe that any graph in $\mathcal{F}$ has order at least 5.

Proposition 2. If $G \in \mathcal{F}$, then there exists a 3RD-function f of G such that $\omega(f) \leq \frac{3|V(G)|}{2}$ and f assigns a weight at least three to each vertex of degree at least 3.

Proof. Let $G \in \mathcal{F}$ be a graph of order n. The proof is by the induction on n. The result is immediate for $n = 5$. Suppose $n \geq 6$ and let the result hold for all graphs in $\mathcal{F}$ of order smaller than n. Let $G \in \mathcal{F}$ be a graph of order $n \geq 6$. Let $V_{\geq 3} = \{v \in V(G) \mid \deg(v) \geq 3\}$ and $V_{\leq 2} = \{v \in V(G) \mid \deg(v) = 2\}$. A path P of G is called a $V_{\geq 3}$-ear path if $V(P) \subseteq V_{\leq 2}$ and each end-vertex of P is adjacent to a vertex of $V_{\geq 3}$. For each $i \geq 1$, let $\mathcal{P}_i = \{P \mid P \text{ is a } V_{\geq 3}\text{-ear path with } |V(P)| = i\}$. Let $\mathcal{P} = \bigcup_{i\geq 1} \mathcal{P}_i$. Note that $V_{\geq 3} \cup \bigcup_{P\in\mathcal{P}} V(P)$ is a partition of $V(G)$. For each $V_{\geq 3}$-ear path $P \in \mathcal{P}$, let $X_P = \{u \in V_{\geq 3} \mid u \text{ is adjacent to an end-vertex of } P\}$. Then $V_{\geq 3} = \bigcup_{P\in\mathcal{P}} X_P$ and $|X_P| = 2$ for each $P \in \mathcal{P}$. Hence $|V_{\geq 3}| \geq 2$. We consider the following cases.

Case 1. $\mathcal{P}_3 \cup \mathcal{P}_5 \cup \mathcal{P}_7 \neq \emptyset$.
Suppose $P = x_1x_2\cdots x_{2k+1} \in \mathcal{P}_3 \cup \mathcal{P}_5 \cup \mathcal{P}_7$ and let $X_P = \{a_1, a_2\}$ where $a_1x_1, a_2x_{2k+1} \in E(G)$. Assume that $G' = (G - (V(P) - \{x_2, x_3, \ldots, x_{2k+1}\})) + \{x_1a_2\}$. Clearly $G' \in \mathcal{F}$. By induction hypothesis, there exists a 3RD-function $f = (V_0, V_1, V_2, V_3, V_4)$ of G' such that $a_1, a_2 \in V_3 \cup V_4$, and $\omega(f) \leq \frac{3(n-2k)}{2}$. Then the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ defined by $g(x_{2i}) = 3$ for $1 \leq i \leq k$, $g(x_{2i+1}) = 0$ for $1 \leq i \leq k$ and $g(x) = f(x)$ otherwise, is a 3RD-function of G such that g assigns the weight at least three to every vertex of degree at least 3, and $\omega(g) = \omega(f) + 3k \leq \frac{3(n-2k)}{2} + 3k = \frac{3n}{2}$.

Case 2. $\mathcal{P}_4 \cup \mathcal{P}_6 \neq \emptyset$.
Suppose $P = x_1x_2 \ldots x_{2k} \in \mathcal{P}_4 \cup \mathcal{P}_6$ and $X_P = \{a_1, a_2\}$ where $a_1x_1, a_2x_{2k} \in E(G)$. Assume that $G' = (G - \{x_2, x_3, \ldots, x_{2k-2}\}) + \{x_1x_{2k-1}\}$. Clearly $G' \in \mathcal{F}$. By induction hypothesis, there exists a 3RD-function f of G' such that $f(a_1), f(a_2) \geq 3$, and $\omega(f) \leq \frac{3(n-(2k-1))}{2}$. To triple Roman dominate x_{2k-1}, we may assume that $f(x_{2k-1}) \geq 3$ and $f(x_1) = f(x_{2k}) = 0$. Then the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ defined by $g(x_2) = g(x_3) = 2$ and $g(x) = f(x)$ otherwise, if $k = 2$, and by $g(x_2) = g(x_3) = 2, g(x_4) = 0$ and $g(x) = f(x)$ otherwise, when $k = 3$, is a 3RDF of G such that g assigns a weight at least three to every vertex in $V_{\geq 3}$, and for $k = 2$ we have $\omega(g) = \omega(f) + 1 \leq \frac{3(n-1)}{2} + 1 < \frac{3n}{2}$ and for $k = 3$ we have $\omega(g) = \omega(f) + 4 \leq \frac{3(n-3)}{2} + 4 < \frac{3n}{2}$.

Case 3. $\bigcup_{i=3}^{7} \mathcal{P}_i = \emptyset$.
Therefore $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Note that $n = |V_{\geq 3}| + m_1 + 2m_2$ and $m_1 + m_2 \geq 3$, where $m_i = |\mathcal{P}_i|$ for $i \in \{1, 2\}$. If $|V_{\geq 3}| = 2$, then let $\mathcal{P}_2 = \{v_1^i v_2^i | 1 \leq i \leq m_2\}$ if $\mathcal{P}_2 \neq \emptyset$ and $\mathcal{P}_1 = \{w_1^j | 1 \leq j \leq m_1\}$ if $\mathcal{P}_1 \neq \emptyset$. If $\mathcal{P}_2 \neq \emptyset$, then the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ defined by $g(x) = 4$ for $x \in V_{\geq 3}$, $g(v_1^i) = g(v_2^i) = 0$ for each $1 \leq i \leq m_2$, $g(w_1^j) = 0$ for each $1 \leq j \leq m_1$, is a 3RD-function of G such that g assigns a weight at least three to every vertex of degree at least 3, and we have $\omega(g) \leq 8 \leq \frac{3(2+2m_2+m_1)}{2} = \frac{3n}{2}$. Assume that $\mathcal{P}_2 = \emptyset$. Then $m_1 \geq 3$ and the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ defined by $g(x) = 3$ for $x \in V_{\geq 3}$, $g(w_1^j) = 0$ for each $1 \leq j \leq m_1$, is a 3RD-function of G such that g assigns a weight at least three to every in $V_{\geq 3}$, and we have $\omega(g) \leq 6 \leq \frac{3(2+m_1)}{2} = \frac{3n}{2}$. Henceforth, we assume $|V_{\geq 3}| \geq 3$. Let u be a vertex in $V_{\geq 3}$ such that $|N(u) \cap (\bigcup_{P\in\mathcal{P}_2} V(P))|$ is maximum. We consider the following subcases.

Subcase 3.1. u is adjacent to at least two ear-paths in $\mathcal{P}_2$.
Let $P' = x_1x_2$ and $P'' = y_1y_2$ be two ear-paths in $\mathcal{P}_2$ such that $ux_1, uy_1 \in E(G)$ and let $ax_2, by_2 \in E(G)$ where $a, b \in V_{\geq 3} \setminus \{u\}$. First let $a \neq b$. Assume that G' is the graph obtained from G by removing the vertices u, y_1, x_1 and joining x_2 to y_2 and joining every vertex x in $N(u) - \{x_1, y_1\}$ to either a or b provided a or b is not adjacent to the end-vertex of the ear-path containing x. Clearly $G' \in \mathcal{F}$. By the induction hypothesis, there exists a 3RD-function $f = (V_0, V_1, V_2, V_3, V_4)$ of G' such that $(V_{\geq 3} - \{u\}) \subseteq V_3 \cup V_4$, and $\omega(f) \leq \frac{3(n-3)}{2}$. We may assume that $f(a) = f(b) = 4$. Define the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(u) = 4$, $g(y_1) = g(x_1) = 0$ and $g(x) = f(x)$ otherwise. Clearly, g is a 3RD-function of G such that g assigns a weight at least three to every vertex in $V_{\geq 3}$, and $\omega(g) = \omega(f) + 4 \leq \frac{3(n-3)}{2} + 4 < \frac{3n}{2}$.
Now let $a = b$. Suppose G' is the graph obtained from $G - \{x_2\}$ by joining x_1 to a. Clearly $G' \in \mathcal{F}$ and by the induction hypothesis, there exists a 3RD-function $f = (V_0, V_1, V_2, V_3, V_4)$ of G' such that $V_{\geq 3} \subseteq V_3 \cup V_4$, and $\omega(f) \leq \frac{3(n-1)}{2}$. We may assume that $f(u) = f(a) = 4$. Define the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(x_2) = 0$ and $g(x) = f(x)$ otherwise. Clearly, g is a 3RD-function of G such that g assigns a weight at least three to every vertex in $V_{\geq 3}$, and $\omega(g) = \omega(f) \leq \frac{3(n-1)}{2} < \frac{3n}{2}$.

Subcase 3.2. u is adjacent to exactly one $V_{\geq 3}$-ear path in $\mathcal{P}_2$.
Let $P' = x_1x_2$ be a $V_{\geq 3}$-ear path in $\mathcal{P}_2$ such that $ux_1 \in E(G)$ and let $ax_2 \in E(G)$ where $a \in V_{\geq 3} - \{u\}$. By the choice of u, we deduce that each vertex in $V_{\geq 3}$ is adjacent to at most one $V_{\geq 3}$-ear path in $\mathcal{P}_2$ and so each vertex in $V_{\geq 3}$ is adjacent to at least two $V_{\geq 3}$-ear paths in $\mathcal{P}_1$. Simple verification on the number of edges between $V_{\geq 3}$ and $\bigcup_{P \in \mathcal{P}_1} V(P)$ imply that $|V_{\geq 3}| \leq m_1$. Let $A' = \{u \in V_{\geq 3} \mid u$ is adjacent to an end-vertex of a $V_{\geq 3}$-ear path in $\mathcal{P}_2\}$ and $A'' = V_{\geq 3} - A'$. Again, by simple verification on the number of edges between A' and $\bigcup_{P \in \mathcal{P}_2} V(P)$, we imply that $|A'| \leq 2m_2$. Define the function $g : V(G) \to \{0, 1, 2, 3, 4\}$ by $g(x) = 4$ for $x \in A'$, $g(x) = 3$ for $x \in A''$ and $g(x) = 0$ otherwise. It is easy to see that g is a 3RD-function of G that assigns a weight at least three to every vertex in $V_{\geq 3}$ and we have

$$\begin{aligned}
\omega(g) &\leq 4|A'| + 3|A''| \\
&= 3|V_{\geq 3}| + |A'| \\
&\leq \tfrac{3}{2}|V_{\geq 3}| + \tfrac{3}{2}m_1 + 2m_2 \\
&< \tfrac{3(|V_{\geq 3}| + m_1 + 2m_2)}{2} \\
&= \tfrac{3n}{2}.
\end{aligned}$$

Subcase 3.3. $\mathcal{P} = \mathcal{P}_1$.
Since $G \in \mathcal{F}$, G is obtained from a connected loopless multigraph G' with minimum degree at least 3, by subdividing each edge of G' once. Clearly $|E(G')| \geq \frac{3}{2}|V(G')|$ and so $n(G) = |V(G')| + |E(G')| \geq \frac{5}{2}|V(G')|$. Define $f : V(G) \to \{0, 1, 2, 3, 4\}$ by $f(x) = 3$ for $x \in V(G')$ and $f(x) = 0$ otherwise. Clearly f is a 3RD-function of G

such that f assigns a weight at least three to every vertex in $V_{\geq 3}$, and

$$\omega(f) = 3|V(G')| < \frac{15}{4}|V(G')| \leq \frac{3n}{2},$$

and this completes the proof. $\square$

Theorem 1. For any connected n-vertex graph G with $\delta(G) \geq 2$,

$$\gamma_{[3R]}(G) \leq \frac{3n}{2}.$$

This bound is sharp for C_4, C_8.

Proof. The proof is by induction on n. If $n = 6$ and $\Delta(G) \neq 3$, then the result follows by Propositions A and B. If $n = 6$, $\Delta(G) = 3$, then let v be a vertex with degree 3 and $V(G) - N[v] = \{a, b\}$. If a and b are not adjacent, then $(V(G) - N(v), \emptyset, \emptyset, N(v), \emptyset)$ is a 3RD-function of weight 9. If a and b are not adjacent, then $(V(G) - \{a, v\}, \emptyset, \emptyset, \emptyset, \{a, v\})$ is a 3RD-function of weight 8. In both cases $\gamma_{[3R]}(G) \leq \frac{3n}{2}$. Suppose $n \geq 7$ and the result holds for all graphs G for order smaller than n with $\delta(G) \geq 2$. Let G be a graph of order $n \geq 7$ and $\delta(G) \geq 2$. Since $\gamma_{[3R]}(G) \leq \gamma_{[3R]}(G - e)$ for every $e \in E(G)$, we may assume that $|E(G)|$ is as small as possible. If G is disconnected and $G_1, G_2, \ldots, G_t$ are the components of G, then it follows from the induction hypothesis that $\gamma_{[3R]}(G_i) \leq \frac{3|V(G_i)|}{2}$ for each i and so $\gamma_{[3R]}(G) = \sum_{i=1}^{t} \gamma_{[3R]}(G_i) \leq \sum_{i=1}^{t} \frac{3|V(G_i)|}{2} = \frac{3n}{2}$. Let G be connected. If $\Delta(G) = 2$, then the result follows by Proposition B. Assume that $\Delta(G) \geq 3$. Let $V_{\geq 3} = \{v \in V(G) \mid \deg(v) \geq 3\}$ and $V_{\leq 2} = \{v \in V(G) \mid \deg(v) = 2\}$. Consider the $V_{\geq 3}$-ear paths and associated notations defined in the proof of by Proposition 2. Note that $V_{\geq 3} = \bigcup_{P \in \mathcal{P}} X_P, V_{\geq 3} \cup \bigcup_{P \in \mathcal{P}} V(P)$ is a partition of $V(G)$ and $1 \leq |X_P| \leq 2$ for each $P \in \mathcal{P}$. We consider the following cases.

Case 1. There exists a $V_{\geq 3}$-ear path P such that $\delta(G - V(P)) \leq 1$.
This implies that $|X_P| = 1$ and since G is simple we have $|V(P)| \geq 2$. Suppose that $X_P = \{a\}$ and $N_G(a) - V(P) = \{b\}$. Then there exists the unique $V_{\geq 3}$-ear path P' such that b is an end-vertex of P'. Let $G' = G - (V(P) \cup V(P') \cup \{a\})$. Then $\delta(G') \geq 2$ and by the induction hypothesis $\gamma_{[3R]}(G') \leq \frac{3|V(G')|}{2}$. On the other hand, since $G'' = G[V(P) \cup V(P') \cup \{a\}] \cong C_{|V(P)|+1, |V(P')|}$, we have $\gamma_{[3R]}(G[V(P) \cup V(P') \cup \{a\}]) \leq \frac{3|V(P) \cup V(P') \cup \{a\}|}{2}$, By Proposition 1. If f is a $\gamma_{[3R]}(G')-$function and g is a $\gamma_{[3R]}(G'')$-function, then the function h defined on $V(G)$ by $h(x) = f(x)$ for $x \in V(G')$ and $h(x) = g(x)$ for $x \in V(G'')$, is a 3RD-function on G and we have

$$\begin{aligned}
\gamma_{[3R]}(G) &\leq \gamma_{[3R]}(G') + \gamma_{[3R]}(G'') \\
&\leq \tfrac{3|V(G')|}{2} + \tfrac{3|V(P) \cup V(P') \cup \{a\}|}{2} \\
&= \tfrac{3n}{2}.
\end{aligned}$$

Case 2. For each $V_{\geq 3}$-ear path $P \in \mathcal{P}$, $\delta(G - V(P)) \geq 2$.
It follows that $|X_P| = 2$ for each $V_{\geq 3}$-ear path $P \in \mathcal{P}$. If $\mathcal{P} = \bigcup_{i=1}^{7} \mathcal{P}_i$, then $G \in \mathcal{F}$ and the result follows from Proposition 2. Assume that $\mathcal{P} \neq \bigcup_{i=1}^{7} \mathcal{P}_i$ and let $\mathcal{Q} \in \mathcal{P} \setminus \bigcup_{i=1}^{7} \mathcal{P}_i$. Suppose $G' = G - V(\mathcal{Q})$. By Proposition C and the induction hypothesis we have $\gamma_{[3R]}(\mathcal{Q}) \leq \frac{3|V(\mathcal{Q})|}{2}$ and $\gamma_{[3R]}(G') \leq \frac{3|V(G')|}{2}$. If f is a $\gamma_{[3R]}(G')$-function and g is a $\gamma_{[3R]}(\mathcal{Q})$-function, then the function h defined on $V(G)$ by $h(x) = f(x)$ for $x \in V(G')$ and $h(x) = g(x)$ for $x \in V(\mathcal{Q})$, is a 3RD-function of G and hence $\gamma_{[3R]}(G) \leq \gamma_{[3R]}(G') + \gamma_{[3R]}(\mathcal{Q}) \leq \frac{3n}{2}$. $\square$

Acknowledgments

H. Abdollahzadeh Ahangar was supported by the Babol Noshirvani University of Technology under research Grant Number BNUT/385001/1401.

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