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***Research Article***

# Normalized distance Laplacian matrices for signed graphs

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**Abstract:** In this paper, we introduce the notion of normalized distance Laplacian matrices for signed graphs corresponding to the two signed distances defined for signed graphs. We characterize balance in signed graphs using these matrices and compare the normalized distance Laplacian spectral radius of signed graphs with that of all-negative signed graphs. Also we characterize the signed graphs having maximum normalized distance Laplacian spectral radius.

**Keywords:** Signed graphs, Weighted signed graphs, Normalized distance Laplacian matrix, Normalized distance Laplacian spectrum

**AMS Subject classification:** 05C12, 05C22, 05C50, 05C75

## 1. Introduction

A signed graph $\Sigma = (G, \sigma)$ is an underlying graph $G = (V, E)$ with a signature function $\sigma : E \to \{1, -1\}$. The concept of the signed distances and the corresponding signed distance matrices are defined by Hameed et al., in [2]. Given a signed graph

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$\Sigma = (G, \sigma)$, the sign of a path $P$ in $\Sigma$ is defined as $\sigma(P) = \prod_{e \in E(P)} \sigma(e)$. The shortest path between two given vertices $u$ and $v$ is denoted by $P_{(u,v)}$ and the collection of all shortest paths $P_{(u,v)}$ by $\mathcal{P}_{(u,v)}$; and $d(u,v)$ denotes the usual distance between $u$ and $v$.

**Definition 1 (Signed distance matrices [2]).** Auxiliary signs are defined as:
(S1) $\sigma_{\max}(u,v) = -1$ if all shortest $uv$-paths are negative, and $+1$ otherwise.
(S2) $\sigma_{\min}(u,v) = +1$ if all shortest $uv$-paths are positive, and $-1$ otherwise.
Signed distances are:
(d1) $d_{\max}(u,v) = \sigma_{\max}(u,v)d(u,v) = \max\{\sigma(P_{(u,v)}) : P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u,v)$.
(d2) $d_{\min}(u,v) = \sigma_{\min}(u,v)d(u,v) = \min\{\sigma(P_{(u,v)}) : P_{(u,v)} \in \mathcal{P}_{(u,v)}\}d(u,v)$.
And the signed distance matrices are:
(D1) $D^{\max}(\Sigma) = (d_{\max}(u,v))_{n \times n}$.
(D2) $D^{\min}(\Sigma) = (d_{\min}(u,v))_{n \times n}$.

**Definition 2.** [2] Two vertices $u$ and $v$ in a signed graph $\Sigma$ are said to be *distance-compatible* (briefly, *compatible*) if $d_{min}(u,v) = d_{max}(u,v)$. And $\Sigma$ is said to be (distance-)compatible if every two vertices are compatible. Then $D^{\max}(\Sigma) = D^{\min}(\Sigma) = D^{\pm}(\Sigma)$.

Corresponding to the signed distances defined for signed graphs, the signed distance Laplacian matrices are defined in [9]. The transmission $tr(v)$ of a vertex $v$ is defined to be the sum of the distances from $v$ to all other vertices in $G$. That is, $tr(v) = \sum_{u \in V(G)} d(v,u)$. The transmission matrix $Tr(G)$ for a graph $G$ is the diagonal matrix with diagonal entries $tr(v_i)$.

**Definition 3.** [9] Signed distance Laplacian matrices for signed graphs is defined as
(L1) $DL^{\max}(\Sigma) = Tr(G) - D^{\max}(\Sigma)$.
(L2) $DL^{\min}(\Sigma) = Tr(G) - D^{\min}(\Sigma)$.
When $\Sigma$ is compatible, $DL^{\max}(\Sigma) = DL^{\min}(\Sigma) = DL^{\pm}(\Sigma)$.

We now define the normalized distance Laplacian matrices for signed graphs as follows.

**Definition 4.** Corresponding to the two signed distance Laplacian matrices for signed graphs, we define the two normalized distance Laplacian matrices $\mathcal{DL}^{\max}(\Sigma) = (\mathcal{DL}^{\max}_{i,j})_{n \times n}$ and $\mathcal{DL}^{\min}(\Sigma) = (\mathcal{DL}^{\min}_{i,j})_{n \times n}$ as

$$\mathcal{DL}^{\max}_{i,j} = \begin{cases} 1 & \text{if } i = j \\ \frac{-d_{\max}(v_i,v_j)}{\sqrt{tr(v_i)tr(v_j)}} & \text{if } i \neq j \end{cases} \text{ and } \mathcal{DL}^{\min}_{i,j} = \begin{cases} 1 & \text{if } i = j \\ \frac{-d_{\min}(v_i,v_j)}{\sqrt{tr(v_i)tr(v_j)}} & \text{if } i \neq j \end{cases}.$$

For a signed graph $\Sigma$, we get, $\mathcal{DL}^{\max}(\Sigma) = Tr(G)^{-1/2} DL^{\max}(\Sigma) Tr(G)^{-1/2}$ and $\mathcal{DL}^{\min}(\Sigma) = Tr(G)^{-1/2} DL^{\min}(\Sigma) Tr(G)^{-1/2}$. By substituting, from the definition of signed Laplacian matrices, it can be seen that $\mathcal{DL}^{\max}(\Sigma) = I - Tr(G)^{-1/2} D^{\max}(\Sigma) Tr(G)^{-1/2}$ and $\mathcal{DL}^{\min}(\Sigma) = I - Tr(G)^{-1/2} D^{\min}(\Sigma) Tr(G)^{-1/2}$. When $\Sigma$ is compatible, $\mathcal{DL}^{\max}(\Sigma) = \mathcal{DL}^{\min}(\Sigma) = \mathcal{DL}^{\pm}(\Sigma)$.

The defintion of the normalized distance Laplacian matrices for signed graphs is analogous to the defintion of the normalized Laplacian matrix defined for signed graphs in [5]. In [6], Li et al. gave the bounds for the frustration number and the frustration index of signed graphs in terms of its least eigenvalue of the normalized Laplacian of signed graphs. The normalized Laplacian matrix for graphs is introduced and studied extensively by Chung in [1] and the normalized distance Laplacian for graphs is explored recently by Reinhart in [8] and by Pirzada in [7].

Throughout this article, unless otherwise mentioned, by a graph we mean a finite, connected, simple graph. For any terms which are not mentioned here, the reader may refer to [3] and [10].

## 2. Normalized Laplacian matrix for weighted signed graphs

In this section, we recall the definition of the adjacency matrix and the Laplacian matrix of a weighted signed graph and then define the normalized Laplacian matrix for a weighted signed graph. A weighted signed graph is denoted by $(\Sigma, w)$ where $\Sigma = (G, \sigma)$ is a signed graph and $w$ is a positive weight function defined on the edges of $\Sigma$. For a weighted signed graph $(\Sigma, w)$, $w(\Sigma)$ is the product of all the weights given to the edges of $\Sigma$. We use the notation $u \sim v$ when the vertices $u$ and $v$ are adjacent and similar notation for the incidence of an edge on a vertex.

**Definition 5.** [9] Let $(\Sigma, w)$ be a weighted signed graph. Its adjacency matrix $A(\Sigma, w) = (a_{ij})_n$ is defined as the square matrix of order $n = |V(G)|$ where

$$a_{ij} = \begin{cases} \sigma(v_i v_j) w(v_i v_j) & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 6.** [9] For a weighted signed graph $(\Sigma, w)$, its weighted Laplacian matrix is defined as $L(\Sigma, w) = D(\Sigma, w) - A(\Sigma, w)$, where the diagonal matrix $D(\Sigma, w)$ is $\operatorname{diag}\left(\sum_{e: v_i \sim e} w(e)\right)$ which is called the weighted degree matrix of $(\Sigma, w)$.

Laplacian matrix $L(\Sigma, w)$ of $(\Sigma, w)$ can be described by means of its quadratic form as

$$x^T L(\Sigma, w) x = \sum_{v_i v_j \in E(G)} w(v_i v_j)(x_i - \sigma(v_i v_j) x_j)^2,$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}_n$. Hence, $L(\Sigma, w)$ is positive semi definite.

**Definition 7.** For a weighted signed graph $(\Sigma, w)$, its normalized Laplacian matrix is defined as $\mathcal{L}(\Sigma, w) = D(\Sigma, w)^{-1/2} L(\Sigma, w) D(\Sigma, w)^{-1/2}$.

For an oriented edge $\vec{e}_j = \overrightarrow{v_i v_k}$, we take $v_i$ as the tail of that edge and $v_k$ as its head and we write $t(\vec{e}_j) = v_i$ and $h(\vec{e}_j) = v_k$.

Analogous to the definition of oriented incidence matrix of a weighted signed graph, we define a matrix $\mathcal{H}(\Sigma, w) = (\eta_{v_i e_j})$ whose rows are indexed by the vertices and columns are indexed by the edges of $G$ where

$$\eta_{v_i e_j} = \begin{cases} \frac{\sigma(e_j)\sqrt{w(e_j)}}{\sqrt{d(v_i)}} & \text{if } t(\vec{e}_j) = v_i, \\ \frac{-\sqrt{w(e_j)}}{\sqrt{d(v_i)}} & \text{if } h(\vec{e}_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for a weighted signed graph $(\Sigma, w)$, $\mathcal{L}(\Sigma, w) = \mathcal{H}(\Sigma, w)\mathcal{H}^T(\Sigma, w)$.

Let $(\Sigma, w)$ be a connected weighted signed graph. Since $\mathcal{L}(\Sigma, w)$ is a real symmetric matrix, the eigenvalues of $\mathcal{L}(\Sigma, w)$ are real. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of its normalized Laplacian matrix $\mathcal{L}(\Sigma, w)$.

Applying the Courant-Fischer theorem to get the eigenvalue $\lambda_k$ of $\mathcal{L}(\Sigma, w)$,

$$\lambda_k = \min_{g_{k+1}, g_{k+2}, \ldots, g_n \in \mathbb{R}_n} \max_{\substack{g \neq 0 \\ g \perp g_{k+1}, g_{k+2}, \ldots, g_n}} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle}$$

Since $(\Sigma, w)$ is connected, $D(\Sigma, w)^{1/2}$ is invertible. For vectors $g$ and $g_j$, define the vectors $f = D(\Sigma, w)^{-1/2} g$ and $f_j = D(\Sigma, w)^{1/2} g_j$.
Note that, $g \perp g_{k+1}, g_{k+2}, \ldots, g_n$ if and only if $f \perp f_{k+1} f_{k+2}, \ldots, f_n$.
Since $D(\Sigma, w)^{1/2}$ is invertible, $g \neq 0$ if and only if $f \neq 0$ and minimizing over vectors $g_{k+1}, g_{k+2}, \ldots, g_n$ is equivalent to minimizing over vectors $f_{k+1}, f_{k+2}, \ldots, f_n$. The vector $f$ is viewed as function $f(u)$ on the vertex set and is often called the harmonic eigenfunction corresponding to $\lambda_k$.
Then,

$$\begin{aligned}
\lambda_k &= \min_{g_{k+1}, g_{k+2}, \ldots, g_n \in \mathbb{R}_n} \max_{\substack{f \neq 0 \\ f \perp f_{k+1}, f_{k+2}, \ldots, f_n}} \frac{\langle f, L f \rangle}{\langle D(\Sigma, w)^{1/2} f, D(\Sigma, w)^{1/2} f \rangle} \\
&= \min_{f_{k+1}, f_{k+2}, \ldots, f_n \in \mathbb{R}_n} \max_{\substack{f \neq 0 \\ f \perp f_{k+1}, f_{k+2}, \ldots, f_n}} \frac{\langle f, L f \rangle}{\langle D(\Sigma, w)^{1/2} f, D(\Sigma, w)^{1/2} f \rangle} \\
&= \min_{f_{k+1}, f_{k+2}, \ldots, f_n \in \mathbb{R}_n} \max_{\substack{f \neq 0 \\ f \perp f_{k+1}, f_{k+2}, \ldots, f_n}} \frac{\displaystyle\sum_{v_i v_j \in E(G)} w(v_i v_j)(f(v_i) - \sigma(v_i v_j) f(v_j))^2}{\displaystyle\sum_{u \in V(G)} f(u)^2 d(u)}.
\end{aligned}$$

The other half of the Courant-Fischer theorem gives

$$\begin{aligned}
\lambda_k &= \max_{g_1, g_2, \ldots, g_{k-1} \in \mathbb{R}_n} \min_{\substack{g \neq 0 \\ g \perp g_1, g_2, \ldots, g_{k-1}}} \frac{\langle g, \mathcal{L} g \rangle}{\langle g, g \rangle} \\
&= \max_{f_1, f_2, \ldots, f_{k-1} \in \mathbb{R}_n} \min_{\substack{f \neq 0 \\ f \perp f_1, f_2, \ldots, f_{k-1}}} \frac{\displaystyle\sum_{v_i v_j \in E(G)} w(v_i v_j)(f(v_i) - \sigma(v_i v_j) f(v_j))^2}{\displaystyle\sum_{u \in V(G)} f(u)^2 d(u)}.
\end{aligned}$$

In particular, we get

$$\lambda_1 = \inf_{f \neq 0} \frac{\displaystyle\sum_{v_i v_j \in E(G)} w(v_i v_j)(f(v_i) - \sigma(v_i v_j) f(v_j))^2}{\displaystyle\sum_{u \in V(G)} f(u)^2 d(u)}$$

and

$$\lambda_n = \sup_{f \neq 0} \frac{\sum\limits_{v_i v_j \in E(G)} w(v_i v_j)(f(v_i) - \sigma(v_i v_j) f(v_j))^2}{\sum\limits_{u \in V(G)} f(u)^2 d(u)}.$$

Thus, the normalized Laplacian matrix for a weighted signed graph is positive semi definite. We denote, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ as the eigenvalues of $\mathcal{L}(\Sigma, w)$.

Our next theorem compares the normalized Laplacian spectral radius of weighted signed graphs with that of all-negative signed graphs and finds a bound for the same. For proving these theorems, we need the following lemmas.

**Lemma 1 ([4]).** *Let $\Sigma = (G, \sigma)$ be a signed graph. Then the following conditions are equivalent:*

(i) *$\Sigma = (G, \sigma)$ is a signed graph such that all odd cycles are negative and all even cycles are positive.*

(ii) *There exists a partition $V(\Sigma) = V_1 \cup V_2$ such that every edge between $V_1$ and $V_2$ is positive and every edge within $V_1$ or $V_2$ is negative.*

(iii) *$(G, \sigma)$ is switching equivalent to $(G, -)$.*

**Lemma 2.** *If $\lambda$ is an eigenvalue of $\mathcal{L}(\Sigma, w)$ corresponding to the eigenvector $g$, then $\lambda$ is also an eigenvalue of $D(\Sigma, w)^{-1} L(\Sigma, w)$ corresponding to the eigenvector $f = D(\Sigma, w)^{-1/2} g$.*

*Proof.* To prove, $\lambda$ is an eigenvalue of $D(\Sigma, w)^{-1} L(\Sigma, w)$, consider

$$\begin{aligned} D(\Sigma, w)^{-1} L(\Sigma, w) f &= D(\Sigma, w)^{-1} L(\Sigma, w) D(\Sigma, w)^{-1/2} g \\ &= D(\Sigma, w)^{-1/2} \mathcal{L}(\Sigma, w) g \\ &= D(\Sigma, w)^{-1/2} \lambda g = \lambda f. \end{aligned}$$

□

For weighted signed graphs $(\Sigma, w)$, since the weight function $w$ is positive, by saying $(\Sigma, w)$ is switched to $(\Sigma^\zeta, w)$ by the switching function $\zeta$, we mean that the corresponding signed graphs $\Sigma$ and $\Sigma^\zeta$ are switching equivalent.

**Lemma 3.** *Let $(\Sigma, w)$ be switched to $(\Sigma^\zeta, w)$. Then $\mathcal{L}(\Sigma, w)$ and $\mathcal{L}(\Sigma^\zeta, w)$ are cospectral.*

*Proof.* If $(\Sigma, w)$ is switched to $(\Sigma^\zeta, w)$, then the weighted Laplacian matrices $L(\Sigma, w)$ and $L(\Sigma^\zeta, w)$ are similar by the switching matrix $S = \mathrm{diag}(\zeta(v_1)), \zeta(v_2)), \ldots, \zeta(v_n)) = S^{-1}$. That is, $L(\Sigma^\zeta, w) = SL(\Sigma, w)S^{-1}$. Thus, we get $\mathcal{L}(\Sigma^\zeta, w) = S\mathcal{L}(\Sigma, w)S^{-1}$ and hence they are cospectral. □

**Theorem 1.** *Let $(\Sigma, w)$ be a connected weighted signed graph of order $n$ and $(\Sigma^-, w)$ be the corresponding all negative weighted signed graph. Then $\lambda_n(\sigma) \leq \lambda_n(-)$ where equality holds if and only if $(\Sigma, w)$ is switching equivalent to $(\Sigma^-, w)$.*

*Proof.* Let $f = (f(v) : v \in V(G))^T$ be the harmonic eigenfunction corresponding to $\lambda_n$. Take $h = (|f(v)| : v \in V(G))^T$. Now,

$$\begin{aligned}
\lambda_n(\sigma) &= \frac{\sum\limits_{v_iv_j \in E(G)} w(v_iv_j)(f(v_i) - \sigma(v_iv_j)f(v_j))^2}{\sum\limits_{u \in V(G)} f(u)^2 d(u)} \\
&\leq \frac{\sum\limits_{v_iv_j \in E(G)} w(v_iv_j)(|f(v_i)| + |f(v_j)|)^2}{\sum\limits_{u \in V(G)} f(u)^2 d(u)} \\
&\leq \lambda_n(-).
\end{aligned}$$

If $\lambda_n(\sigma) = \lambda_n(-)$, then $-\sigma(v_iv_j)w(v_iv_j)f(v_i)f(v_j) = w(v_iv_j)|f(v_i)||f(v_j)|$, for all $v_iv_j \in E(G)$. That is, $-\sigma(v_iv_j)f(v_i)f(v_j) = |f(v_i)||f(v_j)|$, since the edge weights are positive. Now, $\mathcal{L}(\Sigma^-, w)$ is a non negative matrix and hence $f$ has no zero entries. Let $V_1 = \{v : f(v) > 0\}$ and $V_2 = \{v : f(v) < 0\}$. For a negative edge $v_iv_j$, $f(v_i)f(v_j) = |f(v_i)||f(v_j)| > 0$ which imples $v_iv_j$ is either in $V_1$ or in $V_2$. For a postive edge $v_iv_j$, $-f(v_i)f(v_j) = |f(v_i)||f(v_j)|$ which implies $f(v_i)f(v_j) < 0$ and hence $v_iv_j$ is between $V_1$ and $V_2$. Thus by Lemma 1, $(\Sigma, w)$ is switching equivalent to $(\Sigma^-, w)$. Conversely, if $(\Sigma, w)$ is switching equivalent to $(\Sigma^-, w)$, then by Lemma 3 they are cospectral and hence $\lambda_n(\sigma) = \lambda_n(-)$. □

**Theorem 2.** *Let $(\Sigma, w)$ be a connected weighted signed graph of order $n$. Then $\lambda_i \leq 2$, for all $i \leq n$, with $\lambda_n = 2$ if and only if $(\Sigma, w)$ is switching equivalent to $(\Sigma^-, w)$.*

*Proof.* Consider the matrix

$$M = D(\Sigma, w)^{-1/2}\mathcal{L}(\Sigma^-, w)D(\Sigma, w)^{1/2} = D(\Sigma, w)^{-1}L(\Sigma^-, w).$$

By Lemma 2, $\mathcal{L}(\Sigma^-, w)$ and $D(\Sigma, w)^{-1}L(\Sigma^-, w)$ have same set of eigenvalues. Now the row sums of $M$ all equals 2 and hence $\lambda_n(M) = 2$.

Thus, by Theorem 1, $\lambda_n(\sigma) \leq \lambda_n(-) = \lambda_n(M) = 2$ and $\lambda_n(\sigma) = \lambda_n(-) = 2$ if and only if $(\Sigma, w)$ is switching equivalent to $(\Sigma^-, w)$. □

## 3. Normalized Distance Laplacian Matrices for Signed Graphs

The characterization for balance in signed graphs using the signed distance Laplacian matrices is established in [9] as follows.

**Theorem 3 ([9]).** *The following properties of a signed graph $\Sigma$ are equivalent.*

(i) *$\Sigma$ is balanced.*

(ii) *The determinant of signed distance Laplacian matrix,* $\det(DL^{\max}(\Sigma)) = 0$.

(iii) *The determinant of signed distance Laplacian matrix,* $\det(DL^{\min}(\Sigma)) = 0$.

(iv) $DL^{\max}(\Sigma) = DL^{\min}(\Sigma)$ *and the determinant of signed distance Laplacian matrix,* $\det(DL^{\pm}(\Sigma)) = 0$.

A similar characterization for balance in signed graphs can be proved using the normalized distance Laplacian matrices as follows.

**Theorem 4.** *A connected signed graph is balanced if and only if the determinant of its normalized distance Laplacian matrix,* $\det(\mathcal{DL}^{\max}(\Sigma))$ *(or* $\det(\mathcal{DL}^{\min}(\Sigma))$*) is equal to zero.*

*Proof.* $\det(\mathcal{DL}^{\max}(\Sigma)) = \det(Tr(G)^{-\frac{1}{2}}) \det(DL^{\max}(\Sigma)) \det(Tr(G)^{-\frac{1}{2}})$.

For a connected signed graph, $tr(v) \neq 0$ for any vertex $v$. Hence $\det(Tr(G)^{-\frac{1}{2}}) \neq 0$. Thus, $\det(\mathcal{DL}^{\max}(\Sigma)) = 0$ if and only if $\det(DL^{\max}(\Sigma)) = 0$ if and only if $\Sigma$ is balanced.

Similarly, we can prove the case for $\mathcal{DL}^{\min}(\Sigma)$. □

**Theorem 5.** *A signed graph $\Sigma$ is balanced if and only if $\mathcal{DL}^{\max}(\Sigma) = \mathcal{DL}^{\min}(\Sigma) = \mathcal{DL}^{\pm}(\Sigma)$ and $\mathcal{DL}^{\pm}(\Sigma)$ is cospectral with $D^{\mathcal{L}}(G)$, where $D^{\mathcal{L}}(G)$ denotes the normalized distance Laplacian matrix of the underlying graph $G$.*

*Proof.* If $\Sigma$ is balanced, then $\Sigma$ can be switched to the all positive signed graph $\Sigma^{\zeta} = (G, +)$. Then the signed distance Laplacian matrices, $DL^{\max}(\Sigma) = DL^{\min}(\Sigma) = DL^{\pm}(\Sigma)$ and $SDL^{\pm}(\Sigma)S^{-1} = DL^{\pm}(\Sigma^{\zeta})$ by the switching matrix $S$ [9]. Thus, using this same matrix $S$ we get, $\mathcal{DL}^{\pm}(\Sigma)$ is similar to $\mathcal{DL}^{\pm}(\Sigma^{\zeta})$ (which is equal to $D^{\mathcal{L}}(G)$) and hence cospectral.
Conversely, if $\mathcal{DL}^{\pm}(\Sigma)$ is cospectral with $D^{\mathcal{L}}(G)$ then, $\det(\mathcal{DL}^{\pm}(\Sigma)) = \det(D^{\mathcal{L}}(G)) = 0$. Hence by Theorem 4, $\Sigma$ is balanced. □

Hameed et al. in [2] defined the two complete signed graphs from the distance matrices $D^{\max}$ and $D^{\min}$ as follows.

**Definition 8 ([2]).** The associated signed complete graph $K^{D^{\max}}(\Sigma)$ with respect to $D^{\max}(\Sigma)$ is obtained by joining the non-adjacent vertices of $\Sigma$ with edges having signs $\sigma(uv) = \sigma_{\max}(uv)$.
The associated signed complete graph $K^{D^{\min}}(\Sigma)$ with respect to $D^{\min}(\Sigma)$ is obtained by joining the non-adjacent vertices of $\Sigma$ with edges having signs $\sigma(uv) = \sigma_{\min}(uv)$.

Using the defintion of the associated signed complete graphs, we have the following. Let $\Sigma$ be a signed graph. Corresponding to the associated signed complete graph $K^{D^{\max}}(\Sigma)$, we define the weighted signed complete graph $(K^{D^{\max}}(\Sigma), w)$, where $w(e) = d(u, v)$ and $(K^{D^{\min}}(\Sigma), w)$, where $w(e) = d(u, v)$ for an edge $e = uv$.
Then, the normalized Laplacian matrices $\mathcal{L}(K^{D^{\max}}(\Sigma), w)$ is same as the normalized distance Laplacian matrix $\mathcal{DL}^{\max}(\Sigma)$ and $\mathcal{L}(K^{D^{\min}}(\Sigma), w)$ is same as the normalized distance Laplacian matrix $\mathcal{DL}^{\min}(\Sigma)$. That is, $\mathcal{L}(K^{D^{\max}}(\Sigma), w) = \mathcal{DL}^{\max}(\Sigma)$ and $\mathcal{L}(K^{D^{\min}}(\Sigma), w) = \mathcal{DL}^{\min}(\Sigma)$.
Since the normalized Laplacian matrix of a weighted signed graph is positive semi definte, the normalized distance Laplacian matrices $\mathcal{DL}^{\max}(\Sigma)$ and $\mathcal{DL}^{\min}(\Sigma)$ are positive semi definite. We denote $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$ as the eigenvalues of $\mathcal{DL}^{\max}(\Sigma)$ and $0 \leq \mu_1' \leq \mu_2' \leq \cdots \leq \mu_n'$ as the eigenvalues of $\mathcal{DL}^{\min}(\Sigma)$.

**Lemma 4.** *The multiplicity of* $0$ *as an eigenvalue of* $D^{\max}(\Sigma)$ *(or* $D^{\min}(\Sigma)$*) is the multiplicity of* $1$ *as an eigenvalue of* $\mathcal{DL}^{\max}(\Sigma)$ *(or* $\mathcal{DL}^{\min}(\Sigma)$*), the number of negative eigenvalues for* $D^{\max}(\Sigma)$*(or* $D^{\min}(\Sigma)$*) is the number of eigenvalues greater than* $1$ *for* $\mathcal{DL}^{\max}(\Sigma)$

*(or $\mathcal{DL}^{\min}(\Sigma)$), the number of positive eigenvalues fo $D^{\max}(\Sigma)$ (or $D^{\min}(\Sigma)$) is the number of eigenvalues less than 1 for $\mathcal{DL}^{\max}(\Sigma)$ (or $\mathcal{DL}^{\min}(\Sigma)$).*

*Proof.* Two matrices $A$ and $B$ are said to be congruent if there exists an invertible matrix $P$ such that $P^T AP = B$. Sylvester's law of inertia states that any two real symmetric matrices that are congruent have the same number of positive, negative, and zero eigenvalues.

Now, $D^{\max}(\Sigma)$ is congruent to $Tr(G)^{-1/2}D^{\max}(\Sigma)Tr(G)^{-1/2}$ and hence they have same number of positive, negative, and zero eigenvalues. Since $\mathcal{DL}^{\max}(\Sigma) = I - Tr(G)^{-1/2}D^{\max}(\Sigma)Tr(G)^{-1/2}$ we get, 0 is an eigenvalue of $D^{\max}(\Sigma)$ if and only if 1 is an eigenvalue of $\mathcal{DL}^{\max}(\Sigma)$. Also, if $\lambda > 0$ is an eigenvalue of $D^{\max}(\Sigma)$ then $1-\lambda$, which is less than 1, is an eigenvalue of $\mathcal{DL}^{\max}(\Sigma)$ and if $\lambda < 0$ is an eigenvalue of $D^{\max}(\Sigma)$ then $1-\lambda$, which is greater than 1, is an eigenvalue of $\mathcal{DL}^{\max}(\Sigma)$.

Similarly, we can prove the result for $\mathcal{DL}^{\min}(\Sigma)$. □

**Theorem 6.** *For an unbalanced connected signed graph $\Sigma$, $\mu_1 < 1 < \mu_n$ and $\mu_1^{'} < 1 < \mu_n^{'}$.*

*Proof.* We prove the theorem for $\mathcal{DL}^{\max}(\Sigma)$ since the same type of arguments holds for $\mathcal{DL}^{\min}(\Sigma)$.

Note that, $\sum_{i=1}^{n} \mu_i = \text{trace}(\mathcal{DL}^{\max}(\Sigma)) = n$.

Since $\mu_1 > 0$ is the smallest eigenvalue, $n\mu_1 \leq \sum_{i=1}^{n} \mu_i = n$ which implies $\mu_1 \leq 1$.

Similarly, since $\mu_n$ is the largest eigenvalue, $\sum_{i=1}^{n} \mu_i \leq n\mu_n$ which implies $\mu_n \geq 1$.

Now, suppose $\mu_1 = 1$. Then all eigenvalues should also be equal to 1 since $\sum_{i=1}^{n} \mu_i = n$.

That is, 1 is the only eigenvalue of $\mathcal{DL}^{\max}(\Sigma)$ with multiplicity $n$. Then by Lemma 4, 0 is the only eigenvalue of $D^{\max}(\Sigma)$ with multipilicity $n$ which implies, $D^{\max}(\Sigma)$ is a nilpotent matrix. But we have $D^{\max}(\Sigma)$ is real symmetric and hence diagonalisable. Thus $D^{\max}(\Sigma)$ is the zero matrix, which is a contradiction. Thus, $\mu_1 < 1$.

Similarly, if $\mu_n = 1$ following the same line of arguments as above we arrive at a contradiction. Thus $1 < \mu_n$. □

The following theorems are immediate from Theorem 1 and Theorem 2.

**Theorem 7.** *Let $\Sigma = (G, \sigma)$ be a connected signed graph of order $n$. Then $\mu_n(\sigma) \leq \mu_n(-)$ and $\mu_n^{'}(\sigma) \leq \mu_n^{'}(-)$ where the equality holds if and only if $(G, \sigma)$ is switching equivalent to $(G, -)$.*

**Theorem 8.** *Let $\Sigma = (G, \sigma)$ be a connected signed graph of order $n$. Then $\mu_i \leq 2$, for all $i \leq n$, with $\mu_n = 2$ if and only if $(G, \sigma)$ is switching equivalent to $(G, -)$. Similarly, $\mu_i^{'} \leq 2$, for all $i \leq n$, with $\mu_n^{'} = 2$ if and only if $(G, \sigma)$ is switching equivalent to $(G, -)$.*

A signed graph $\Sigma$ is $t$-transmission regular if $tr(v) = \sum_{u \in V(G)} d(v, u) = t$ for all $v \in V(G)$. Odd cycles $C_{2k+1}$ are $k(k+1)$ transmission regular and even cycles $C_{2k}$ are $k^2$ transmission regular.

**Theorem 9.** *If the signed graph $\Sigma$ is $t$-transmission regular, then the normalized distance Laplacian eigenvalues of $\mathcal{DL}^{\max}(\Sigma)$(or $\mathcal{DL}^{\min}(\Sigma)$) are $\frac{1}{t}\lambda$, where $\lambda$ is an eigenvalue of $DL^{\max}(\Sigma)$(or $DL^{\min}(\Sigma)$).*

The odd unbalanced cycle $C_n^-$ is compatible and the signed distance Laplacian spectrum of $C_n^-$ is given in [9]. Thus, we get, the normalized distance Laplacian spectrum of $C_n^-$ as an immediate corollary.

**Theorem 10.** *For an odd unbalanced cycle $C_n^-$, where $n = 2k+1$, the spectrum of $\mathcal{L}^{\pm}$ is*

$$\begin{pmatrix} 1 - \frac{1}{k(k+1)}\left(k(-1)^k - \dfrac{1-(-1)^k}{2}\right) & 1 - \frac{1}{k(k+1)}\left(\dfrac{k(-1)^j}{\sin((2j+1)\frac{\pi}{2n})} - \dfrac{\sin^2((2j+1)\frac{k\pi}{2n})}{\sin^2((2j+1)\frac{\pi}{2n})}\right) \\ 1 & 2 \quad (j = 0, 1, 2, \ldots, k-1) \end{pmatrix}.$$

## Conclusion

In this paper, we dealt with the normalized distance Laplacian matrices for a signed graph. We also defined the normalized Laplacian matrix for weighted signed graphs and studied the spectral properties of the same in Section 2. Theorem 7 compares the normalized distance Laplacian spectral radius of signed graphs with that of all-negative signed graphs and Theorem 8 characterize those signed graphs having maximum normalized distance Laplacian spectral radius.

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