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Research Article

On the total monophonic number of a graph

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Abstract: Let $G = (V, E)$ be a connected graph of order n. A path P in G which does not have a chord is called a monophonic path. A subset S of V is called a monophonic set if every vertex v in V lies in a x-y monophonic path where $x, y \in S$. If further the induced subgraph $G[S]$ has no isolated vertices, then S is called a total monophonic set. The total monophonic number $m_t(G)$ and the upper total monophonic number $m_t^+(G)$ are respectively the minimum cardinality of a total monophonic set and the maximum cardinality of a minimal total monophonic set. In this paper we determine the value of these parameters for some classes of graphs and establish bounds for the same. We also prove the existence of graphs with prescribed values for $m_t(G)$ and $m_t^+(G)$.

Keywords: total geodetic set, total monophonic set, total monophonic number, minimal total monophonic set, upper total monophonic number

AMS Subject classification: 05C12

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected and connected graph with neither loops nor multiple edges. The order $|V|$ and size $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology we refer to [2, 11].

For any vertex $v \in V$, the set $N(v) = \{u \in V : uv \in E\}$ is called the open neighborhood of v and $|N(v)|$ is the degree of v. A vertex of degree 1 is called a pendant vertex and a vertex which is adjacent to a pendant vertex is called a support vertex. If the induced subgraph $G[N(v)]$ is complete, then v is called an extreme vertex. The distance $d(u, v)$ between two vertices u and v is the length of a shortest u-v path in G. A u-v path of length $d(u, v)$ is called a u-v geodesic. A subset S of V is called a geodetic set if every vertex v in V lies on a x-y geodesic for some $x, y \in S$. The minimum cardinality of a geodetic set of G is called the geodetic number of G and is denoted by $g(G)$. This parameter was introduced in [9] and further investigated in [3–5, 12, 13]. A geodetic set $S \subseteq V(G)$ is a *total geodetic set* if the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total geodetic set is the *total geodetic number* $g_t(G)$. The total geodetic number of a graph was introduced and studied in [1].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P in G which does not have a chord is called a monophonic path. A subset S of V is called a monophonic set if every vertex v in V lies in a x-y monophonic path where $x, y \in S$. The *monophonic number* $m(G)$ of G is the minimum cardinality of a monophonic set in G. The monophonic number of a graph was studied in detail [14] and further studied in [10]. A monophonic set S is called a total monophonic set if the induced subgraph $G[S]$ has no isolated vertices. The total monophonic number $m_t(G)$ is the minimum cardinality of a total monophonic set of G. This parameter was introduced in [8] and further investigated in [6, 7, 15]. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is the length of a longest $u - v$ monophonic path in G. The monophonic diameter $diam_m(G)$ is defined as $\max\{d_m(u, v) : u, v \in V\}$.

The concept of monophonic path and monophonic number have applications in networks. Consider a communication network or a social network in which a few selected vertices provide a service to the remaining vertices. If a path is required for a vertex v to provide service to a vertex u, then a shortest u-v path is a natural choice. However, there are several graphs in which the number of monophonic u-v paths is much larger than the number of shortest u-v paths. Let G be the graph obtained from the θ-graph $\theta(a_1, a_2, \ldots, a_k)$ consisting of k-internally disjoint x-y paths of lengths $a_1, a_2, \ldots, a_k$ where $2 = a_1 < a_2 \leq a_3 \leq \cdots \leq a_k$, by attaching a pendant vertex u adjacent to x and a pendant vertex v adjacent to y.

Clearly the number of shortest u-v paths in G is 1 and the number of monophonic u-v paths is k. These monophonic paths can be used by v for providing service in parallel.

The following theorem will be used in the sequel.

Theorem 1. [8] *All the extreme vertices and support vertices of a connected graph G belong to every total monophonic set of G. If the set S of all extreme vertices and support vertices of G is a total monophonic set, then it is the unique minimum total monophonic set of G.*

In this paper we present several results on total monophonic number of a graph.

2. Total monophonic number of a graph

Since every geodesic path is a monophonic path, it follows that every total geodetic set of G is a total monophonic set of G. Hence it follows that $2 \leq m_t(G) \leq g_t(G) \leq n$.

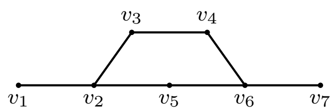

Figure 1. **A graph G with $m_t(G) = 4$ and $g_t(G) = 5$**

Remark 1. The above bounds are sharp. For the complete graph K_2, $g_t(K_2) = m_t(K_2) = 2$ and for the star $K_{1,n-1}(n \geq 2)$, $g_t(K_{1,n-1}) = m_t(K_{1,n-1}) = n$. For the graph G given in Figure 1, clearly no 2-element or 3-element subset of $V(G)$ is a total monophonic set of G. Also $S = \{v_1, v_2, v_6, v_7\}$ is the unique minimum total monophonic set of G and so $m_t(G) = 4$. It is easily verified that $S' = \{v_1, v_2, v_3, v_6, v_7\}$ is a minimum total geodetic set of G and so $g_t(G) = 5$. Thus, we have $2 < m_t(G) < g_t(G) < n$.

Theorem 2. *Let a and b be positive integers with $4 \leq a \leq b$. Then there exists a connected graph G such that $m_t(G) = a$ and $g_t(G) = b$.*

Proof. If $a = b$, then $G = K_{1,a-1}$ is the required graph. Suppose $a < b$. Let H be the graph obtained from the path $P_4 = (v_1, v_2, v_3, v_4)$ by attaching $a - 3$ pendant vertices $u_1, u_2, \ldots, u_{a-3}$ at v_4. Let G be the graph obtained from H by adding $b - a$ edges $x_i y_i, 1 \leq i \leq b - a$, and joining each x_i with v_2 and each y_i with v_4. The graph G is shown in Figure 2. Clearly $S = \{v_1, u_1, u_2, \ldots, u_{a-3}, v_2, v_4\}$ is the set of all pendant vertices and support vertices. Also S is a total monophonic set of G and hence by Theorem 1, $m_t(G) = |S| = a$. Now any total geodetic set of G contain S, any vertex $w \notin S \cup \{v_3\}$ does not lie on any x-y geodesic of G where $x, y \in S$. Also $S' = S \cup \{x_1, x_2, \ldots, x_{b-a}\}$ is a minimum total geodetic set of G and hence $g_t(G) = |S'| = b$. $\square$

3. Upper total monophonic number of a graph

In this section we introduce the concept of the upper total monophonic number of a graph G and investigate its properties.

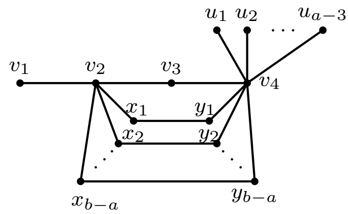

Figure 2. **A graph G with $m_t(G) = a$ and $g_t(G) = b$**

Definition 1. A total monophonic set S of G is called a minimal total monophonic set if no proper subset of S is a total monophonic set of G. The upper total monophonic number of G, denoted by $m_t^+(G)$, is defined as the maximum cardinality of a minimal total monophonic set of G.

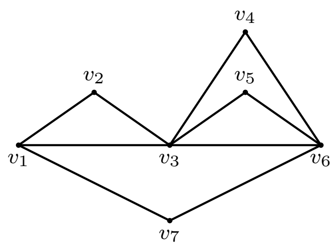

Figure 3. **A graph G with $m(G) = 3, m_t(G) = 4$ and $m_t^+(G) = 5$**

Example 1. For the graph G given in Figure 3, the set $S = \{v_2, v_4, v_5\}$ is the unique minimum monophonic set of G and so $m(G) = 3$. Also $S_1 = S \cup \{v_3\}$ is a total monophonic set of G and $m_t(G) = 4$. The minimal total monophonic sets of G are S_1 and $S_2 = S \cup \{v_1, v_6\}$. Hence the upper total monophonic number of G is 5. Thus the monophonic number, the total monophonic number and the upper total monophonic number of G are different.

Remark 2. Let G be a connected graph of order n. It follows from definition that $2 \leq m(G) \leq m_t(G) \leq m_t^+(G) \leq n$. For the complete graph K_n all the parameters are equal to n and for the graph G given in Figure 3 all of them are distinct. It can be easily verified that $m_t(G) = n$ if and only if $m_t^+(G) = n$. Also if $m_t(G) = n - 1$, then $m_t^+(G) = n - 1$. An example of a graph with $m_t(G) = n - 2$ and $m_t^+(G) = n - 1$ is given in Figure 4.

Theorem 3. *Let G be the complete bipartite graph $K_{r,s}$ with $2 \leq r \leq s$. Then $m_t^+(G) = s + 1$.*

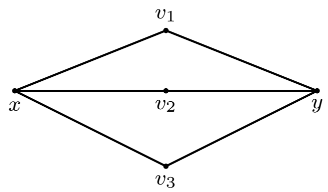

Figure 4. **A graph G with $m_t(G) = n-2$ and $m_t^+(G) = n-1$**

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{w_1, w_2, \ldots, w_s\}$ be the bipartition of G. Since (w_i, v_j, w_k) is a monophonic path, V_2 is a monophonic set of G. Hence $V_2 \cup \{v_1\}$ is a total monophonic set of G and no proper subset of $V_2 \cup \{v_1\}$ is a total monophonic set. Hence $m_t^+(G) \geq s+1$.
Suppose there exists a minimal total monophonic set S with $|S| \geq s+2$. Since $V_1 \cup \{w_j\}$ and $V_2 \cup \{v_i\}$ are total monophonic sets, S does not contain a subset of the above form. Hence $S = A \cup B$ where A is a proper subset V_1 and B is a proper subset of V_2. Let $v_1 \notin A$ and $w_1 \notin B$. If $|A| = 1$, then there does not exist a x-y monophonic path P such that $x, y \in S$ and w_1 lies on P. Hence $|A| \geq 2$. Similarly $|B| \geq 2$. Choose $v_i, v_j \in A$ and $w_k, w_\ell \in B$. Then $T = \{v_i, v_j, w_k, w_\ell\}$ is a total monophonic set of G and hence S is not a minimal total monophonic set, which is a contradiction. Hence $m_t^+(G) = s+1$. □

Theorem 4. *Let a and b be two positive integers with $3 \leq a \leq b$. Then there exists a graph G with $m_t(G) = a$ and $m_t^+(G) = b$.*

Proof. If $a = b$, then $G = K_{1,a-1}$ has the desired property. If $a = 3$, let $G = K_{2,b-1}$. Clearly $m_t(G) = 3$ and it follows from Theorem 3 that $m_t^+(G) = b$. Now let $a \geq 4$. Consider the bipartite graph $K_{2,b-a+2}$ with bipartition $V_1 = \{v_1, v_2\}$ and $V_2 = \{w_1, w_2, \ldots, w_{b-a+2}\}$. Let G be the graph obtained from $K_{2,b-a+2}$ by attaching $a-3$ pendant vertices $u_1, u_2, \ldots, u_{a-3}$ at w_1. It follows from Theorem 1 that any total monophonic set contains the set $S = \{w_1, u_1, u_2, \ldots, u_{a-3}\}$. Also $S_1 = S \cup \{v_1, v_2\}$ is a minimum total monophonic set of G and hence $m_t(G) = a$. We now claim that $T = S \cup \{w_2, w_3, \ldots, w_{b-a+2}, v_1\}$ is a minimal total monophonic set of G. Suppose there exists a proper subset T_1 of T such that T_1 is a total monophonic set. Since $S \subseteq T_1$, it follows that either $v_1 \notin T_1$ or $w_i \notin T_1$ for some $i, 2 \leq i \leq b-a+2$. If $v_1 \notin T_1$, then $G[T_1]$ has isolated vertices. If $w_i \notin T_1$, then there does not exist a $x-y$ monophonic path P such that $x, y \in T_1$ and w_i lies on P. Thus T_1 is not a total monophonic set which is a contradiction. Hence T is a minimal total monophonic set. Since $|T| = b$ and $|V(G)| = b+1$, it follows that $m_t^+(G) = b$. □

In the following theorem we prove the existence of a graph G with prescribed order, monophonic diameter and upper total monophonic number.

Theorem 5. *Let n, d and k be positive integers such that $n \geq d + k - 2$, $4 \leq d \leq n - 2$ and $5 \leq k < n$. Then there exists a connected graph G of order n with monophonic diameter d and $m_t^+(G) = k$.*

Proof. Let G be the graph obtained from the path $P_{d+1} = (v_1, v_2, \ldots, v_{d+1})$ by adding the vertices $u_1, u_2, \ldots, u_{k-3}, w_1, w_2, \ldots, w_{n-d-k+2}$ such that $N(u_i) = \{v_1, v_2, v_3\}$ and $N(w_j) = \{v_1, v_d\}$. Clearly G is a graph of order n. Clearly P_{d+1} is the longest monophonic path in G and hence the monophonic diameter of G is d. Now any total monophonic set of G contains v_d and v_{d+1}. Also $S = \{v_d, v_{d+1}, v_2, u_1, u_2, \ldots, u_{k-3}\}$ is a minimal total monophonic set of maximum cardinality. Hence $m_t^+(G) = k$. □

4. Conclusion and Scope

In this paper we have initiated a study of the upper total monophonic number of a graph. The following problems arise naturally from our study.

Problem 6. Obtain a characterization of minimal total monophonic sets.

Problem 7. Determine the upper total monophonic number for other families of graphs.

Problem 8. Given four positive integers a, b, c and d with $a \leq b \leq c \leq d$, under what conditions there exists a graph G of order d with $m(G) = a, m_t(G) = b$ and $m_t^+(G) = c$?

Acknowledgement

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