Communications in Combinatorics and Optimization
Vol. 8, No. 4 (2023) pp. 631-637
DOI: 10.22049/CCO.2022.27871.1376

CCO
Commun. Comb. Optim.

Research Article

Domination parameters of the splitting graph of a graph

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Received: 8 May 2022; Accepted: 7 July 2022
Published Online: 10 July 2022

Abstract: Let $G = (V, E)$ be a graph of order n and size m. The graph $Sp(G)$ obtained from G by adding a new vertex v' for every vertex $v \in V$ and joining v' to all neighbors of v in G is called the splitting graph of G. In this paper, we determine the domination number, the total domination number, connected domination number, paired domination number and independent domination number for the splitting graph $Sp(G)$.

Keywords: splitting graph, domination, total domination, connected domination, paired domination, independent domination

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [4].

The concept of graph operator, which is the process of constructing a new graph from a given graph, plays a major role in almost all aspects of graph theoretic research. One of the richest and most studied graph operators is the line graph $L(G)$ of a graph G. The study of line graphs and its variants has witnessed an explosive growth and a recent monograph on line graphs and line digraphs by Beineke and Bagga [3] provides an exhaustive account of primary results on this topic. Domination in line graphs is simply edge domination which has been extensively investigated. A variant of edge domination has been investigated in [1].

The Mycielskian $\mu(G)$ of a graph G is a graph operator and was introduced primarily to prove the existence of triangle-free graphs with arbitrarily large chromatic number. For a study of domination parameters in Mycielskian, we refer to [9].

The main focus of this paper is a study of some of the domination related parameters in the splitting graph $Sp(G)$ of G. The splitting graph is a graph operator which was introduced by Sampathkumar and Walikar [10]

Definition 1. [10] The graph $Sp(G)$ obtained from a graph G by adding a vertex v' for every $v \in V(G)$ and joining v' to all the neighbors of v is called the splitting graph of G.

We observe that $N(v) = N(v')$ and the vertices v and v' are called twin vertices. Also if G is a disconnected graph with k components, then $Sp(G)$ is also a disconnected graph with k components. Hence in the rest of the paper we assume that G is a connected nontrivial graph.

Domination in graphs is one of the major research areas within graph theory. For an excellent treatment of fundamentals of domination in graphs we refer to the book by Haynes et al. [5]. We need the following definition.

Definition 2. Let $G = (V, E)$ be a graph of order n. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. If S is both a dominating set and an independent set, then S is called an independent dominating set. The independent domination number $i(G)$ is the minimum cardinality of an independent dominating set of G. For a graph G without isolated vertices, a dominating set S of G is called a total dominating set if the induced subgraph $G[S]$ has no isolated vertices. If $G[S]$ has a perfect matching, then S is called a paired dominating set of G. If G is connected and the induced subgraph $G[S]$ is connected, then S is called a connected dominating set of G. The total domination number $\gamma_t(G)$, paired domination number $\gamma_{pr}(G)$ and the connected domination number $\gamma_c(G)$ are the minimum cardinality of a total dominating set, paired dominating set and a connected dominating set respectively.

For recent results on paired-domination, total domination and connected domination we refer to [6–8].

In this paper we determine the values of domination related parameters for the splitting graph $Sp(G)$ of a graph G.

2. Main Results

The domination number of the splitting graph $Sp(G)$ is the total domination number of G, as shown in the following theroem.

Theorem 1. *Let G be a connected nontrivial graph. Then $\gamma(Sp(G)) = \gamma_t(G)$.*

Proof. Let D be a total dominating set of G with $|D| = \gamma_t(G)$. Let $v' \in V(Sp(G)) - V(G)$. Since D is a total dominating set of G, $N(v) \cap D \neq \emptyset$. Let $w \in N(v) \cap D$. Since $N(v') = N(v)$, it follows that v' is dominated by w. Thus D is a dominating set of $Sp(G)$ and so $\gamma(Sp(G)) \leq \gamma_t(G)$.
Now, let A be any dominating set of $Sp(G)$ with $|A| = \gamma(Sp(G))$. Suppose A has an isolated vertex x. Then $N(x) \cap A = \emptyset$. Hence $N(y) \cap A = \emptyset$, where y is the twin of x and so $y \in A$. Thus x and y are both isolated vertices in A. We may assume, without loos of generality, that $x \in V(G)$ and $y \in V(Sp(G)) - V(G)$. Cleary, $A_1 = (A - \{y\}) \cup \{w\}$ where w is any vertex in $N(x)$ is a dominating set of $Sp(G)$ in which x is not an isolated vertex. By repeating this process we obtain a dominating set D of $Sp(G)$ such that $|A| = |D|, D \subseteq V(G)$ and D has no isolated vertices. Hence $\gamma(Sp(G)) = |A| = |D| \geq \gamma_t(G)$. Thus $\gamma(Sp(G)) = \gamma_t(G)$. □

Corollary 1. *Let G be a connected nontrivial graph. Then $\gamma(Sp(G)) = \gamma(G)$ if and only if $\gamma(G) = \gamma_t(G)$.*

Example 1. The complete bipartite graph $K_{r,s}$ where $r \geq s \geq 2$, the bistar $B(r, s)$ and the corona $G \circ K_1$ where G is any connected graph of order n, are examples of graphs with $\gamma(G) = \gamma_t(G) = \gamma(Sp(G))$.

The following theorem shows that the total domination number γ_t remains unaltered under the graph operator which transforms G to $Sp(G)$.

Theorem 2. *Let G be a graph of order n having no isolated vertices. Then $\gamma_t(Sp(G)) = \gamma_t(G)$.*

Proof. Let D be a total dominating set of G with $|D| = \gamma_t(G)$. Since D has no isolated vertices, it follows from Theorem 2.1 that D is a dominating set of $Sp(G)$. Hence D is a total dominating set of $Sp(G)$ and so $\gamma_t(Sp(G)) \leq \gamma_t(G)$. Also $\gamma_t(Sp(G)) \geq \gamma(Sp(G)) = \gamma_t(G)$. Hence $\gamma_t(Sp(G)) = \gamma_t(G)$. □

Corollary 2. *Let G be any connected nontrivial graph. Then $\gamma(Sp(G)) = \gamma_t(Sp(G))$.*

Proof. Follows from Theorem 2.1 and Theorem 2.4 □

Observation 3. Let P be a graph theoretic property. If any graph G can be embedded as an induced subgraph of a graph H where H has property P, then a forbidden subgraph characterization of graphs with property P is not possible. Now, let P denote the property $\gamma(G) = \gamma_t(G)$. Corollary 2 shows that G can be embedded as an induced subgraph of $Sp(G)$ and $\gamma(Sp(G)) = \gamma_t(Sp(G))$. Hence there is no forbidden subgraph characterization of graphs satisfying $\gamma(G) = \gamma_t(G)$. Obtaining a necessary condition or a sufficient condition in terms of forbidden subgraphs for graphs satisfying $\gamma(G) = \gamma_t(G)$ is an interesting direction for further research. Results in this direction for graphs with $\gamma(G) = i(G)$, where $i(G)$ is the independent domination number, have been reported in [2] and [11].

The following theorem shows that the connected domination number is also unaltered except for one case, under the splitting graph operator.

Theorem 4. *Let G be a connected graph of order n. Then*

$$\gamma_c(Sp(G)) = \begin{cases} 2 & \text{if } \gamma_c(G) = 1 \\ \gamma_c(G) & \text{if } \gamma_c(G) \geq 2. \end{cases}$$

Proof. We consider two cases.
Case 1. $\gamma_c(G) = 1$.
Then $\Delta(G) = n - 1$. Also the order of $Sp(G)$ is $2n$ and there is no vertex of degree $2n-1$ in $Sp(G)$. Also $\{u, v\}$ where u and v are two adjacent vertices in G is a connected dominating set of $Sp(G)$. Hence $\gamma_c(Sp(G)) = 2$.
Case 2. $\gamma_c(G) \geq 2$.
Let D be a γ_c-set of G. Since $|D| \geq 2, D$ has no isolated vertices and hence D is a connected dominating set of $Sp(G)$. Hence $\gamma_c(Sp(G)) \leq \gamma_c(G)$. Now, let A be a connected dominating set of $Sp(G)$ such that $|A| = \gamma_c(Sp(G))$. Let $A \cap V(G) = A_1$ and $A \cap (V(Sp(G)) - V(G)) = A_2$. Let $v' \in A_2$. Since A is a connected dominating set there exists $u \in A_1$ such that $u \in N_{Sp(G)}(v')$. Since v' is adjacent to u, it follows that $u \in N_G(v)$. If $v \in A$, then v and v' dominate the same set of vertices and v is adjacent to u. Hence $A - \{v'\}$ is a connected dominating set of $Sp(G)$ and $|A - \{v'\}| = \gamma_c(Sp(G)) - 1$, which is a contradiction. Hence $v \notin A$. Now $(A - \{v'\}) \cup \{v\}$ is a connected dominating set of $Sp(G)$. By repeating this process for every vertex in A_2, we obtain a connected dominating set B of $Sp(G)$ such that $B \subseteq V(G)$ and $|B| = |A| = \gamma_c(Sp(G))$. Now B is a connected dominating set of G and hence $\gamma_c(G) \leq |B| = \gamma_c(Sp(G))$.
Thus $\gamma_c(G) \leq \gamma_c(Sp(G))$ and therefore $\gamma_c(G) = \gamma_c(Sp(G))$. □

We now proceed to prove that paired domination number $\gamma_{pr}(G)$ is also unaltered by the splitting graph operator.

Theorem 5. *Let G be a graph of order n without isolated vertices. Then $\gamma_{pr}(G) = \gamma_{pr}(Sp(G))$.*

Proof. Let D be a paired dominating set of G. Since D has no isolated vertices, D is a dominating set of $Sp(G)$. Hence $\gamma_{pr}(Sp(G)) \leq \gamma_{pr}(G)$. Now, let A be a paired dominating set of $Sp(G)$ such that $|A| = \gamma_{pr}(Sp(G))$ and $|A \cap V(G)|$ is as large as possible. Let $A_1 = A \cap V(G)$ and $A_2 = A \cap (V(Sp(G) - V(G))$. Suppose there exists a vertex $w \in A_1$ such that w is paired with a vertex x' in A_2 under the perfect matching in A. Suppose $x \notin A_1$. Then $B = (A_1 \cup \{x\}) \cup (A_2 - \{x'\})$ is a paired dominating set of $Sp(G)$ in which w is matched with x. Also $|B \cap V(G)| > |A \cap V(G)|$, which is a contradiction. Hence $x \in A_1$. If all neighbors of w in G belong to A_1, then clearly

$A - \{w, x'\}$ is a paired dominating set of $Sp(G)$, which is a contradiction. Hence w has a neighbor w' in $N_G(w) - A_1$. Now it can be easily seen that $B = (A - \{x'\}) \cup \{w\}$ is a paired dominating set of $Sp(G)$ such that $|B \cap V(G)| > |A \cap V(G)|$, which is a contradiction. Thus each vertex in A_1 is matched with a vertex in A_1 and $A_2 = \phi$. Hence $A = A_1$ and A is a paired dominating set of G. Therefore $\gamma_{pr}(Sp(G)) = |A| \geq \gamma_{pr}(G)$ and so $\gamma_{pr}(Sp(G)) = \gamma_{pr}(G)$.

□

Sampthkumar and Walikar [10] have proved that for any graph G the independence number $\beta_o(Sp(G))$ is twice the independence number of G. We now proceed to investigate the validity of a similar result for the independent domination number $i(G)$. If G is a connected bipartite graph of order n with bipartition V_1 and V_2, then both V_1 and V_2 are independent dominating sets of G. Hence $i(G) \leq \min\{|V_1|, |V_2|\} \leq \frac{n}{2}$. However, for nonbipartite graphs $i(G)$ can exceed $\frac{n}{2}$. Consider the graph G given in Figure 1.

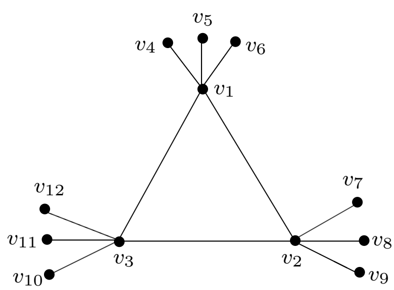

Figure 1. **A graph G with $i(G) > \frac{n}{2}$.**

Clearly G is a nonbipartite graph of order 12 and any independent dominating set of G contains at most one vertex of the triangle (v_1, v_2, v_3, v_1). Hence $S = \{v_1, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is an independent dominating set of minimum cardinality and $i(G) = |S| = 7 > \frac{n}{2}$. The above example can be generalized to construct infinite families of graphs of order n with $i(G) > \frac{n}{2}$. In the following theorem we determine $i(Sp(G))$, which depends on whether $i(G) > \frac{n}{2}$ or not.

Theorem 6. *Let G be a connected graph of order n. Then*

$$i(Sp(G)) = \begin{cases} 2i(G) & \text{if} \quad i(G) < \frac{n}{2} \\ n & \text{otherwise.} \end{cases}$$

Proof. We consider two cases.

Case 1. $i(G) < \frac{n}{2}$.

Let S be an independent dominating set of G with $|S| = i(G)$. Let $S' = \{v' : v \in S\}$. Since S is independent, for any $v \in S, N(v) \cap S = \emptyset$ and hence $N(v') \cap S = \emptyset$.

Thus the vertices of S' are not dominated by S. Now let $w \in V(G) - S$. Then w is adjacent to a vertex v in S and hence w' is also adjacent to v. Hence w' is dominated by S. Thus the only vertices which are not dominated by S are the vertices in S'. Also S' is independent and no vertex of S' is adjacent to a vertex in S. Hence $S \cup S'$ is an independent dominating set of $Sp(G)$ and $|S \cup S'| = 2|S| = 2i(G)$. Thus $i(Sp(G)) \leq 2i(G)$.

Now, let A be any independent dominating set of $Sp(G)$ with $|A| = i(Sp(G))$. Let $A \cap V(G) = A_1$ and $A \cap (V(Sp(G)) - V(G)) = A_2$. We choose A such that $|A_2|$ is minimum. Now let $v \in A_1$. Since A_1 is independent, v' is not adjacent to any vertex in A_1. Hence if $v' \notin A_2$, then v' is not dominated by A, which is a contradiction. Hence $v' \in A_2$. Thus $A'_1 = \{v' : v \in A_1\} \subseteq A_2$. Now let $w \in V(G) - A_1$. If w is not adjacent to a vertex in A_1, then w is not adjacent to any vertex in $A_1 \cup A'_1$. Hence w is dominated by a vertex x' where $x' \in A'_2 - A'$. Hence w' is adjacent to x. Hence $B = (A - \{x'\}) \cup \{x\}$ is also an independent dominating set of $Sp(G), |B| = |A| = i(Sp(G))$ and $|B_2| = |A_2| - 1$, which is a contradiction to the assumption that $|A_2|$ is minimum. Hence $A_2 = A'_1$ and thus $A = A_1 \cup A'_1$ where A_1 is an independent dominating set of G. Thus $i(Sp(G)) = |A| = 2|A_1| \geq 2i(G)$.

Hence $i(Sp(G)) = 2i(G)$.

Case 2. $i(G) \geq \frac{n}{2}$.

Clearly $D = V(Sp(G)) - V(G)$ is an independent dominating set of $Sp(G)$ and $|D| = n$. Hence $i(Sp(G)) \leq n$. Now let A be any independent dominating set of $Sp(G)$ with $|A| = i(Sp(G))$. Let $A \cap V(G) = A_1$ and $A \cap (V(Sp(G)) - V(G)) = A_2$. If $A_1 = \emptyset$, then $A = V(Sp(G) - V(G))$ and $|A| = n$. If $A_1 \neq \emptyset$, then proceeding as in Case 1, we get $A = A_1 \cup A'_1$ where A_1 is an independent dominating set of G. Hence $|A_1| \geq \frac{n}{2}$ and so $i(Sp(G)) = |A| \geq n$.

Thus $i(Sp(G)) = n$. □

Example 2. For the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, the set $\{v_1, v_4\}$ is an independent dominating set of minimum cardinality and $i(C_6) = 2$. Also $\{v_1, v_4, v'_1, v'_4\}$ is an independent dominating set of $Sp(G)$ and $i(Sp(G)) = 4$.

3. Conclusion and Scope

For any graph G we have investigated how the values some of the domination parameters for G and its splitting graph $Sp(G)$ are related. A similar study for other graph theoretic paraemters is an interesting direction for further research.

Acknowledgement

The authors are thankful to the anonymous reviewers for their helpful suggestions, which resulted in substantial improvement in the paper.

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