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Weak Roman domination stable graphs upon edge addition

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Abstract: A Roman dominating function (RDF) on a graph G is a function $f: V(G) \rightarrow \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. A vertex u with $f(u) = 0$ is said to be undefended if it is not adjacent to a vertex with $f(v) > 0$. The function $f: V(G) \rightarrow \{0,1,2\}$ is a weak Roman dominating function (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function $f': V(G) \rightarrow \{0,1,2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. A graph G is said to be Roman domination stable upon edge addition, or just γ_R-EA-stable, if $\gamma_R(G + e) = \gamma_R(G)$ for any edge $e \notin E(G)$. We extend this concept to a weak Roman dominating function as follows: A graph G is said to be weak Roman domination stable upon edge addition, or just γ_r-EA-stable, if $\gamma_r(G + e) = \gamma_r(G)$ for any edge $e \notin E(G)$. In this paper, we study γ_r-EA-stable graphs, obtain bounds for γ_r-EA-stable graphs and characterize γ_r-EA-stable trees which attain the bound.

Keywords: Weak Roman dominating function, weak Roman domination, stable

AMS Subject classification: 05C69

1. Introduction

Cockayne *et al.* [6] defined a *Roman dominating function* (RDF) in a graph G to be a function $f\colon V(G) \rightarrow \{0,1,2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function of a graph G is called the *Roman domination number*

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of G and denoted by $\gamma_R(G)$. For more details on Roman domination and its variations we refer the reader to the recent two book chapters [2, 5] and survey paper [3, 4].
Henning *et al.* [9] defined a *weak Roman dominating function* as follows: For a graph G, let f: $V(G) \to \{0, 1, 2\}$ be a function. A vertex u with $f(u) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex v with the positive weight. A function f: $V(G) \to \{0, 1, 2\}$ is said to be a *weak Roman dominationg function* (WRDF) if each vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) > 0$ such that the function f': $V(G) \to \{0, 1, 2\}$ defined by $f'(u) = 1$, $f'(v) = f(v) - 1$ and $f'(w) = f(w)$ if $w \in V - \{u, v\}$, has no undefended vertex. We say that v *defends* u. The weight $w(f)$ of f is defined to be $\sum_{u \in V} f(u)$. The minimum weight of a weak Roman dominating function of a graph G is called the *weak Roman domination number* of G and denoted by $\gamma_r(G)$. A WRDF with weight $\gamma_r(G)$ is called a $\gamma_r(G)$-function. This concept of weak Roman domination as suggested by Henning et al. [9] is an attractive alternative for Roman domination as it further reduces the weight of the Roman dominating function. Weak Roman domination in graphs has been studied in [10–12]. A weak Roman dominating function f can also be written as $f = (V_0, V_1, V_2)$ where $V_i = \{v \mid f(v) = i\}$, $i = 0, 1, 2$. Let $v \in V_1 \cup V_2$. A vertex $w \in N(v) \cap V_0$ is said to be in the *dependent set* of v, denoted by $D_G(v)$ if w is defended by v alone.
M. Chellali and N. J. Rad [1] introduced the concept of *Roman domination stable graphs upon edge addition* or just γ_R-EA-*stable*, if addition of any extra edge does not affect the Roman domination number, that is $\gamma_R(G + e) = \gamma_R(G)$ for any edge $e \notin E(G)$. We extend this concept to a weak Roman dominating function as follows. A graph G is said to be *weak Roman domination stable upon edge addition*, or just γ_r-EA-stable, if $\gamma_r(G+e) = \gamma_r(G)$ for any edge $e \notin E(G)$. It is clear that $\gamma_r(G)-1 \leq \gamma_r(G + e) \leq \gamma_r(G)$. In this paper, we study γ_r-EA-stable graphs, obtain bounds for γ_r-EA-stable graphs and characterize γ_r-EA-stable trees which attain the bound.

2. Notation

For notation and graph theoretic terminology, we in general follow [7, 8]. Throughout this paper, we consider only simple and connected graphs. Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by n. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in a graph G is the number of edges that are incident to the vertex v and is denoted by $\deg(v)$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$. A set S of vertices is called *independent* if no two vertices in S are adjacent. A simple graph in which every pair of distinct vertices are adjacent is called a *complete graph*. A *clique* of a simple graph G is a subset S of V such that $G[S]$ is complete. A connected graph with exactly one cycle is called an *unicyclic* graph. For two positive integers m, n, the *complete bipartite* graph $K_{m,n}$ is the graph with partition $V(G) = V_1 \cup V_2$ such that $|V_1| = m$, $|V_2| = n$ and such that $G[V_i]$ has no edges for $i = 1, 2$, and every two vertices belonging to different partition sets are adjacent to each other. A *maximal path* is a path in which no vertex can be added further to make it longer.

3. Some Standard Graphs

In this section we investigate paths, cycles and complete bipartite graphs that are γ_r-EA-stable. We state the following theorem proved in [9]

Theorem 1. [9] *For $n \geq 4$, $\gamma_r(C_n) = \gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$.*

In order to investigate paths and cycles that are γ_r-EA-stable, we first define a family $\mathcal{G}$ of unicyclic graphs and subsequently prove two lemmas. A unicyclic graph $G \in \mathcal{G}$ if the following holds.

(i) $\Delta(G) = 3$.

(ii) At most two vertices in G are of degree 3.

(iii) If two vertices are of degree 3, then both are in the cycle and are adjacent.

We also define two subfamilies $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{G}$ as follows. A unicyclic graph G with cycle C_k is in $\mathcal{G}_1$ if $k = n - 2$ and is in $\mathcal{G}_2$ if $k = n - 1$.

Lemma 1. *Let $G \in \mathcal{G}_1$. Then $\gamma_r(G) = \lceil \frac{3n}{7} \rceil$.*

Proof. It is a simple exercise to verify the result for $n \leq 14$. Suppose that $n \geq 15$. Let $V(G) = \{v_1, v_2, \ldots, v_k, x, y\}$ where v_i, $1 \leq i \leq k$ are on the cycle C_k and x, y are not in C_k and are adjacent to v_1 and v_k respectively. Let f be the γ_r-function of G. Since P_n is a spanning subgraph of G, $\gamma_r(G) \leq \gamma_r(P_n)$. Thus, $\gamma_r(G) \leq \lceil \frac{3n}{7} \rceil$. Now to safeguard the vertices v_i, $1 \leq i \leq 6$ and v_j, $k - 5 \leq j \leq k$ and x, y, f will assign a total weight of at least 6. Hence, $\gamma_r(G) \geq 6 + \gamma_r(P_{k-12}) \geq \left\lceil \frac{3(k-12)}{7} \right\rceil + 6 \geq \left\lceil \frac{3(n-14)}{7} \right\rceil + 6 = \lceil \frac{3n}{7} \rceil$. Thus, $\gamma_r(G) = \lceil \frac{3n}{7} \rceil$. □

Lemma 2. *Let $G \in \mathcal{G}_2$. Then* $\gamma_r(G) = \begin{cases} \lfloor \frac{3n}{7} \rfloor, & \text{if } n \equiv 5 \pmod 7, n \geq 12 \\ \lceil \frac{3n}{7} \rceil, & \text{if } n \not\equiv 5 \pmod 7, n \geq 11. \end{cases}$

Proof. It is a simple exercise to verify the result for $n \leq 11$. Suppose that $n \geq 12$. Let $V(G) = \{v_1, v_2, \ldots, v_k, x\}$, where v_i, $1 \leq i \leq k$, are on the cycle C_k and x is not in C_k adjacent to v_1. Since P_n is a spanning subgraph of G, $\gamma_r(G) \leq \lceil \frac{3n}{7} \rceil$. Let f be a γ_r-function of G. Now, to safeguard the vertices v_i, $1 \leq i \leq 6$, v_j, $k - 4 \leq j \leq k$

and x, f will assign a total weight of at least 5. Hence, $\gamma_r(G) \geq 5 + \gamma_r(P_{k-11}) \geq 5 + \left\lceil \frac{3(k-11)}{7} \right\rceil \geq 5 + \left\lceil \frac{3(n-12)}{7} \right\rceil$.
When $n \equiv 5 \pmod 7$, $\gamma_r(G) \geq \lfloor \frac{3n}{7} \rfloor$ and when $n \not\equiv 5 \pmod 7$, $\gamma_r(G) \geq \lceil \frac{3n}{7} \rceil$. Hence,

$$\gamma_r(G) = \begin{cases} \lfloor \frac{3n}{7} \rfloor, & \text{if } n \equiv 5 \pmod 7, n \geq 12 \\ \\ \lceil \frac{3n}{7} \rceil, & \text{if } n \not\equiv 5 \pmod 7, n \geq 11. \end{cases}$$

□

Lemma 3. *Let $G \in \mathcal{G} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$, where $n \equiv 0, 2, 4, 6 \pmod 7$. Then $\gamma_r(G) = \lceil \frac{3n}{7} \rceil$*

Proof. We prove the result by induction on n. It is a simple exercise to verify that the result is true for graphs with $n \leq 11$. Suppose that the result is true for graphs of order at most $n-1$, $n \geq 12$. Let G be a graph of order n. Since P_n is a spanning subgraph of G, $\gamma_r(G) \leq \lceil \frac{3n}{7} \rceil$.
Case (i). $n \equiv 0 \pmod 7$.
In this case, $\gamma_r(G) \leq \frac{3n}{7}$. Remove a leaf vertex from G to obtain a graph G'. Then, $|V(G')| = n - 1 \equiv 6 \pmod 7$ and $G' \in \mathcal{G}$ or G' is a cycle. If either $G' \in \mathcal{G}_1 \cup \mathcal{G}_2$ or G' is a cycle, then by Theorem 1, $\gamma_r(G') = \left\lceil \frac{3(n-1)}{7} \right\rceil$. If $G' \in \mathcal{G} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$, then by induction hypothesis, $\gamma_r(G') = \left\lceil \frac{3(n-1)}{7} \right\rceil$. Hence, $\gamma_r(G) \geq \left\lceil \frac{3(n-1)}{7} \right\rceil = \frac{3(n-1)+3}{7} = \frac{3n}{7}$. Thus, $\gamma_r(G) = \frac{3n}{7}$.
Case (ii). $n \equiv 2 \pmod 7$.
In this case, $\gamma_r(G) \leq \frac{3n+1}{7}$. Remove a leaf vertex and a vertex adjacent to it from G to obtain a graph G'. Then, $|V(G')| = n - 2 \equiv 0 \pmod 7$ and $G' \in \mathcal{G}$ or G' is a cycle. If either $G' \in \mathcal{G}_1 \cup \mathcal{G}_2$ or G' is a cycle, then by Theorem 1, $\gamma_r(G') = \left\lceil \frac{3(n-2)}{7} \right\rceil$. If $G' = \mathcal{G} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$, then by induction hypothesis, $\gamma_r(G') = \left\lceil \frac{3(n-2)}{7} \right\rceil$. Hence, $\gamma_r(G) \geq \left\lceil \frac{3(n-2)}{7} \right\rceil + 1 = \frac{3n+1}{7}$. Thus, $\gamma_r(G) = \frac{3n+1}{7} = \lceil \frac{3n}{7} \rceil$.
Case (iii). $n \equiv 4 \pmod 7$.
In this case, $\gamma_r(G) \leq \frac{3n+2}{7}$. As discussed in Case (ii), we obtain a graph G' by removing a leaf vertex and a vertex adjacent to it. Also, $\gamma_r(G) \geq \gamma_r(G') + 1 \geq \left\lceil \frac{3(n-2)}{7} \right\rceil + 1 = \frac{3n+2}{7}$. Thus, $\gamma_r(G) = \frac{3n+2}{7} = \lceil \frac{3n}{7} \rceil$.
Case (iv). $n \equiv 6 \pmod 7$.
In this case, $\gamma_r(G) \leq \frac{3n+2}{7}$. A similar argument as in Case (ii) holds and hence $\gamma_r(G) \geq \gamma_r(G') + 1 \geq \left\lceil \frac{3(n-2)}{7} \right\rceil + 1 = \frac{3n+3}{7}$. Thus, $\gamma_r(G) = \frac{3n+3}{7} = \lceil \frac{3n}{7} \rceil$. □

Theorem 2. *Paths P_n are γ_r-EA-stable if and only if $n \equiv 0, 2, 4, 6 \pmod 7$.*

Proof. Let $n \equiv 1, 3, 5 \pmod 7$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Clearly, P_3 is not γ_r-EA-Stable. When $n = 5, 8, 10$, join the vertices v_1 and v_3. Clearly, $\gamma_r(P_n + v_1v_3) = 2, 3$ or 4 according as $n = 5, 8$ or 10. Thus, $\gamma_r(P_n + v_1v_3) < \gamma_r(P_n)$ which implies that P_n is not γ_r-EA-stable. When $n \geq 11$, join the vertices v_2 and v_n. Then $P_n + v_2v_n \in \mathcal{G}_2$ and $\gamma_r(P_n + v_2v_n) < \lfloor \frac{3n}{7} \rfloor < \gamma_r(P_n)$. Thus, P_n is not γ_r-EA-stable. Let $n \equiv 0, 2, 4, 6 \pmod 7$. Joining any two vertices of P_n by an edge e will result in a graph which will be in $\mathcal{G}$. If $P_n + e \in \mathcal{G}_1 \cup \mathcal{G}_2$, then by Lemma 1 and Lemma 2, $\gamma_r(P_n + e) = \gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$. If $P_n + e \in \mathcal{G} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$, then by Lemma 3 we have $\gamma_r(P_n + e) = \gamma_r(P_n) = \lceil \frac{3n}{7} \rceil$. Thus, P_n is γ_r-EA-stable when $n \equiv 0, 2, 4, 6 \pmod 7$. □

Theorem 3. *Cycles C_n are γ_r-EA-stable if and only if $n \equiv 0, 2, 4, 6 \pmod 7$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. If $n \equiv 1, 3, 5 \pmod 7$, join the vertices v_1 and v_{n-1} by an edge e. Then, $\gamma_r(C_n) = \lceil \frac{3n}{7} \rceil$. In $C_n + e$, any γ_r-function of $C_n + e$ will assign a total weight of 1 to the vertices v_1, v_n, v_{n-1}. Considering the path $Q = (v_n, v_1, v_2, \ldots, v_{n-2})$ on $n-1$ vertices, any γ_r-function of $C_n + e$ will assign a total weight of $\left\lceil \frac{3(n-1)}{7} \right\rceil$ to Q. Thus, $\gamma_r(C_n + e) = \left\lceil \frac{3(n-1)}{7} \right\rceil = \frac{3(n-1)}{7}$ or $\frac{3(n-1)+1}{7}$ or $\frac{3(n-1)+2}{7}$. That is $\gamma_r(C_n + e) = \frac{3n-3}{7}$ or $\frac{3n-2}{7}$ or $\frac{3n-1}{7}$ according as $n \equiv 1$ or 3 or $5 \pmod 7$. But $\gamma_r(C_n) = \frac{3n+4}{7}$ or $\frac{3n+5}{7}$ or $\frac{3n+6}{7}$ according as $n \equiv 1$ or 3 or $5 \pmod 7$. Thus, $\gamma_r(C_n + e) < \gamma_r(C_n)$ when $n \equiv 1, 3, 5 \pmod 7$.
Let $n \equiv 0, 2, 4, 6 \pmod 7$. Join any two non adjacent vertices of P_n by an edge e. Since C_n is a spanning subgraph of $C_n + e$, $\gamma_r(C_n + e) \leq \lceil \frac{3n}{7} \rceil$.

Case (i). $n \equiv 0 \pmod 7$.
In this case $\gamma_r(C_n + e) \leq \frac{3n}{7}$. Remove a vertex of degree 2 from $C_n + e$ to obtain a graph G'. Then, $|V(G')| = n - 1 \equiv 6 \pmod 7$ and $G' \in \mathcal{G}$ or G' is C_{n-1}. By Lemma 1, Lemma 2 and Theorem 1, $\gamma_r(G') = \left\lceil \frac{3(n-1)}{7} \right\rceil$. Hence, $\gamma_r(C_n + e) \geq \left\lceil \frac{3(n-1)}{7} \right\rceil = \frac{3(n-1)+3}{7} = \dfrac{3n}{7}$. Thus, $\gamma_r(C_n + e) = \frac{3n}{7}$.

Case (ii). $n \equiv 2 \pmod 7$.
In this case $\gamma_r(C_n + e) \leq \frac{3n+1}{7}$. Remove two adjacent vertices of degree two in $C_n + e$ to obtain a graph G'. Then $|V(G')| = n - 2 \equiv 0 \pmod 7$ and $G' \in \mathcal{G}$ or G' is C_{n-2}. By Lemma 1, Lemma 2 and Theorem 1, $\gamma_r(G') = \left\lceil \frac{3(n-2)}{7} \right\rceil$. Hence, $\gamma_r(C_n + e) \geq \left\lceil \frac{3(n-2)}{7} \right\rceil + 1 \geq \frac{3(n-2)}{7} + 1 = \frac{3n+1}{7}$. Thus, $\gamma_r(C_n + e) = \frac{3n+1}{7}$.
A similar argument holds for $n \equiv 4, 6 \pmod 7$. When $n \equiv 4 \pmod 7$, $\gamma_r(C_n) = \gamma_r(C_n + e) = \frac{3n+2}{7}$. When $n \equiv 6 \pmod 7$, $\gamma_r(C_n) = \gamma_r(C_n + e) = \frac{3n+3}{7}$. This completes the proof. □

Theorem 4. *The complete bipartite graphs $G = K_{m,n}$, $m \leq n$, $m + n \geq 4$ are γ_r-EA-stable if and only if $m \neq 3, 4$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be a bipartition of $V(G)$. Now, $\gamma_r(G) = 3$ if $m = 3$ and $\gamma_r(G) = 4$ if $m = 4$. Adding the edge $e = x_1x_2$ in G, we see that $\gamma_r(G+e) = 2$ if $m = 3$ and $\gamma_r(G+e) = 3$ if $m = 4$. Thus, $\gamma_r(G+e) < \gamma_r(G)$ and G is not γ_r-EA-stable.
Suppose that $m \leq 2$. Then, $\gamma_r(G) = 2$. Since $m + n \geq 4$, $G \neq P_3$. Thus, adding any edge in $K_{m,n}$ will not result in a complete graph. Thus, G is γ_r-EA-stable. If $m \geq 5$, $\gamma_r(G) = 4$ and adding any edge in G will not decrease the value of $\gamma_r(G)$. Hence $\gamma_r(G+e) = \gamma_r(G)$ for every $e \in E(G)$. Thus G is γ_r-EA-stable. □

Theorem 5. *If G is a γ_r-EA-stable graph of order $n \geq 3$, then $\gamma_r(G) \leq \frac{n}{2}$.*

Proof. Let G be a γ_r-EA-stable graph of order $n \geq 3$. Then, clearly $|D_G(x)| \geq 3$ for every $x \in V_2$. Hence, $|V_0| \geq 3|V_2| + |V_1|$. Thus

$$n = |V_2| + |V_0| + |V_1| \geq |V_2| + 3|V_2| + 2|V_1| \geq 2(2|V_2| + |V_1|) \geq 2\gamma_r(G)$$

which leads to the desired bound. □

Theorem 6. *Paths P_n and cycles C_n are γ_r-EA-stable with $\gamma_r(G) = \frac{n}{2}$ if and only if $n = 4, 6$.*

Proof. Suppose that the given graphs are γ_r-EA-stable with $\gamma_r(G) = \frac{n}{2}$. Since $\gamma_r(P_n) = \gamma_r(C_n) = \left\lceil \frac{3n}{7} \right\rceil$, $4 \leq n \leq 12$. By Theorems 2 and 3, we see that $n = 4, 6$. For $n = 4, 6$, P_n and C_n are clearly γ_r-EA-stable and $\gamma_r(P_n) = \gamma_r(C_n) = \frac{n}{2}$. □

4. Split Graphs

In this section we characterize split graphs which are γ_r-EA-stable. A graph G with bipartition (X, Y), where X forms a complete graph and the vertices in Y are independent is called a *split graph*. We also assume that $|X| = r$ and $|Y| = s$. For convenience we define the following: Two vertices u, v in X with $N(u) \cap Y = \{u_1, u_2, u_3\}$ and $N(v) \cap Y = \{v_1, v_2, v_3\}$ are said to be *associate vertices* if the following holds (Refer Figure 1).

(i) Exactly one vertex in $N(u) \cap Y$ say u_1 and exactly two vertices in $N(v) \cap Y$ say v_1 and v_2 have a common neighbor in X.

(ii) $N(u_2) = N(u_3)$ and each vertex in $N(u_2) \setminus \{u\}$ is of degree $r+1$ and each vertex in $N(v_3) \setminus \{v\}$ is of degree r.

(iii) $N(u_1) \setminus \{u\} = N(v_1) \setminus \{v\} = N(v_2) \setminus \{v\}$ and each vertex of $N(u_1) \setminus \{u\}$ is of degree $r+2$.

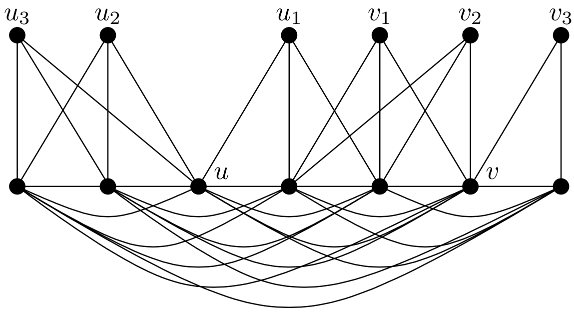

Figure 1. **A split graph illustrating associate vertices**

First, we define a family $\mathcal{G}_3$ of split graphs as follows. Let $G = G_1 = (X, Y_1)$ be a split graph with $|X| = r$, $\Delta(G_1) \geq r + 2$ and no associate vertices. Let $x_1 \in X$ in G_1 with $\deg(x_1) = \Delta(G_1)$. Remove all the neighbors of x_1 in Y_1. Let $G_2 = (X, Y_2)$ be the resulting graph. Let $x_2 \in X$ in G_2 with $\deg(x_2) = \Delta(G_2) \geq r + 2$. Remove all the neighbors of x_2 in Y_2 to obtain a graph $G_3 = (X, Y_3)$. Repeat the process until we get a graph G_k such that $\Delta(G_k) < r + 2$. Then $G \in \mathcal{G}_3$ if G_k is K_r.

Theorem 7. *Let G be a split graph with $\Delta(G) \geq r + 2$. Then G is γ_r-EA-stable if and only if $G \in \mathcal{G}_3$.*

Proof. Let G be γ_r-EA-stable and let f be a γ_r-function of G. Suppose that G has a pair of associate vertices say u, v with $N(u) \cap Y = \{u_1, u_2, u_3\}$ and $N(v) \cap Y = \{v_1, v_2, v_3\}$ where u_i, v_i, $i = 1, 2, 3$ satisfy the conditions given in the definition of associate vertices. Now f will assign a total weight of 4 to the vertices u_i, v_i, $i = 1, 2, 3$ and their neighbors in X. Now join u_2 and u_3 in G. Then define a function $g : V(G + u_2u_3) \to \{0, 1, 2\}$ by $g(u) = g(v) = g(z) = 1$, where $z \in N(u_1) \setminus \{u\}$ and $g(x) = 0$ if $x \in \{u_i, v_i, N(u_i) \setminus \{u\}, N(v_i) \setminus \{v\}\}$ and $g(x) = f(x)$ otherwise. Now u defends u_2, u_3 and all their neighbors in X, v defends v_3 and all its neighbors in X and z defends u_1, v_1, v_2 and all their neighbors in X. Hence $\gamma_r(G + u_2u_3) < \gamma_r(G)$, which implies that G is not γ_r-EA-stable, a contradiction. Hence G has no associate vertices. Now remove the vertices successively as described in the procedure. Let $G_k = (X_k, Y_k)$ be the final graph. We claim that $G_k = K_r$. Equivalently, we prove that $Y = \emptyset$ in G_k. Suppose to the contrary that $G_k \neq K_r$. Suppose that there exists a vertex x in X such that $\deg_{G_k}(x) = r + 1$. Let y_1, y_2 be the neighbors of x in Y_k. Then, there exists a γ_r-function f of G_k such that $f(x) + f(y_1) + f(y_2) = 2$. Since $\Delta(G) \geq r + 2$, there is a vertex in X say z such that $\deg_G(z) \geq r + 2$ and $f(z) = 2$. Hence by adding an edge e between z and y_1 or z and y_2, we see that $\gamma_r(G + e) < \gamma_r(G)$. Hence, G is not γ_r-EA-stable, a contradiction.
Suppose that $\deg_{G_k}(x) \leq r$ for every $x \in X$. Let $x \in X$ be such that $\deg_{G_k}(x) = r$ and y be its neighbor in Y_k. Then for any γ_r-function f will assign a weight 1 either

to x or to y. In any case adding an edge e between y and z (as mentioned earlier) we see that $\gamma_r(G+e) < \gamma_r(G)$. Hence G is not γ_r-EA-stable, a contradiction. Thus, $G_k = K_r$ and hence $G \in \mathcal{G}_3$.
Conversely, suppose that $G \in \mathcal{G}_3$. From the description of $\mathcal{G}_3$, one can easily observe that every time the neighbors of a vertex $x \in X$ in Y with $\deg(x) \geq r+2$ are removed, x is adjacent to at least three vertices in Y. Therefore, any γ_r-function f will assign 2 to x and 0 to the neighbors of x which are removed. Hence adding a single edge between any two non adjacent vertices will not alter the γ_r-value of G. Hence G is γ_r-EA-stable. □

Theorem 8. *Let G be a split graph with $\Delta(G) = r+1$ and $n \geq 4$. Then G is γ_r-EA-stable if and only if the following holds.*

(i) *If some component H of $G[X,Y]$ is either a P_3 or a $K_{2,t}$, $t \geq 2$ then $G[X,Y] = H$.*

(ii) *$G[X,Y]$ does not contain maximal paths P_5 (with both ends in Y), P_7 (with both ends in X) and P_6.*

(iii) *If a maximal path P_5 (with both ends in X) exists in $G[X,Y]$, then Y does not contain a vertex, where all its neighbors in X are of degree r.*

Proof. Suppose that G is γ_r-EA-stable. Let f be a γ_r-function of G. To prove (i), suppose that H of $G[X,Y]$ is either a P_3 or a $K_{2,t}$, $t \geq 2$. Choose f such that $f(v) = 2$, where v is a vertex of the P_3 or $K_{2,t}$ which is in X. Suppose that X contains a vertex of degree $r-1$. If some vertex in $X \setminus \{v\}$ is assigned the value 2 by f, then joining the two vertics of P_3 or $K_{2,t}$ in X by an edge e, we see that $\gamma_r(G+e) = \gamma_r(G) - 1$ which implies that G is not γ_r-EA-stable. Otherwise some vertex of $X \setminus \{v\}$, say x is assigned the value 1 by f such that $|D_G(x)| = 1$. Let $D_G(x) = \{z\}$. If x is not a guarding vertex, then joining z and v by an edge we see that $\gamma_r(G+e) < \gamma_r(G)$, as any γ_r-function g of $G+e$ will assign 0 to x and $g(w) = f(w)$ for every vertex $w \in V(G) \setminus \{x\}$. Hence G is not γ_r-EA-stable, a contradiction. If x is a guarding vertex then some vertex, say y in X exists such that $|D_G(y)| = 2$. Then joining y and a vertex of P_3 or $K_{2,t}$, say u which is in Y by an edge e, we see that $\gamma_r(G+e) < \gamma_r(G)$, as any γ_r-function of $G+e$ will assign 0 to u and 1 to v and $g(w) = f(w)$ for every $w \in V(G) \setminus \{u, v\}$. Hence G is not γ_r-EA-stable, a contradiction. Suppose that X contains no vertex of degree $r-1$, then by joining the 2 vertices of P_3 or $K_{2,t}$ in Y by an edge we see that $\gamma_r(G+e) = \gamma_r(G) - 1$ which implies that G is not γ_r-EA-stable, a contradiction. Thus, $G[X,Y] = H$ and hence (i) is proved.
To prove (ii), suppose to the contrary that either a maximal path P_5 (with both ends in Y) or a maximal path P_7 (with both ends in X) exist in $G[X,Y]$. Then f will assign a total weight of 3 to the vertices of P_5 or P_7. Joining the $2nd$ and $5th$ vertices in P_5 or joining the $3rd$ and $6th$ vertices of $P_7(P_6)$ will reduce the total weight of these vertices to 2. Hence G is not γ_r-EA-stable, a contradiction. Thus, (ii) is proved.
To prove (iii), suppose to the contrary that a maximal path P_5 (with both ends in X) exists and Y contains a vertex z such that all its neighbors in X are of degree r. Now f will assign a total weight 2 to the vertices of P_5. Choose f such that $f(v) = 2$,

where v is the central vertex of P_5 which is in X. Now f will assign a total weight 1 to all its neighbors in X. Now joining v and z we see that the value of $\gamma_r(G+e)$ will reduce by 1 as v defends z and all its neighbors in X. Thus G is not γ_r-EA-stable, a contradiction. Hence (iii) is proved.
Conversely suppose the given conditions hold. One can choose a γ_r-function $f = (V_0, V_1, V_2)$ of G such that $V_2 = \emptyset$ and $D_G(x) \neq \emptyset$ for every $x \in V_1$. Hence G is γ_r-EA-stable. □

Theorem 9. *Let G be a split graph with $\Delta(G) = r$. Then, G is γ_r-EA-stable if and only if either each vertex of X is of degree r or at least two vertices in X are of degree $r-1$.*

Proof. If every vertex of X is of degree r, we are through. Otherwise, at least one vertex of X is of degree $r-1$. Since $\Delta(G) = r$, every vertex $y \in Y$ along with its neighbors will induce a complete graph and the vertices in X of degree $r-1$ will induce a complete graph. Hence, clearly, $\gamma_r(G) = |Y| + 1$. If exactly one vertex in X is of degree $r-1$, then joining that vertex to any vertex in Y by an edge e, we see that $\gamma_r(G+e) = |Y|$. Thus, G is not γ_r-EA-stable, a contradiction. Thus, the condition given in theorem holds.
Conversely, suppose that one of the conditions hold. Then, it is clear that addition of any edge will not alter the value of $\gamma_r(G)$. Hence, G is γ_r-EA-stable. □

5. Trees

In this section we characterize γ_r-EA-stable trees T with $\gamma_r(T) = \frac{n}{2}$. For this purpose we first define a family $\mathcal{A}$ of trees as follows. A tree $T \in \mathcal{A}$ if T satisfies the following conditions.

(i) A strong support vertex is adjacent to at most three leaf vertices.

(ii) The length of a pendant path is at most 4 and the length of a non-pendant path is at most 5.

(iii) The non leaf neighbor of a strong support vertex of degree three is not a support vertex.

(iv) The non leaf neighbor of a weak support vertex of degree two is not a strong support vertex.

We next define a family $\Im$ of trees as follows. Let $T = T_1 \in \mathcal{A}$. We perform the following operations successively in T_1.
$\mathcal{O}_1$**:** Consider a weak support vertex w of degree two. Remove the edge between w and its non-leaf neighbor.
$\mathcal{O}_2$**:** Consider a strong support vertex w of degree 3. Remove all the edges incident with its non-leaf neighbor (except the edge which is incident with w).

$\mathcal{O}_3$**:** Consider a strong support vertex w which is adjacent to exactly 3 leaf vertices where at least one neighbor of w is a non strong support adjacent to exactly three leaf vertices. Remove all the non pendant edges incident with w such that the other end of these edges are non strong supports adjacent to exactly three leaf vertices.
If some component of the resulting graph, say T_2 is either not in $\mathcal{A}$ or a path P_m, $m \neq 2, 4$, then we stop the process. Also if some component of T_2 is a $H \circ K_1$, then operation $\mathcal{O}_1$ is not performed in that component. We repeat the process until no such edge (the edges which are mentioned in the operations) remains. Let T_k be the final graph. Then $T \in \Im$ if each component of T_k is either a K_2 or a $H \circ K_1$ or a $H \circ 3K_1$ subject to the following conditions.

(1) A leaf vertex of a $K_{1,3}$ is not adjacent to the head vertex of a $K_{1,3}$.

(2) For a $K_{1,3}$, at least one leaf vertex is not adjacent to a vertex in a K_2.

(3) A vertex in a $H \circ K_1$ is not adjacent to the head vertex of a $K_{1,3}$. Further, a leaf vertex of a $H \circ K_1$ is not adjacent to a leaf vertex of a $K_{1,3}$.

(4) If for some K_2 with $V(K_2) = \{a, b\}$, a is adjacent to a vertex of another K_2, then every neighbor of b is a vertex of some K_2. None of the vertices of a K_2 is adjacent to the vertex of a $K_{1,3}$.

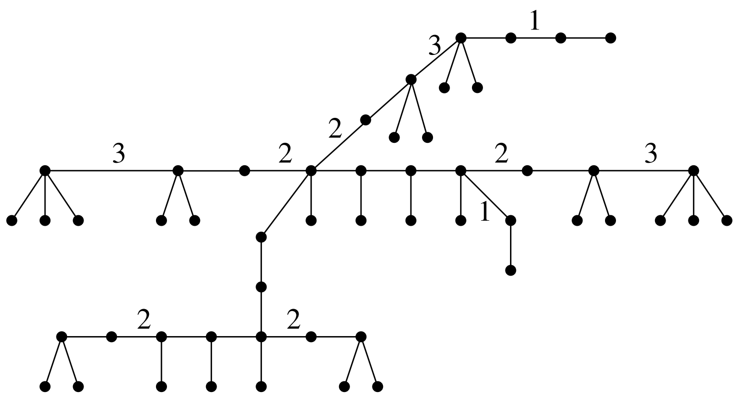

Figure 2. **A tree** $T \in \Im$

In the above figure, the edges that are labeled 1 are removed first and secondly the edges that are labeled 2 are removed and finally the edges that are labeled 3 are removed.

Theorem 10. *Let T be a tree of order n. Then T is γ_r-EA-stable with $\gamma_r(T) = \frac{n}{2}$ if and only if $T \in \Im$.*

Proof. Let T be a tree which is γ_r-EA-stable and $\gamma_r(T) = \frac{n}{2}$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_r(T)$-function. First, we claim that $T \in \mathcal{A}$. Now, we prove the following claims.

Claim 1. A strong support junction vertex x is adjacent to at most three leaf vertices. Suppose to the contrary that x is adjacent to at least four leaf vertices, then $|D_T(x)| \geq 4$, which implies that $\gamma_r(T) \leq \frac{n}{2}$, a contradiction.

Claim 2. The length of a non-pendant path is at most 5 and the length of a pendant path is at most 4.

Let $Q = (x, x_1, x_2, \ldots, x_m, y)$ be a non-pendant path. Suppose to the contrary that $l(Q) \geq 6$, then $m+1 \geq 6$. Let $f(x) = f(y) = 2$. It is clear that x and y can defend x_1 and x_m respectively. Since $\gamma_r(T) = \frac{n}{2}$, $\sum_{i=2}^{m-1} f(x_i) = \left\lceil \frac{3(m-2)}{7} \right\rceil = \frac{m-2}{2}$ which implies that $m = 2j+2$, $0 \leq j \leq 6$. Since, $m \geq 5$,$2 \leq j \leq 6$. If $x_1 \notin D_T(x)$, then when $m = 2j$, $3 \leq j \leq 7$, we see that $x_1 \notin D_T(w)$ for any $w \in V_1 \cup V_2$ which implies that $\gamma_r(T) < \frac{n}{2}$, a contradiction. Thus, $x_1 \in D_T(x)$. Similarly, $x_m \in D_T(y)$. Let z_1 and z_2 be the members of $D_T(x)$ not in Q. Now, join the vertices z_1 and z_2 and let g be a γ_r-function of the resulting graph. Then, $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(z_1) + f(z_2) = \left\lceil \frac{3(m-2)}{7} \right\rceil + 4$ and $\sum_{i=1}^{m} g(x_i) + g(x) + g(y) + g(z_1) + g(z_2) = \left\lceil \frac{3(m+1)}{7} \right\rceil + 2$ as x will receive the weight 1 under g. Now, for $m = 2j+2$, $2 \leq j \leq 6$, the above weights will be respectively $\{6,5\}, \{7,6\}, \{8,7\}, \{9,8\}, \{10,9\}$. Hence, we see that the value of $\gamma_r(T)$ changes upon the addition of the edge z_1, z_2. Hence, T is not γ_r-EA-stable, a contradiction.

Suppose that $f(x) = 2, f(y) = 1$. It is clear that x can defend x_1. As before only 2 members of $D_T(x)$ are not in Q. Since $\gamma_r(T) = \frac{n}{2}$, $|D_T(y)| = 1$ and clearly the members w of $D_T(y)$ is not in Q. Further $\sum_{i=2}^{m} f(x_i) + f(y) + f(w) = \left\lceil \frac{3(m+1)}{7} \right\rceil = \frac{m+1}{2}$ implies that $m = 2j-1$, $0 \leq j \leq 6$. Since $m \geq 5$, $3 \leq j \leq 6$. Now, join the vertices z_1 and z_2 and let g be a γ_r-function of the resulting graph. Then, $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(z_1) + f(z_2) + f(w) = \left\lceil \frac{3(m+1)}{7} \right\rceil + 2$ and $\sum_{i=1}^{m} g(x_i) + g(x) + g(y) + g(z_1) + g(z_2) + g(w) = \left\lceil \frac{3(m+4)}{7} \right\rceil$ as x will receive the weight 1 under g. Now, for $m = 2j+1$, $2 \leq j \leq 5$, the above weight will be $\{5,4\}, \{6,5\}, \{7,6\}, \{8,7\}$ respectively. Thus, we see that $\gamma_r(T)$ reduces upon the addition of the edge $z_1 z_2$. Hence, T is not γ_r-EA-stable, a contradiction.

Suppose that $f(x) = f(y) = 1$. Since, $\gamma_r(T) = \frac{n}{2}$, $|D_T(y)| = |D_T(x)| = 1$ and clearly the members say w_1, w_2 of $D_T(x)$ and $D_T(y)$ respectively are not in Q. Further $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(w_1) + f(w_2) = \left\lceil \frac{3(m+4)}{7} \right\rceil = \frac{m+4}{2}$ implies that $m = 2j-4$, $0 \leq j \leq 6$. Since $m \geq 5$, $j = 5, 6$. Now, join the vertices x_m and w_2 and let g be a γ_r-function of the resulting graph. Then $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(w_1) + f(w_2) = \left\lceil \frac{3(m+4)}{7} \right\rceil$ and $g(x) + g(y) + g(w_1) + g(w_2) + \sum_{i=1}^{m} g(x_i) = \left\lceil \frac{3(m+3)}{7} \right\rceil$ where y will defend both w_2 and x_m under the function g. Now, for $m = 2j-4$, $j = 5, 6$, the above weights will be $\{5,4\}, \{6,5\}$ respectively. Thus, we see that $\gamma_r(T)$ reduces upon the addition of the edge $x_m w_2$. Hence, T is not γ_r-EA-stable, a contradiction.

Suppose that $f(x) = 2$ and $f(y) = 0$. Then x defends x_1 and choose f such that $f(x_m) = 1$ and x_m defends y. (If some vertex not in Q defends y, then one can choose f such that $f(y) = 1$ which has already been discussed). Since $\gamma_r(T) = \frac{n}{2}$, $\sum_{i=2}^{m} f(x_i) + f(y) = \left\lceil \frac{3m}{7} \right\rceil = \frac{m}{2}$ which implies that $m = 2j$, $3 \leq j \leq 6$. Now, join the vertices z_1 and z_2 and let g be a γ_r-function of the resulting graph. Then, $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(z_1) + f(z_2) = \lceil \frac{3m}{7} \rceil + 2$ and $\sum_{i=1}^{m} g(x_i) + g(x) + g(y) + g(z_1) + g(z_2) = \left\lceil \frac{3(m+3)}{7} \right\rceil$. Now for $m = 2j$, $3 \le j \le 6$, the above weights will be respectively $\{5,4\},\{6,5\},\{7,6\},\{8,7\}$ respectively. Hence $\gamma_r(T + z_1z_2) < \gamma_r(T)$. Thus, we see that $\gamma_r(T)$ reduces upon the addition of the edge z_1z_2. Hence, T is not γ_r-EA-stable, a contradiction.

Suppose that $f(x) = 1$ and $f(y) = 0$. Since $\gamma_r(T) = \frac{n}{2}$, $|D_T(x)| = 1$ and clearly the member $w \in D_T(x)$ is not in Q. Also choose f such that $f(x_m) = 1$ and x_m defends y. Since $\gamma_r(T) = \frac{n}{2}$, $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(w) = \left\lceil \frac{3(m+3)}{7} \right\rceil = \frac{m+3}{2}$ implies that $m = 2j - 3$, $4 \le j \le 6$. Now, join the vertices w and x_1 and let g be a γ_r-function of the resulting graph. Then $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) + f(w) = \left\lceil \frac{3(m+3)}{7} \right\rceil$, $\sum_{i=1}^{m} g(x_i) + g(x) + g(y) + g(w) = \left\lceil \frac{3(m+2)}{7} \right\rceil$ and $g(v) = f(v)$ for the rest of the vertices. Now for $m = 2j - 3$, $4 \le j \le 6$, the above weights are $\{4,3\},\{5,4\}$ and $\{6,5\}$ respectively. Hence, $\gamma_r(T + wx_1) < \gamma_r(T)$ and thus T is not γ_r-EA-stable, a contradiction.

Suppose that $f(x) = f(y) = 0$. Choose f such that x_1 and x_m defends x and y respectively. Since $\gamma_r(T) = \frac{n}{2}$, $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) = \left\lceil \frac{3(m+2)}{7} \right\rceil = \frac{m+2}{2}$ implies that, $m = 2j-2$, $4 \le j \le 6$. Now, join the vertices x_2 and x and let g be a γ_r-function of $T + xx_2$. Then, $\sum_{i=1}^{m} f(x_i) + f(x) + f(y) = \left\lceil \frac{3(m+2)}{7} \right\rceil$, $\sum_{i=1}^{m} g(x_i) + g(x) + g(y) = \left\lceil \frac{3(m+1)}{7} \right\rceil$ and $g(v) = f(v)$ for the rest of the vertices. Now, for $m = 2j-2$, $4 \le j \le 6$, the above weights are $\{4,3\},\{5,4\}$ and $\{6,5\}$ respectively. Hence, $\gamma_r(T + wx_1) < \gamma_r(T)$ and thus T is not γ_r-EA-stable, a contradiction.

Next, we claim that the length of a pendant path is at most 4.

Let $Q = (x, x_1, x_2, \ldots, x_m = y)$ be a pendant path incident at x, where x is a junction vertex and y is a leaf vertex. We claim that $l(Q) \le 4$. That is $m \le 4$. Suppose to the contrary that $m \ge 5$. Let $f(x) = 2$, then as discussed earlier $x_1 \in D_T(x)$. Let z_1, z_2 be the members of $D_T(x)$, not in Q. Since $\gamma_r(T) = \frac{n}{2}$, $\sum_{i=1}^{m} f(x_i) = \left\lceil \frac{3(m-1)}{7} \right\rceil = \frac{m-1}{2}$ implies that $m = 2j + 1$, $2 \le j \le 6$. Now join z_1 and z_2 and let g be a γ_r-function of $T + z_1z_2$. Then, $\sum_{i=1}^{m} f(x_i) + f(x) + f(z_1) + f(z_2) = \left\lceil \frac{3(m-1)}{7} \right\rceil + 2$ and $\sum_{i=1}^{m} g(x_i) + g(x) + g(z_1) + g(z_2) = \left\lceil \frac{3(m+2)}{7} \right\rceil$ as x will receive the weight 1 under g, and f and g coincide at all other vertices. Now, for $m = 2j + 1$, $1 \le j \le 6$, the above weights will be $\{4,3\},\{5,4\}$ and $\{6,5\},\{7,6\},\{8,7\}$ respectively. Hence, $\gamma_r(T + z_1z_2) < \gamma_r(T)$ and thus T is not γ_r-EA-stable, a contradiction.

Suppose that $f(x) = 1$, since $\gamma_r(T) = \frac{n}{2}$, $|D_T(x)| = 1$ and clearly, the member $w \in D_T(x)$ is not in Q. Again $\sum_{i=1}^{m} f(x_i) + f(x) + f(w) = \left\lceil \frac{3(m+2)}{7} \right\rceil = \frac{m+2}{2}$ implies that $m = 2j - 2$, $4 \le j \le 6$. Now, join the vertices w and x_1 and let g be a γ_r-function of $T + wx_1$. Then, $\sum_{i=1}^{m} f(x_i) + f(x) + f(w) = \left\lceil \frac{3(m+2)}{7} \right\rceil$ and $\sum_{i=1}^{m} g(x_i) + g(x) + g(w) = \left\lceil \frac{3(m+1)}{7} \right\rceil$ and $g(v) = f(v)$ for the rest of the vertices. Now, for $m = 2j - 2$, $4 \le j \le 6$, the above weights are $\{4,3\},\{5,4\}$ and $\{6,5\}$ respectively.

Hence, $\gamma_r(T + wx_1) < \gamma_r(T)$ and thus T is not γ_r-EA-stable, a contradiction.
If $f(x) = 0$, then some vertex not in Q defends x and one can choose f such that $f(x) = 1$ which has already been discussed.

Claim 3. If x is a strong support vertex of degree 3, then its non-leaf neighbor is not a support vertex.
Suppose to the contrary that x is adjacent to a support vertex y, then join the two leaf vertices of x by an edge e. Clearly, $\gamma_r(T + e) < \gamma_r(T)$, a contradiction. which implies that T is not γ_r-EA-stable.

Claim 4. If x is a weak support vertex of degree 2, then its non-leaf neighbor is not a strong support vertex.
Suppose to the contrary that x is adjacent to a strong support vertex y, then add an edge e between the leaf vertex incident with x and the head vertex of y. Clearly, $\gamma_r(T+e) < \gamma_r(T)$, a contradiction. which implies that T is not γ_r-EA-stable. Hence, $T = T_1 \in \mathcal{A}$.
Now we perform the operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ in T_1. Let T_2 be the resulting graph. Suppose that some component of T_2 say, T^* is such that either $T^* \notin \mathcal{A}$ or $T^* = P_m$, $m \neq 2, 4$. If $T^* \notin \mathcal{A}$, then either there exist two non adjacent vertices x and y such that $\gamma_r(T^* + xy) < \gamma_r(T^*)$ or $\gamma_r(T^*) < \frac{n}{2}$. Hence, either $\gamma_r(T + xy) < \gamma_r(T)$ or $\gamma_r(T) < \frac{n}{2}$. Thus, in either case we get a contradiction. Suppose that $T^* = P_m$, $m \neq 2, 4$. Let $P_m = (u_1, u_2, \ldots, u_m)$. If m is odd and $m \geq 7$, then clearly $\gamma_r(T) < \frac{n}{2}$, a contradiction. If $m = 3$ or 5, then joining u_1 and u_3 by an edge e, we see that $\gamma_r(T + e) < \gamma_r(T)$, a contradiction. Suppose that m is even and $m \geq 8$. If $m \geq 14$, then as $\gamma_r(P_m) = \lceil \frac{3m}{7} \rceil$, we see that $\gamma_r(T) < \frac{n}{2}$, a contradiction. If $m = 6$, then one end of P_m say z is either adjacent to a vertex in a K_2 or a vertex of a $K_{1,3}$. If z is adjacent to a vertex in a K_2 with $V(K_2) = \{a, b\}$, where a and z are adjacent, then $\gamma_r(T + zb) < \gamma_r(T)$, a contradiction. If z is adjacent to the head vertex of a $K_{1,3}$, say a, then there exists a vertex in P_6, say b such that $b \in V_1$ and $D_T(b) = \emptyset$. Now $\gamma_r(T + ab) < \gamma_r(T)$, a contradiction. If z is adjacent to the leaf vertex of a $K_{1,3}$, then $\gamma_r(T + ab) < \gamma_r(T)$, where a and b are the leaf vertices not adjacent to z, a contradiction.
If $8 \leq m \leq 12$, then by Theorem 3, P_m is not γ_r-EA-stable which implies that T is not γ_r-EA-stable, a contradiction. Thus, each component of T_2 is in $\mathcal{A}$. Again we perform the operations $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$ in T_2 to obtain a graph T_3 and check whether each component of T_3 is in $\mathcal{A}$ and none of the components of T_3 is a P_m, $m \neq 2, 4$. If so, as before either $\gamma_r(T) < \frac{n}{2}$ or T is not γ_r-EA-stable. Otherwise, we repeat the process until no such edges remain (as mentioned in the operations). Let T_k be the final graph. We claim that T_k is either a $H \circ 3K_1$ or a $H \circ K_1$ or a K_2.
Suppose to the contrary that some component of T_k is a P_m, $m \neq 2, 4$. If $m \neq 1$, then as before we get a contradiction. Suppose that $m = 1$. Let $V(P_1) = \{w\}$. If $f(w) = 1$, then every neighbor of w is either a leaf vertex of a $H \circ 3K_1$ or a K_2. Then joining w to a leaf vertex of a K_2 or a head vertex of the $K_{1,3}$, we see that $\gamma_r(T + e)$ reduces by 1 and hence $\gamma_r(T + e) < \gamma_r(T)$, a contradiction. Suppose that $f(w) = 0$. If w is adjacent to the head vertex of the $K_{1,3}$, then $\gamma_r(T) < \dfrac{n}{2}$, a contradiction.

Otherwise there exists another P_1 in T_k with $V(P_1) = \{z\}$ such that $f(z) = 0$ and both w and z are adjacent to a leaf vertex of a $K_{1,3}$. Then joining the other two leaf vertices of the said $K_{1,3}$ will reduce the γ_r-value by 1, a contradiction. Hence, each component of T_k is either a $H \circ 3K_1$ or a $H \circ K_1$ or K_2.
Now we claim that at least one leaf vertex of a $K_{1,3}$ is not adjacent to a vertex in a K_2. If not, all the leaf vertices are adjacent to a K_2 and the head vertex of the said $K_{1,3}$ will receive a weight 1 and all its leaf vertices will receive a weight 0 under f which implies that $\gamma_r(T) < \dfrac{n}{2}$, a contradiction.
Next we claim that a leaf vertex of a $K_{1,3}$, say H is not adjacent to the head vertex of a $K_{1,3}$. If so, then joining the two leaf vertices of H by an edge e, we see that $\gamma_r(T + e) < \gamma_r(T)$, as the sum of the weights of the vertices in H is 2 under f and in $T + e$ the above said weight will be 1 under any γ_r-function of $T + e$. Hence T is not γ_r-EA-stable, a contradiction.
Next we claim that a vertex in a $H \circ K_1$ is not adjacent to the head vertex of a $K_{1,3}$. Suppose to the contrary that a vertex in a $H \circ K_1$, say x is adjacent to a head vertex of a $K_{1,3}$, say y. Let z be the leaf neighbor or support neighbor of x according as x is a support vertex or a leaf vertex of $H \circ K_1$. Then $\gamma_r(T + zy) < \gamma_r(T)$, a contradiction.
Next we claim that a leaf vertex of a $H \circ K_1$ is not adjacent to a leaf vertex of a $K_{1,3}$. If so, join the two leaf vertices of $K_{1,3}$ by an edge e and any γ_r-function of the resultant graph will assign 1 the leaf vertex of $H \circ K_1$ and to the head vertex of the said $K_{1,3}$ and 0 to the corresponding support vertex of $H \circ K_1$ and to all the leaf vertices of the said $K_{1,3}$ which implies that $\gamma_r(T + e) < \gamma_r(T)$, which is a contradiction to the fact that T is γ_r-EA-stable.
Finally, we claim that if for some K_2 with $V(K_2) = \{a, b\}$, a is adjacent to a vertex of another K_2, then every neighbor of b is a vertex of some K_2. Suppose to the contrary, that some neighbor say, w of b is not a vertex of a K_2. Then, w is a vertex of some $K_{1,3}$. If w is the head vertex of a $K_{1,3}$, then joining w and a, we see that $\gamma_r(T + wa) < \gamma_r(T)$, as $f(a) + f(b) + f(w) = 3$ and in $T + wa$, this weight will be reduced by 1. Hence T is not γ_r-EA-stable, a contradiction. If w is a leaf vertex of a $K_{1,3}$, then joining the other two leaf vertices of the said $K_{1,3}$ by an edge e, we see that $\gamma_r(T + e) < \gamma_r(T)$ as $f(a) + f(b) + f(w) = 3$ and in $T + e$ this weight will be 2 under any γ_r-function of $T + e$. Hence T is not γ_r-EA-stable, a contradiction. Thus, $T \in \Im$.
Conversely, suppose that $T \in \Im$. Let f be a γ_r-function of T. Each time we perform the operations $\mathcal{O}_1$ and $\mathcal{O}_2$, we see that either a subgraph $K_{1,3}$ or a K_2 is removed and a weight of 2 or 1 is associated with these subgraphs. Since each component of T_k is either a K_2 or a $H \circ K_1$ or a $H \circ 3K_1$, then clearly a weight of half the order of each component is associated. Hence $f(V) = \frac{n}{2}$ which implies that $\gamma_r(T) = \frac{n}{2}$. Further since $T \in \Im$, $|D_T(x)| = 3$ for every $x \in V_2$ and $|D_T(x)| = 1$ for every $x \in V_1$. Hence T is γ_r-EA-stable. □

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