Communications in Combinatorics and Optimization
Vol. ?? No. ??, ???? pp.1-10
DOI: 10.22049/CCO.2022.27700.1318

CCO
Commun. Comb. Optim.

Research Article

Signed total Italian domination in digraphs

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Received: 25 February 2022; Accepted: 2 June 2022
Published Online: 5 June 2022

Abstract: Let D be a finite and simple digraph with vertex set $V(D)$. A signed total Italian dominating function (STIDF) on a digraph D is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$, where $N^-(v)$ consists of all vertices of D from which arcs go into v, and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. The weight of an STIDF f is $\sum_{v\in V(D)} f(v)$. The signed total Italian domination number $\gamma_{stI}(D)$ of D is the minimum weight of an STIDF on D. In this paper we initiate the study of the signed total Italian domination number of digraphs, and we present different bounds on $\gamma_{stI}(D)$. In addition, we determine the signed total Italian domination number of some classes of digraphs.

Keywords: Digraph, Signed total Italian domination number, signed total Roman domination number

AMS Subject classification: 05C69

1. Terminology and introduction

Let G be a simple graph with vertex set $V(G)$, and let $N(v) = N_G(v)$ be the neighborhood of the vertex v.

A *signed total Roman dominating function* on a graph G is defined in [14] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{x\in N_G(v)} f(x) \geq 1$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of a signed Roman dominating function f on a graph G is $\sum_{v\in V(G)} f(v)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ of G is the minimum weight of a signed total Roman dominating function on G.

A *signed total Italian dominating function* on a graph G is defined in [17] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that (i) $\sum_{x\in N(v)} f(x) \geq 1$ for every $v \in V(G)$ and (ii) every vertex u for which $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or adjacent to two vertices w and z with $f(w) = f(z) = 1$. The weight of a signed

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total Italian dominating function f on a graph G is $\sum_{v\in V(G)} f(v)$. The *signed total Italian domination number* $\gamma_{stI}(G)$ of G is the minimum weight of a signed total Italian dominating function on G. Clearly, $\gamma_{stI}(G) \leq \gamma_{stR}(G)$.

Let now D be a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n(D) = |V(D)|$ and $m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. We write $d^+_D(v) = d^+(v)$ for the *out-degree* of a vertex v and $d^-_D(v) = d^-(v)$ for its *in-degree*. The *minimum* and *maximum in-degree* are $\delta^-(D)$ and $\Delta^-(D)$ and the *minimum* and *maximum out-degree* are $\delta^+(D)$ and $\Delta^+(D)$. The sets $N^+_D(v) = N^+(v) = \{x \,|\, (v,x) \in A(D)\}$ and $N^-_D(v) = N^-(v) = \{x \,|\, (x,v) \in A(D)\}$ are called the *out-neighborhood* and *in-neighborhood* of the vertex v. Likewise, $N^+_D[v] = N^+[v] = N^+(v) \cup \{v\}$ and $N^-_D[v] = N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x,y) \in A(D)$, the vertex y is an *out-neighbor* of x and x is an *in-neighbor* of y, and we also say that x *dominates* y or y *is dominated by* x. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v\in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v\in S} f(v)$, so $w(f) = f(V(D))$. Consult [7] or [6] for notation and terminology which are not defined here.

We define a set $S \subseteq V(D)$ to be a *total dominating set* of D if for all $v \in V(D)$, there exists a vertex $u \in S$ such that v is dominated by u. The minimum cardinality of a total dominating set in D is the *total domination number* $\gamma_t(D)$.

In this paper, we continue the study of signed (total) Roman and Italian domination in graphs and digraphs (see, for example the survey article [2, 3] and [1, 4, 5, 8, 9, 11–17]).

A *signed total Roman dominating function* (abbreviated STRDF) on D is defined in [15] as a function $f : V(D) \longrightarrow \{-1, 1, 2\}$ such that $f(N^-(v)) = \sum_{x\in N^-(v)} f(x) \geq 1$ for every $v \in V(D)$ and every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The weight of an STRDF f on a digraph D is $w(f) = \sum_{v\in V(D)} f(v)$. The *signed total Roman domination number* $\gamma_{stR}(D)$ of D is the minimum weight of an STRDF on D. A $\gamma_{tsR}(D)$-function is a signed total Roman dominating function on D of weight $\gamma_{stR}(D)$.

A *signed total Italian dominating function* (abbreviated STIDF) on D is defined as a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N^-(v)} f(x) \geq 1$ for each $v \in V(D)$, and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$ or two in-neighbors w and z with $f(w) = f(z) = 1$. The weight of an STIDF f is $\omega(f) = \sum_{v\in V(D)} f(v)$. The *signed total Italian domination number* $\gamma_{stI}(D)$ of D is the minimum weight of an STIDF on D. A $\gamma_{stI}(D)$-function is a signed total Italian dominating function on D of weight $\gamma_{stI}(D)$. For an STIDF f on D, let $V_i = V_i(f) = \{v \in V(D) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Italian dominating function $f : V(D) \longrightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(D)$.

A signed total Italian dominating function on a digraph combines the properties of both a signed total dominating function (see [10, 18]) and signed total Roman dominating function (see [14, 15]). The signed total Italian domination number and

the signed total Roman domination number exist when $\delta^- \geq 1$, because the function $f : V(D) \to \{-1, 1, 2\}$ with $f(x) = 1$ for each vertex $x \in V(D)$ is an STRDF as well as an STIDF on D of weight $n(D)$ and hence $\gamma_{stR}(D) \leq n(D)$. Thus we assume throughout this paper that $\delta^-(D) \geq 1$. Our purpose in this work is to initiate the study of the signed total Italian domination number in digraphs. We present basic properties and sharp bounds for the signed total Italian domination number of digraphs. In particular, we show that many lower bounds on $\gamma_{stR}(D)$ are also valid for $\gamma_{stI}(D)$. In addition, we show that the difference $\gamma_{stR}(D) - \gamma_{stI}(D)$ can be arbitrarily large, and we determine the signed total Italian domination number of some classes of digraphs. The definitions lead to $\gamma_{stI}(D) \leq \gamma_{stR}(D) \leq n(D)$. Therefore each lower bound of $\gamma_{stI}(D)$ is also a lower bound of $\gamma_{stR}(D)$.
The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 1. *If $D(G)$ is the associated digraph of a graph G, then $\gamma_{stR}(D(G)) = \gamma_{stR}(G)$ and $\gamma_{stI}(D(G)) = \gamma_{stI}(G)$.*

Let K_n and K_n^* be the complete graph and complete digraph of order n, respectively. In [17], the author determines the signed total domination number of complete graphs.

Proposition 1. ([17]) If $n \geq 2$, then $\gamma_{stI}(K_n) = 2$ when n is even and $\gamma_{stI}(K_n) = 3$ when n is odd.

Using Observation 1 and Proposition 1 , we obtain the signed total Italian domination number of complete digraphs.

Corollary 1. If $n \geq 2$, then $\gamma_{stI}(K_n^*) = 2$ when n is even and $\gamma_{stI}(K_n^*) = 3$ when n is odd.

Let $K_{p,q}$ be the complete bipartite graph with partite sets X and Y, where $|X| = p$ and $|Y| = q$, and let $K_{p,q}^*$ its associated digraph.

Proposition 2. ([17]) If $n \geq 3$, then $\gamma_{stI}(K_{1,n-1}) = 3$ and if $p, q \geq 2$, then $\gamma_{stI}(K_{p,q}) = 2$.

Using Observation 1 and Proposition 2, we obtain the signed total Italian domination number of complete bipartite digraphs.

Corollary 2. If $n \geq 3$, then $\gamma_{stI}(K_{1,n-1}^*) = 3$ and if $p, q \geq 2$, then $\gamma_{stI}(K_{p,q}^*) = 2$.

Proposition 3. ([17]) If C_n is a cycle of length $n \geq 3$, then $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stI}(C_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stI}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

The next result follows from Observation 1 and Proposition 3.

Corollary 3. Let C_n^* be the associated digraph of the cycle C_n of length $n \geq 3$. Then $\gamma_{stI}(C_n^*) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stI}(C_n^*) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stI}(C_n^*) = (n+6)/2$ when $n \equiv 2 \pmod 4$.

2. Preliminary results and first bounds

In this section we present basic properties and some first bounds on the signed total Italian domination number of digraphs.

Proposition 4. If $f = (V_{-1}, V_1, V_2)$ is an STIDF on a digraph D of order n with $\delta^-(D) \geq 1$, then

(a) $|V_{-1}| + |V_1| + |V_2| = n$.

(b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.

(c) Every vertex in V_{-1} is dominated by one vertex of V_2 or two vertices of V_1.

(d) $V_1 \cup V_2$ is a total dominating set of D.

Proof. Since (a), (b) and (c) are immediate, we only prove (d). By the definition, every vertex of V_{-1} has an in-neighbor in $V_1 \cup V_2$. Thus $V_1 \cup V_2$ dominates V_{-1}. Suppose that $V_1 \cup V_2$ contains a vertex v without an in-neighbor in $V_1 \cup V_2$. As $\delta^-(D) \geq 1$, the vertex v has an in-neighbor in V_{-1} and all its in-neighbors are in V_{-1}. This leads to the contradiction $f(N^-(v)) \leq -1$. Consequently, $V_1 \cup V_2$ is a total dominating set of D. □

The proof of the next proposition is identically with the proof of Proposition 9 in [15] and is therefore omitted.

Proposition 5. Assume that $f = (V_{-1}, V_1, V_2)$ is an STIDF on a digraph D of order n with $\delta^-(D) \geq 1$. If $\Delta^+(D) = \Delta^+$ and $\delta^+(D) = \delta^+$, then

(a) $(2\Delta^+ - 1)|V_2| + (\Delta^+ - 1)|V_1| \geq (\delta^+ + 1)|V_{-1}|$.

(b) $(2\Delta^+ + \delta^+)|V_2| + (\Delta^+ + \delta^+)|V_1| \geq (\delta^+ + 1)n$.

(c) $(\Delta^+ + \delta^+)\omega(f) \geq (\delta^+ - \Delta^+ + 2)n + (\delta^+ - \Delta^+)|V_2|$.

(d) $\omega(f) \geq (\delta^+ - 2\Delta^+ + 2)n/(2\Delta^+ + \delta^+) + |V_2|$.

A digraph D is *out-regular* or *r-out-regular* if $\delta^+(D) = \Delta^+(D) = r$, and D is *r-regular* if $\delta^+(D) = \Delta^+(D) = \delta^-(D) = \Delta^-(D) = r$. As an application of Proposition 5 (c), we obtain a lower bound on the signed total Italian domination number for r-out-regular digraphs.

Corollary 4. If D is an r-out-regular digraph of order n with $r \geq 1$, then $\gamma_{stI}(D) \geq n/r$.

Therefore $\gamma_{stR}(D) \geq \gamma_{stI}(D) \geq n/r$ for each r-out-regular digraph of order n with $r \geq 1$ (see [15]). Using Corollary 4 and Observation 1, we obtain the next known results.

Corollary 5. ([14, 17]) If G is an r-regular graph of order n with $r \geq 1$, then $\gamma_{stR}(G) \geq \gamma_{stI}(G) \geq n/r$.

If H is a 1-regular digraph of order n, then it follows from Corollary 4 that $\gamma_{stI}(H) \geq n$ and so $\gamma_{stI}(H) = n$. Thus Corollary 4 is sharp for $r = 1$. Corollary 2 implies that $\gamma_{stI}(K^*_{p,p}) = 2$ for $p \geq 2$. If $n \equiv 0 \pmod 4$, then it follows from Corollary 3 that $\gamma_{stI}(C^*_n) = n/2$. These are further examples which show that Corollary 4 is sharp.
Let $n = 2r + 1$ with an integer $r \geq 1$. We define the *circulant tournament* $CT(n)$ with n vertices as follows. The vertex set of $CT(n)$ is $V(CT(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.

Theorem 2. Let $n = 2r + 1$ with an integer $r \geq 1$. Then $\gamma_{stI}(CT(n)) = 3$ when $r = 2p + 1$ is odd and $\gamma_{stI}(CT(n)) = 4$ when $r = 2p$ is even.

Proof. Since $CT(n))$ is r-regular, Corollary 4 implies that $\gamma_{stI}(CT(n)) \geq (2r+1)/r$ and thus $\gamma_{stI}(CT(n)) \geq 3$.
First let $r = 2p + 1$ for an integer $p \geq 0$. If $p = 0$, then we have seen above that $\gamma_{stI}(CT(3)) = 3$. Let now $p \geq 1$, and define the function $g : V(CT(n)) \longrightarrow \{-1, 1, 2\}$ by $g(u_1) = g(u_2) = \ldots = g(u_p) = -1$, $g(u_{2p+2}) = g(u_{2p+3}) = \ldots = g(u_{3p+1}) = -1$ and $g(x) = 1$ otherwise. Then it is straighforward to verify that g is an STIDF on $CT(n)$ of weight 3. Therefore $\gamma_{stI}(CT(n)) \leq 3$ and hence $\gamma_{stI}(CT(n)) = 3$ in this case.
Let now $r = 2p$ for an integer $p \geq 1$, and let f be a $\gamma_{stI}(CT(n))$-function. If $f(x) = 1$ for each $x \in V(CT(n))$, then $\omega(f) = n \geq 5$. Let now $f(w) = -1$ for a vertex $w \in V(CT(n))$. Assume first that there exist a vertex, say u_r, such that $f(u_r) = 2$. Consider the sets $N^-(u_0) = \{u_{r+1}, u_{r+2}, \ldots, u_{2r}\}$ and $N^-(u_r) = \{u_0, u_1, \ldots, u_{r-1}\}$. As f is an STIDF on $CT(n)$, we deduce that

$$\omega(f) = f(N^-(u_0)) + f(N^-(u_r)) + f(u_r) \geq 1 + 1 + 2 = 4.$$

Assume next that $f(x) = 1$ or $f(x) = -1$ for each vertex $x \in V(CT(n))$, and let, without loss of generality, $f(u_r) = 1$. As $f(N^-(u_0)) \geq 1$, $f(N^-(u_r)) \geq 1$ and $f(N^-(u_0)$ as well as $f(N^-(u_r))$ are even, we conclude that $f(N^-(u_0)) \geq 2$ and $f(N^-(u_r)) \geq 2$. It follows that

$$\omega(f) = f(N^-(u_0)) + f(N^-(u_r)) + f(u_r) \geq 2 + 2 + 1 = 5.$$

Altogether, we have $\omega(f) \geq 4$.
If $p = 1$, then define the function $g : V(CT(5)) \longrightarrow \{-1, 1, 2\}$ by $g(u_0) = g(u_1) = g(u_3) = 2$ and $g(u_2) = g(u_4) = -1$. Obviously, g is an STIDF on $CT(5)$ of weight 4 and thus $\gamma_{stI}(CT(5)) \leq 4$ and so $\gamma_{stI}(CT(5)) = 4$. If $p \geq 2$, then define the function $g : V(CT(n)) \longrightarrow \{-1, 1, 2\}$ by $g(u_0) = g(u_1) = g(u_{2p+1}) = 2$, $g(u_2) = g(u_3) = \ldots = g(u_p) = 1$, $g(u_{p+1}) = g(u_{p+2}) = \ldots = g(u_{2p}) = -1$, $g(u_{2p+2}) = g(u_{2p+3}) = \ldots = g(u_{3p}) = 1$ and $g(u_{3p+1}) = g(u_{3p+2}) = \ldots = g(u_{4p}) = -1$. Then it is straighforward to verify that g is an STIDF on $CT(n)$ of weight 4. Therefore $\gamma_{stI}(CT(n)) \leq 4$ and hence $\gamma_{stI}(CT(n)) = 4$ in that case. □

If D is not out-regular, then the next lower bound on the signed total Italian domination number is valid.

Corollary 6. Let D be a digraph of order n with $\delta^-(D) \geq 1$, minimum out-degree δ^+ and maximum out-degree Δ^+. If $\delta^+ < \Delta^+$, then

$$\gamma_{stI}(D) \geq \left(\frac{2\delta^+ + 3 - 2\Delta^+}{2\Delta^+ + \delta^+}\right) n.$$

Proof. Multiplying both sides of the inequality in Proposition 5 (d) by $\Delta^+ - \delta^+$ and adding the resulting inequality to the inequality in Proposition 5 (c), we obtain the desired lower bound. □

Because of $\gamma_{stR}(D) \geq \gamma_{stI}(D)$, the bound of Corollary 6 is also valid for $\gamma_{stR}(D)$ (see [15]).
Since $\Delta^+(D(G)) = \Delta(G)$ and $\delta^+(D(G)) = \delta(G)$, Corollary 6 and Observation 1 lead to the next known corollary.

Corollary 7. ([14, 17]) Let G be a graph of order n, minimum degree $\delta \geq 1$ and maximum degree Δ. If $\delta < \Delta$, then

$$\gamma_{stR}(G) \geq \gamma_{stI}(G) \geq \left(\frac{2\delta + 3 - 2\Delta}{2\Delta + \delta}\right) n.$$

Example 11 in [14] demonstrates that Corollary 7 is sharp. This example together with Observation 1 shows that Corollary 6 is sharp too.

3. Further bounds

Proposition 6. If D is a digraph of order n with $\delta^-(D) \geq 1$, then

$$\gamma_{stI}(D) \geq \Delta^-(D) + 1 - n.$$

Proof. Let $w \in V(D)$ be a vertex of maximum in-degree, and let f be a $\gamma_{stI}(D)$-function. Then the definitions imply

$$\begin{aligned}\gamma_{stI}(D) &= \sum_{x \in V(D)} f(x) = \sum_{x \in N^-(w)} f(x) + \sum_{x \in V(D)\setminus N^-(w)} f(x) \\ &\geq 1 + \sum_{x \in V(D)\setminus N^-(w)} f(x) \geq 1 - (n - (\Delta^-(D))) = 1 + \Delta^-(D) - n,\end{aligned}$$

and the proof of the desired lower bound is complete. □

Example 1. *Let F be an arbitray digraph of order $t \geq 2$ with $\delta^-(F) \geq 1$, and let $H = K_t^*$. Let $Q = F \cup H$ such that each vertex of F dominates each vertex of H. Then $n(Q) = 2t$ and $\Delta^-(Q) = 2t - 1$. It follows from Proposition 6 that $\gamma_{stI}(Q) \geq \Delta^-(Q) + 1 - n(Q) = 0$. Now define $f : V(Q) \longrightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each $v \in V(F)$ and $f(x) = -1$ for each $x \in V(H)$. Then f is an STIDF of Q of weight 0 and hence $\gamma_{stI}(Q) \leq 0$. Therefore $\gamma_{stI}(Q) = 0$. This example shows that Proposition 6 is sharp.*

Proposition 7. If D is a digraph of order n with $\delta^-(D) \geq 1$, then

$$\gamma_{stI}(D) \geq \delta^-(D) + 3 - n.$$

Proof. Let f be a $\gamma_{stI}(D)$-function. Clearly, there exists a vertex w with $f(w) \geq 1$. Now the definitions imply

$$\begin{aligned}\gamma_{stI}(D) &= \sum_{x \in V(D)} f(x) = f(w) + \sum_{x \in N^-(w)} f(x) + \sum_{x \in V(D)\setminus N^-[w]} f(x) \\ &\geq 1 + 1 + \sum_{x \in V(D)\setminus N^-[w]} f(x) \geq 2 - (n - (d^-(w) + 1)) \geq 3 + \delta^-(D) - n.\end{aligned}$$

□

Proposition 8. If D is a digraph of order n with $\delta^-(D) \geq 1$, then $\gamma_{stI}(D) \geq 2\gamma_t(D) - n$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}(D)$-function. Then it follows from Proposition 4 that

$$\gamma_{stI}(D) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_t(D) - n, \quad (1)$$

and the desired bound is proved. □

If C_n^o is an oriented cycle of order n, then $\gamma_t(C_n^o) = \gamma_{stI}(C_n^o) = n$, and thus $\gamma_{stI}(C_n^o) = 2\gamma_t(C_n^o) - n$. This example shows that Proposition 8 is sharp.
If in Proposition 8 there exists a $\gamma_{stI}(D)$-function $g = (V_{-1}, V_1, V_2)$ with $|V_2| \geq 1$, then the proof of this proposition shows that $\gamma_{stI}(D) \geq 2\gamma_t(D) + 1 - n$.

Let F_n be the digraph of order $n \geq 3$ with vertex set $\{u, v, x_1, x_2, \ldots, x_{n-2}\}$ such that u dominates $v, x_1, x_2, \ldots, x_{n-2}$ and v dominates $u, x_1, x_2, \ldots, x_{n-2}$. Let $A = \{x_1, x_2, \ldots, x_{n-2}\}$. We define the family $\mathcal{F}_n$ as follows. The digraph F_n belongs to $\mathcal{F}_n$. There is no arc from A to $\{u, v\}$. In addition, there are admissible arcs between vertices of A such that $d^-(x_i) \leq 3$ for $1 \leq i \leq n-2$.

Theorem 3. Let D be a digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$. Then $\gamma_{stI}(D) \geq 4 - n$, with equality if and only if D is a member of $\mathcal{F}_n$.

Proof. Proposition 7 leads to the desired bound $\gamma_{stI}(D) \geq \delta^-(D) + 3 - n \geq 4 - n$. Let $\gamma_{stI}(D) = 4 - n$, and let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}(D)$-function. Then there exist at least two vertices u and v such that $f(u), f(v) \geq 1$. Since $\gamma_{stI}(D) = 4 - n$, we observe that $f(u) = f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v\} = \{x_1, x_2, \ldots, x_{n-2}\}$. By the definitions, u and v dominate x_i for $1 \leq i \leq n-2$, u dominates v and v dominates u. If there is an arc from $\{x_1, x_2, \ldots, x_{n-2}\}$ to $\{u, v\}$, then we obtain the contradiction $f(N^-(u)) \leq 0$ or $f(N^-(v)) \leq 0$. Hence there is no arc from $\{x_1, x_2, \ldots, x_{n-2}\}$ to $\{u, v\}$. If $d^-(x_j) \geq 4$ for an index $j \in \{1, 2, \ldots, n-2\}$, then we obtain the contradiction $f(N^-(x_j)) \leq 0$. We deduce that $d^-(x_i) \leq 3$ for $1 \leq i \leq n-2$, and thus D is a member of $\mathcal{F}_n$.
Conversely, assume that D is a member of $\mathcal{F}_n$. Define the function $g : V(D) \longrightarrow \{-1, 1, 2\}$ by $g(u) = g(v) = 1$ and $g(x_i) = -1$ for $1 \leq i \leq n-2$. Then g is an STIDF on D of weight $4 - n$ and therefore $\gamma_{stI}(D) \leq 4 - n$ and thus $\gamma_{stI}(D) = 4 - n$. □

If Q is a member of $\mathcal{F}_n$, then $4 - n = \gamma_{stI}(Q) = 2\gamma_t(Q) - n$, and therefore equality in Proposition 8.
If H is a member of $\mathcal{F}_n$ such that $\Delta^-(H) = 3$, then $4 - n = \gamma_{stI}(Q) = \Delta^-(H) + 1 - n$, and therefore equality in Proposition 6.
These are further examples for the sharpness of Propositions 6 and 8.

Let H_n be the digraph of order $n \geq 3$ with vertex set $\{u, v, x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t\}$ such that $s + t + 2 = n$, u dominates $v, x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_t$ and v dominates $u, y_1, x_2, \ldots, x_t$. Let $A = \{x_1, x_2, \ldots, x_s\}$ and $B = \{y_1, y_2, \ldots, y_t\}$. We define the family $\mathcal{H}_n$ as follows. The digraph H_n belongs to $\mathcal{H}_n$. There is no arc from $A \cup B$ to u and at most one arc from $A \cup B$ to v. In addition, there are admissible arcs between vertices of $A \cup B$ such that $d^-(x_i) \leq 2$ for $1 \leq i \leq s$ and $d^-(y_j) \leq 4$ for $1 \leq j \leq t$. If $s = 0$, then there is an arc from B to v or there exists a vertex y_k such that $d^-(y_k) = 4$.

Theorem 4. Let D be a digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$ such that D is not a member of $\mathcal{F}_n$. Then $\gamma_{stI}(D) \geq 5-n$, with equality if and only if D is a member of $\mathcal{H}_n$.

Proof. Theorem 3 implies the desired bound $\gamma_{stI}(D) \geq 5-n$.
Let $\gamma_{stI}(D) = 5-n$, and let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}(D)$-function. Then there exist at least two vertices u and v such that $f(u), f(v) \geq 1$. Since $\gamma_{stI}(D) = 5-n$, we observe that, without loss of generality, $f(u) = 2$, $f(v) = 1$ and $f(x) = -1$ for $x \in V(D) \setminus \{u, v\}$. By the definitions, u dominates v, v dominates u and each vertex $x \in V(D) \setminus \{u, v\}$ is dominated only by u or by u and v. Let $A = \{x_1, x_2, \ldots, x_s\} \subseteq V(D) \setminus \{u, v\}$ be the set of vertices only dominated by u and $B = \{y_1, y_2, \ldots, y_t\} \subseteq V(D) \setminus \{u, v\}$ be the set of vertices dominated by u and v. If there is an arc from $A \cup B$ to u, then we obtain the contradiction $f(N^-(u)) \leq 0$. Hence there is no arc from $A \cup B$ to u. If there are two arcs from $A \cup B$ to v, then we obtain the contradiction $f(N^-(v)) \leq 0$. Hence there exists at most one arc from $A \cup B$ to v. If $d^-(x_j) \geq 3$ for an index $j \in \{1, 2, \ldots, s\}$, then we obtain the contradiction $f(N^-(x_j)) \leq 0$. We deduce that $d^-(x_i) \leq 2$ for $1 \leq i \leq s$. If $d^-(y_j) \geq 5$ for an index $j \in \{1, 2, \ldots, t\}$, then we obtain the contradiction $f(N^-(y_j)) \leq 0$. We deduce that $d^-(y_i) \leq 4$ for $1 \leq i \leq t$. If $s = 0$, then the hypothesis that D is not a member of $\mathcal{F}_n$ shows that there is an arc from B to v or there exists a vertex y_k such that $d^-(y_k) = 4$. Altogether, we deduce that D is a member of $\mathcal{H}_n$.
Conversely, assume that D is a member of $\mathcal{H}_n$. Define the function $g : V(D) \longrightarrow \{-1, 1, 2\}$ by $g(u) = 2$, $g(v) = 1$ and $g(x) = -1$ for $x \in A \cup B$. Then g is an STIDF on D of weight $5-n$ and therefore $\gamma_{stI}(D) \leq 5-n$ and thus $\gamma_{stI}(D) = 5-n$. □

The next example will demonstrate that the difference $\gamma_{stR}(D) - \gamma_{stI}(D)$ can be arbitrarily large.

Example 2. *Let J be an arbitray digraph of order $t \geq 1$, and for each vertex $v \in V(J)$ add a vertex-disjoint copy of a complete digraph K_s^* with $s \geq 6$ even and identify the vertex v with one vertex of the added complete digraph. Let Q denote the resulting digraph. Furthermore, let $Q_1, Q_2, \ldots, Q_t$ be the added copies of K_s^*. For $i = 1, 2, \ldots, t$ let v_i be the vertex of Q_i that is identified with a vertex of J.*
First we construct an STIDF on Q as follows. For each $i = 1, 2, \ldots, t$, let $f_i : V(Q_i) \longrightarrow \{-1, 1, 2\}$ be an STIDF on the complete digraph of weight 2 (see Corollary 1) such that $f_i(v_i) \geq 1$. Now let $f : V(Q) \longrightarrow \{-1, 1, 2\}$ be the function defined by $f(v) = f_i(v)$ for each $v \in V(H_i)$. Then f is an STIDF of Q of weight $2t$ and hence $\gamma_{stI}(Q) \leq 2t$.
Now let g be a $\gamma_{stR}(Q)$-function. We show that $g(V(Q_i)) \geq 3$ for each $1 \leq i \leq t$. If $g(x) = -1$ for at most one $x \in V(Q_i)$, then $g(V(Q_i)) \geq s-2 \geq 4$. Hence assume that there are at least two vertices $u, v \in V(Q_i)$ such that $g(u) = g(v) = -1$. This implies that there exists a vertex $w \in V(Q_i)$ with $g(w) = 2$. If $w \neq v_i$, then we deduce that $g(V(Q_i)) = g(w) + g(N^-(w)) \geq 2+1 = 3$. Next we assume that $w = v_i$ and $g(x) = 1$ or $g(x) = -1$ for $x \in V(Q_i) \setminus \{w\}$. Since $g(N^-(u)) \geq 1$ and $s \geq 6$, we observe that there exists a vertex $z \in V(Q_i)$ with $g(z) = 1$. Assume that z has exactly j in-neighbors of weight 1 and $s-j-2$ in-neighbors of weight -1. We deduce that $g(N^-(z)) = 2 + j - (s-j-2) = 4 + 2j - s \geq 1$, and since s is even, it follows that $g(N^-(z)) = 4 + 2j - s \geq 2$. Thus $g(V(Q_i)) = g(z) + g(N^-(z)) \geq 1 + 2 = 3$, and we obtain $\gamma_{stR}(Q) = g(V(Q)) = \sum_{i=1}^{t} g(V(Q_i)) \geq 3t$. Consequently, we see that $\gamma_{stR}(Q) - \gamma_{stI}(Q) \geq 3t - 2t = t$.

References

[1] H. Abdollahzadeh Ahangar, M.A. Henning, C. Löwenstein, Y. Zhao, and V. Samodivkin, *Signed Roman domination in graphs*, J. Comb. Optim. **27** (2014), no. 2, 241–255.

[2] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *A survey on Roman domination parameters in directed graphs*, J. Combin. Math. Combin. Comput. **115** (2020), 141–171.

[3] ———, *The Roman domatic problem in graphs and digraphs: A survey*, Discuss. Math. Graph Theory **42** (2022), no. 3, 861–891.

[4] N. Dehgardi and L. Volkmann, *Signed total Roman k-domination in directed graphs*, Commun. Comb. Optim. **1** (2016), no. 2, 165–178.

[5] G. Hao, X. Chen, and L. Volkmann, *Bounds on the signed Roman k-domination number of a digraph.*, Discuss. Math. Graph Theory **39** (2019), no. 1, 67–79.

[6] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Domination in Graphs, Advanced Topics*, Marcel Dekker, Inc., New York, 1998.

[7] ———, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.

[8] M.A. Henning and L. Volkmann, *Signed Roman k-domination in trees*, Discrete Appl. Math. **186** (2015), 98–10.

[9] ———, *Signed Roman k-domination in graphs*, Graphs Combin. **32** (2016), no. 1, 175–190.

[10] S.M. Sheikholeslami, *Signed total domination numbers of directed graphs*, Util. Math. **85** (2011), 273–279.

[11] S.M. Sheikholeslami, A. Bodaghli, and L. Volkmann, *Twin signed Roman domination numbers in directed graphs*, Tamkang J. Math. **47** (2016), no. 3, 357–371.

[12] S.M. Sheikholeslami and L. Volkmann, *Signed Roman domination in digraphs*, J. Comb. Optim. **30** (2015), no. 3, 456–467.

[13] L. Volkmann, *Signed Roman k-domination in digraphs*, Graphs Combin. **32** (2016), no. 6, 1217–1227.

[14] ———, *Signed total Roman domination in graphs*, J. Comb. Optim. **32** (2016), no. 3, 855–871.

[15] ———, *Signed total Roman domination in digraphs*, Discuss. Math. Graph Theory **37** (2017), no. 1, 261–272.

[16] ———, *Signed total Roman k-domination in graphs*, J. Combin. Math. Combin. Comput. **105** (2018), 105–116.

[17] ———, *Signed total Italian domination in graphs*, J. Combin. Math. Combin. Comput. **115** (2020), 291–305.

[18] B. Zelinka, *Signed total domination nnumber of a graph*, Czechoslovak Math. J. **51** (2001), no. 2, 225–229.