Communications in Combinatorics and Optimization
Vol. 8, No. 2 (2023), pp. 411-421
DOI: 10.22049/CCO.2022.27502.1300

CCO
Commun. Comb. Optim.

Research Article

$2S3$ transformation for dyadic fractions in the interval $(0, 1)$

K.G. Sreekumar[1]*, K. Manilal[2]†, John. K. Rajan[2]‡

[1]Department of Mathematics, University of Kerala, Thiruvananthapuram, India
*sreekumar3121@gmail.com

[2]Department of Mathematics, University College, University of Kerala, Thiruvananthapuram, India
†manilalvarkala@gmail.com
‡johnkrajan@yahoo.co.in

Received: 5 January 2022; Accepted: 3 May 2022
Published Online: 7 May 2022

Abstract: The $2S3$ transformation, which was first described for positive integers, has been defined for dyadic rational numbers in the open interval $(0, 1)$ in this study. The set of dyadic rational numbers is a Prüfer 2-group. For the dyadic $2S3$ transformation $T_{ds}(x)$, the restricted multiplicative and additive properties have been established. Graph parameters are used to generate more combinatorial outcomes for these properties. The relationship between the SM dyadic sum graph's automorphism group and the symmetric group has been investigated.

Keywords: SM sum graphs, Bipartite Kneser type-1 graphs, $2S3$ transformation, Dyadic fractions, Dyadic $2S3$ transformation

AMS Subject classification: 05C25, 05C60

1. Introduction

In group theory, Prüfer p-groups were introduced by Henz Prüfer, a German Mathematician of the early 20^{th} century. The Prüfer p-group or the p-quasi cyclic group for a prime number p is the unique p-group in which every element has p different p^{th} roots. These are countable abelian groups and are used in the classification of infinite abelian groups. They form the smallest building blocks of all divisible groups. The graphs considered here are finite and simple, unless explicitly stated. Let $G = (V, E)$ be a graph. We denote by V and E the vertex set and the edge set of G, respectively. A homomorphism from a graph G to a graph H is a mapping f from the vertex set of

* *Corresponding Author*

G to that of H such that the mapping preserves edges, that is, if for any edge (u, v) of G, $(f(u), f(v))$ is an edge of H. If the mapping f is a bijective, then it is called an isomorphism. An automorphism [3] of graph G is an isomorphism with itself.

Modern computers use different number systems like binary number systems and balanced ternary number systems. The "Setun computer" built in the year 1958 by a Soviet university in Russia was based on the balanced ternary number system. The combinatorial structures of these two number systems were studied [9] using graph-theoretical methods. These combinatorial structures were established as two families of SM sum graphs and SM balancing graphs. For a fixed positive integer n, consider the set $\mathscr{P}_n = \{2^m : m \text{ is an integer}, \ 0 \leq m \leq n-1\}$. Let $M_n = \{1, 2, 3, \ldots, 2^n - 1\}$, then $\mathscr{P}_n^c = M_n - \mathscr{P}_n$. Any positive integer $x < 2^n$ and not in $\mathscr{P}_n$ can be expressed as the sum of two or more distinct elements of $\mathscr{P}_n$. If $x = \sum x_i$ with distinct $x_i \in \mathscr{P}_n$, then each x_i is called an additive component of x. The simple graph $SM(\sum_n)$ [9] is defined as a graph with vertex set $\{u_1, u_2, \ldots, u_{2^n-1}\}$ and adjacency of vertices defined by: two distinct vertices u_i and u_j are adjacent if either i is an additive component of j or j is an additive component of i. The combinatorial structure of the binary number system is established distinctively by using the graph $SM(\sum_n)$. Moreover, the low weight polynomial form [2] of integers that was used in elliptical curve cryptography is related to the polynomial form of x. The Hamming weight of a string was defined as the number of 1's in the string of 0 and 1. The number of zeros in the binary representation of $x \in \mathscr{P}_n^c = 1 + \left\lfloor \dfrac{\ln x}{\ln 2} \right\rfloor - \deg_{v_x \in V} v_x$. Also, consider the set $\mathscr{T}_n = \{3^m : m \text{ is an integer}, 0 \leq m \leq n-1\}$ for a fixed integer $n \geq 2$. Any positive integer $y \leq \frac{1}{2}(3^n - 1)$, which is not a power of 3 can be expressed as a linear combination of two or more distinct elements of the set $\mathscr{T}_n$ with coefficients -1, 0 *or* 1. The relation between y and the elements of $\mathscr{T}_n$ is utilized to form a new class of graphs called $n^{th} SM$ balancing graphs denoted as $SM(B_n)$ [9].

The discrete $2S3$ transformation [8] was defined by using these two graphs. It possesses typical multiplicative and additive properties. The SM dyadic sum graph $SMF(\sum_n)$ and the SM dyadic balancing graph $SMF(B_n)$ are defined in the same way as SM sum graphs and SM balancing graphs in this work, with the goal of merging these graphs with some graph operations to cover the whole real number system in future work. The concept of discrete $2S3$ transformation is extended for dyadic rational numbers in $(0, 1)$. In 2011, a study of the Hausdorff dimension of the maximal run-length in dyadic expansion was done by Ruibiao and Zou [12]. Also, a study on Null sets for doubling and dyadic doubling measures was done by Jang-Mei and Wu [11]. A Balanced ternary adder using recharged semi-floating gate devices have been presented by Henning Gundersen and Yngvar Berg in 2006 [4].

2. Basic definitions and results

A dyadic fraction or dyadic rational number in a real number system whose denominator is a power of two when the ratio is in minimum (co-prime) terms. That is, a number of the form $\dfrac{x}{y}$, where x is an integer and y is a natural number of the form 2^n for some positive integer n. Examples are $\frac{1}{2^2}$ or $\frac{3}{2^4}$, but not $\frac{1}{3\times 2^3}$. These dyadic numbers are precisely the numbers possessing a finite binary representation. Their binary expansions are not unique; each has a finite and infinite representation. However, only their finite binary representations are considered here. A study on integer partitions and Bell number was done by Kok et. al [5]. Now let us see the definition of the $2S3$ transformation.

Definition 1 ([8]). Let $\mathscr{P}_n = \{2^m : m \text{ is an integer}, 0 \leq m \leq n-1\}$ for a fixed integer $n \geq 2$. Let $x < 2^n$ be a positive integer. Then $x = \sum_1^n x_i$, with $x_i = 0$ or 2^m, for some integer m, $0 \leq m \leq n-1$ and x_i's are distinct. Each $x_i \neq 0$ and $x_i \in \mathscr{P}_n$ is an additive component of x. Let $\mathbb{N}$ be the set of all natural numbers. We define a transformation $T_s : N' \to \mathbb{N}$ such that

$$T_s(x) = \sum_1^n x_i^*, \tag{1}$$

where $N' = \{1, 2, 3, 4, ..., n\}$ and each x_i^* is obtained by converting x_i from the base 2 to base 3. This transformation is called $2S3$ *transformation*.

Some examples of the $2S3$ transformation are given below.

Example 1. Let $x = 6 = 2^1 + 2^2$. Then $T_s(x) = 3^1 + 3^2 = 12$. Also, when $x = 35 = 2^0 + 2^1 + 2^5$, then $T_s(x) = 3^0 + 3^1 + 3^5 = 247$.

Definition 2. [8] Let $x_i, x_j \neq 0$ and $x_i, x_j \in \mathscr{P}_n$ be additive components. Let $a = \sum_{i=1}^t a_i x_i$ and $b = \sum_{j=1}^r b_j x_j$ be two positive integers with $a_i, b_j \in \{0, 1\}$, for some positive integers t and r. The two integers a and b are called *component independent numbers* if $a_i b_j \neq a_k b_s$ for all $i \neq k$, $j \neq s$. Otherwise, they are called *component dependent numbers*.

Two integers a and b are component independent numbers if there is no term with coefficient $2(a_j b_{k-j})$ in the multiplied form of $a \times b$.

Definition 3. Let $S = \{x \in \mathbb{R} : 0 \leq x < 1\}$. Let $\circ : S \times S \to S$ be the operation defined as $x \circ y = x + y - \lfloor x + y \rfloor$. The operation $\circ$ is the binary operation addition modulo 1.

Let $\mathscr{S}_n = \{1, 2, \cdots, n\}$, for an integer $n > 1$. For any two integers $k \geq 1$ and $n \geq 2k + 1$, the bipartite Kneser graph [6] $H(n, k)$ has all the k-element subsets and all the $(n - k)$-element subsets of $\mathscr{S}_n$ as vertices, and two vertices are adjacent if and only if one of them is a subset of the other. Here we define the bipartite Kneser type-1 graph as follows.

Definition 4. [10] Let $\mathscr{S}_n = \{1, 2, 3, \ldots, n\}$ for a fixed integer $n > 1$. Let $\phi(\mathscr{S}_n)$ be the set of all non-empty subsets of $\mathscr{S}_n$. Let V_1 be the set of 1- element subsets of $\mathscr{S}_n$ and $V_2 = \phi(\mathscr{S}_n) - V_1$. Define a bipartite graph with adjacency of vertices as: a vertex $A \in V_1$ is adjacent to a vertex $B \in V_2$ if and only if $A \subset B$. This graph is called a bipartite Kneser type-1 graph.

3. SM Dyadic sum graph and SM Dyadic balancing graph

We define the SM dyadic sum graph and SM dyadic balancing graph for dyadic fractions and some other rational numbers in the interval $(0, 1)$. The concept can then be applied to all real numbers with a finite binary representation. Also, we consider rational fractions in their simplest form in (0,1), where the numerator can be written as a sum or difference of powers of 3.

Definition 5. If $x \in (0, 1)$ is a dyadic fraction of the form $\dfrac{r}{2^n}$ (when the ratio is in minimal terms) where the numerator is a positive integer which is not a power of 2 and $r < 2^n$, n is a positive integer, then $x = \sum\limits_1^n x_i$, with $x_i = 0$ or $\frac{1}{2^m}$, for some integer m, $1 \leq m \leq n$ and x_i's are distinct. Here we call each $x_i \neq 0$ as a *dyadic additive component* (Dac) of x.

Definition 6. For a fixed integer $n \geq 2$, define a simple graph $SMF(\sum_n)$, called the *SM dyadic sum graph*, with vertex set $\{v_i : \ i = \frac{k}{2^n},\ k = 1, 2, 3, \ldots, 2^n - 1\}$ and adjacency of vertices defined by: two distinct vertices v_i and v_j are adjacent if either i is a dyadic additive component of j, or j is a dyadic additive component of i.

We can see that the graph $SMF(\sum_n)$ is a connected bipartite graph. The graph $SMF(\sum_n)$ is isomorphic to the bipartite Kneser type-1 graph for each n. The chromatic number [7] of this graph is 2. Let $V^n = \{i : \ i = \frac{k}{2^n},\ k = 1, 2, 3, \ldots, 2^n - 1\}$ be the vertex set(on relabelling) of the graph $SMF(\sum_n)$. Then $V^n \bigcup \{0\}$ is a Prüfer 2-group for each $n > 2$ under the operation addition modulo 1. The independence number of the graph $SMF(\sum_n)$ is $2^n - n - 1$ for all $n \geq 2$.

Definition 7. Let $W_n = \{\dfrac{1}{3^m} :\ m \text{ is an integer}, 1 \leq m \leq n\}$ for a fixed integer $n \geq 2$. Let $S = \{-1, 0, 1\}$. Let x be any number in $(0, 1)$, $x \leq \dfrac{1}{2 \times 3^n}(3^n - 1)$, of the form $\dfrac{f}{3^n}$ (when the ratio is in minimal terms) in which numerator f is an integer and is not a power of 3. Then x can be expressed as

$$x = \sum_{j=1}^{n} \beta_j y_j \ , \tag{2}$$

where $\beta_j \in S$ and $y_j \in W_n$, y_j's are distinct. Each y_j such that $\beta_j \neq 0$ is called a dyadic balancing component of x. Let $\Theta_n = \{x : \text{equation (2) holds}\}$ and $\Theta_n^+ = \{x : \text{equation (2) holds and } \beta_j \in \{0, 1\}\}$. If $W_n^0 = \{\dfrac{1}{3^m} :\ m \text{ is an integer}, 0 \leq m \leq n\}$, then equation (2) holds for more numbers in (0,1).

Definition 8. For a fixed integer $n \geq 2$, let $W_n = \{\frac{1}{3^m} : m$ is an integer, $1 \leq m \leq n\}$. Consider the simple digraph $G = (V, E)$, where the vertex set $V = \{v_i : i = \frac{k}{3^n}, k = 1, 2, 3, \ldots, \frac{1}{2}(3^n - 1)\}$ and adjacency of vertices defined by: for two distinct vertices v_x and v_{y_j}, $(v_x, v_{y_j}) \in E$ if equation (2) holds, $\beta_j = -1$, and $(v_{y_j}, v_x) \in E$ if equation (2) holds, $\beta_j = 1$. This digraph G is called the *SM dyadic balancing digraph, $SMFD(B_n)$*. Its underlying undirected graph is called the SM dyadic balancing graph, $SMF(B_n)$.

Definition 9. Let $W_n = \{\frac{1}{3^m} : m$ is an integer, $1 \leq m \leq n\}$ for a fixed integer $n \geq 2$. Consider the simple digraph G=(V,E), where the vertex set $V = \{v_i : i \in \Theta_n^+\} \bigcup W_n$ and adjacency of vertices defined by, for two distinct vertices v_x and v_{y_j}, v_x is adjacent to v_{y_j} if equation (2) holds and $\beta_j = 1$. This directed graph is called the SM dyadic P balancing digraph, $SMFD(B_n^+)$. The underlying undirected graph is called SM dyadic P balancing graph, $SMF(B_n^+)$.

Lemma 1. *If $G = SMF(\sum_n)$, $\mathscr{D}_n = \{\frac{1}{2^m} : m$ is an integer, $1 \leq m \leq n\}$, then*

$$d(v_i, v_j) = \begin{cases} 1 & , \textit{if } i \textit{ is a Dac of } j \textit{ or } j \textit{ is a Dac of } i. \\ 2 & , \textit{if } i, j \in \mathscr{D}_n \textit{ or } i, j \notin \mathscr{D}_n,\ i \textit{ and } j \textit{ have at least one common Dac.} \\ 3 & , \textit{if neither } i \textit{ nor } j \textit{ is a Dac but exactly one of them belongs to } \mathscr{D}_n. \\ 4 & , \textit{if } i, j \notin \mathscr{D}_n,\ i \textit{ and } j \textit{ have no common Dac.} \end{cases}$$

Proof. It is easy to see that the graphs $SMF(\sum_n)$ and $SM(\sum_n)$ are isomorphic for each $n \geq 2$. Since the isomorphism preserves the distances in the graphs, the lemma is proved. □

Theorem 1. *Suppose $G = SMF(\sum_n)$ for all $n \geq 2$, then the diameter of G is n when $n = 2$ or 3 and the diameter of G is 4 when $n \geq 4$.*

Proof. The proof follows from Lemma 1. □

Proposition 1. *Let $G = SMF(\sum_n)$ be an n^{th} SM dyadic sum graph. Then the number of unordered pairs of vertices for which $d(v_i, v_j) = 4$ is given by* $\delta_n = \frac{1}{2}\sum_{r=2}^{n-2}\left[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}\right]$.

Proof. From Lemma 1, we have that $d(v_i, v_j) = 4$ when $i, j \notin \mathscr{D}_n$, i and j have no common dyadic additive components. The number of such unordered pairs is the same as the number of pairs of pairwise disjoint subsets of $\mathscr{D}_n$ excluding the empty set and singleton sets. Hence the result. □

Theorem 2. *The automorphism group of $SMF(\sum_n)$ is isomorphic to the symmetric group S_n for all $n \geq 3$.*

Proof. Let $G = SMF(\sum_n)$, $n \geq 3$. Let $Q = \{i : v_i \in V(G)\}$. The graph G has bipartite partition sets $V_1 = \{v_i : i \in \mathscr{D}_n\}$ and $V_2 = \{v_j : j \in \mathscr{D}_n^c\}$, where $\mathscr{D}_n = \{\frac{1}{2^m} :\ m \text{ is an integer}, 1 \leq m \leq n\}$ and $\mathscr{D}_n^c = Q - \mathscr{D}_n$. All the vertices of V_1 are of the same degree $2^{n-1} - 1$ and the vertices of V_2 are not of the same degree and has a degree sequence $\{2_{(\binom{n}{2})}, 3_{(\binom{n}{3})}, \ldots, n_{(\binom{n}{n})}\}$, for $n > 2$. There are $\binom{n}{2}$ vertices of degree 2 and $\binom{n}{3}$ vertices of degree 3 and so on. Also, each vertex in V_1 has a neighbourhood with degree sequence $\{2_{(\binom{n-1}{1})}, 3_{(\binom{n-1}{2})}, \ldots, n_{(\binom{n-1}{n-1})}\}$. Here $\mathscr{D}_n = \{\frac{1}{2^m} : m \text{ is an integer}, 1 \leq m \leq n\}$ is one of the orbits. If we fix any permutation of the vertices of V_1, this fixes how the vertices from V_2 must be permuted to give an automorphism. So we get as many automorphisms as the number of permutations of elements of V_1 which results in $n!$ automorphisms including the trivial automorphism. On the other hand, no automorphisms can result from swapping the vertex from the first bipartite set and second bipartite set because unless such a swap is done in its entirety, the adjacency will be lost. A swap can be done in entirety only if $|V_1| = |V_2|$ which is not the case here as G is not a complete bipartite graph as well. Therefore we get that the number of automorphisms is $n!$.
But for $n = 2$, Aut(G) is an abelian group.

Claim. The automorphism group of G is non-abelian for all $n > 2$.
Now we have to prove that $Aut(G)$ is a non-abelian group for all $n > 2$. Let α and β be any two non-trivial distinct automorphisms of G as follows: $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 2 & 1 & 4 & \cdots & n \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 3 & 4 & \cdots & n \end{pmatrix}$, then $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 1 & 4 & \cdots & n \end{pmatrix}$ and $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 3 & 1 & 2 & 4 & \cdots & n \end{pmatrix}$. Therefore $Aut(G)$ is non-abelian. So we get that the automorphism group of $SMF(\sum_n)$ is isomorphic to S_n for all $n \geq 3$. Hence the theorem. □

4. Dyadic $2S3$ transformations and their properties

Only positive integers were used to define the $2S3$ transformation. We are now extending the idea of this discrete transformation for all dyadic fractions in the interval (0,1). For the dyadic $2S3$ transformation $Tds(x)$, the restricted multiplicative and additive properties have been determined. To generate more combinatorial outcomes for these features, graph parameters are used. The automorphism group of the SM dyadic sum graph and the symmetric group have previously been explored in the preceding section.

Definition 10. Let $\mathscr{D}_n = \{\frac{1}{2^m} :\ m \text{ is an integer}, 1 \leq m \leq n\}$ for a fixed integer $n \geq 2$. Let N_d be a set of all dyadic rational numbers in the open interval $(0, 1)$. Then for each $x \in N_d$, $x = \sum_1^n x_i$, with $x_i =\ 0$ or $\frac{1}{2^m}$, for some integer m, $1 \leq m \leq n$ and x_i's are distinct. Each $x_i \neq 0$ and $x_i \in \mathscr{D}_n$ is a dyadic additive component of x. We define a transformation $T_{ds} : N_d \to (0, 1)$ such that $T_{ds}(x) = \sum_1^n x_i^*$, where each x_i^* is obtained by changing the base $\frac{1}{2}$ of x_i to base $\frac{1}{3}$. This transformation function is called *dyadic* $2S3$ *transformation* .

The following example describes the dyadic $2S3$ transformation.

Example 2. Let $x = 0.75 = \frac{1}{2} + \frac{1}{2^2}$. Then $T_{ds}(x) = \frac{1}{3} + \frac{1}{3^2} = \frac{4}{9}$.
If $x = \dfrac{7}{8} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}$, then $T_{ds}(x) = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} = \frac{13}{27}$.

Definition 11. Let $x,\ y \in (0,1)$ be two dyadic fractions, then for some positive integers k and r, $x = \sum\limits_{1}^{k} x_i$ and $y = \sum\limits_{1}^{r} x_i$ with $x_i =\ 0$ or $\frac{1}{2^m}$, for some $1 \leq m \leq n$, for a fixed integer $n \geq 2$ and x_i's are distinct. Here we call each $x_i \neq 0$ as a dyadic additive component of x or y accordingly. If the terms in the dyadic additive component expansion of x are different from that of y, then x and y are called *additive distinct dyadic fractions.*

Definition 12. Let $x_i, x_j \neq 0$ and $x_i, x_j \in \mathscr{D}_n$ be the dyadic additive components. Let $\chi_1 = \sum\limits_{i=1}^{t} \alpha_i x_i$ and $\chi_2 = \sum\limits_{j=1}^{r} \beta_j x_j$ be two dyadic fractions in $(0,1)$ with $\alpha_i = 0\, or\, 1$ and $\beta_j = 0\, or\, 1$, for some positive integers t and r. Two numbers χ_1 and χ_2 are called *component independent dyadic rational numbers* if $\alpha_i\beta_j \neq \alpha_k\beta_s$ for all $i \neq k$, $j \neq s$. Otherwise, they are called *component dependent dyadic rational numbers.*

Example 3. Consider two numbers $0.75 = \frac{1}{2} + \frac{1}{2^2}$ and $0.5 = \frac{1}{2}$.
When we multiply component-wise, $0.75 \times 0.5 = (\frac{1}{2} + \frac{1}{2^2}).\frac{1}{2} = \frac{1}{2^2} + \frac{1}{2^3}$. Since there is no term of the form $2(\alpha_j\beta_{k-j})$, the dyadic fractions 0.75 and 0.5 are component independent dyadic rational numbers.
Now consider another two numbers 0.75 and 0.375. Then $0.75 = \frac{1}{2} + \frac{1}{2^2}$ and $0.375 = \frac{1}{2^2} + \frac{1}{2^3}$. On multiplying component-wise, we get, $0.75 \times 0.375 = (\frac{1}{2} + \frac{1}{2^2}).(\frac{1}{2^2} + \frac{1}{2^3}) = \frac{1}{2^3} + 2.\frac{1}{2^4} + \frac{1}{32}$. Since there exists a term (second term) of the form $2(\alpha_j\beta_{k-j})$, the dyadic numbers 0.75 and 0.375 are component dependent dyadic rational numbers.

Dyadic fractions have two different binary expansions. It would be more precise to say that these dyadic fractions have binary representations which are eventually constant [1]. The set of all dyadic fractions is dense in the real line. Now let us discuss some properties of the dyadic $2S3$ transformation. We establish that T_{ds} has a restricted multiplicative property and additive property. The following theorem gives some of the properties of the dyadic $2S3$ transformation.

Theorem 3. *Let $T_{ds}(x)$ be a dyadic $2S3$ transformation. Then the following holds.*

(i) *T_{ds} is a strictly increasing function.*

(ii) *If $T_s(x) = a$, $x < 2^n$, then $T_{ds}(\frac{x}{2^n}) = \frac{a}{3^n}$, for all positive integers n.*

(iii) *(Product rule). If x and y are component independent dyadic rational numbers, then $T_{ds}(xy) = T_{ds}(x).T_{ds}(y)$.*

(iv) $T_{ds}(x+y) = T_{ds}(x) + T_{ds}(y)$ *only when x and y are additive distinct dyadic fractions.*

Proof. Let us consider the set $\mathscr{D}_n = \{\frac{1}{2^m} : \ m \text{ is an integer}, 1 \leq m \leq n\}$ for a fixed positive integer $n \geq 2$. Let N_d be set of all dyadic rational numbers in the open interval $(0,1)$. Then for each $x \in N_d$, $x = \sum\limits_1^n x_i$, with $x_i = 0$ or $\frac{1}{2^m}$, for some integer m, $1 \leq m \leq n$ and x_i's are distinct. Each $x_i \neq 0$ and $x_i \in \mathscr{D}_n$ is a dyadic additive component of x.
We have $T_{ds}(x) = \sum\limits_1^n x_i^*$, where each x_i^* is obtained by changing the base $\frac{1}{2}$ of x_i to base $\frac{1}{3}$. Suppose, $x > y$ and $x, y \in N_d$. By definition, we get $T_{ds}(x) > T_{ds}(y)$ for all $x, y \in N_d$. This gives that $T_{ds}(x)$ is a strictly increasing function. Therefore T_{ds} is an injective function too. By definitions of T_{ds} and T_s , result (ii) holds for all positive integers n. For proving (iii), we take two dyadic fractions in (0,1), $x = \sum\limits_{k=0}^{r} \alpha_k \frac{1}{2^k}$ and $y = \sum\limits_{k=0}^{m} \beta_k \frac{1}{2^k}$, where $\alpha_k = 0 \,\text{or}\, 1$ and $\beta_k = 0 \,\text{or}\, 1$. Therefore, we get the product as $x \times y = \sum\limits_{k=0}^{m+r} \sum\limits_{j=0}^{k} (\alpha_j \beta_{k-j}) \frac{1}{2^k}$. Then there are two cases to consider.

Case 1. When x and y are component independent dyadic rational numbers in (0,1), here we get,

$$x \times y = \sum_{k=0}^{r} \alpha_k \frac{1}{2^k} . \sum_{k=0}^{m} \beta_k \frac{1}{2^k} = \sum_{k=0}^{m+r} \sum_{j=0}^{k} (\alpha_j \beta_{k-j}) \frac{1}{2^k}. \tag{3}$$

Since x and y are component independent dyadic rational numbers, there will not be any term with coefficient $2(\alpha_j \beta_{k-j})$ in the multiplied form of the equation (3). So we can apply the dyadic $2S3$ transformation. Using the definition of $T_{ds}(x)$, we get

$$T_{ds}(xy) = \sum_{k=0}^{m+r} \sum_{j=0}^{k} (\alpha_j \beta_{k-j}) \frac{1}{3^k}. \tag{4}$$

Again

$$T_{ds}(x).T_{ds}(y) = \sum_{k=0}^{r} \alpha_k \frac{1}{3^k} . \sum_{k=0}^{m} \beta_k \frac{1}{3^k} = \sum_{k=0}^{m+r} \sum_{j=0}^{k} (\alpha_j \beta_{k-j}) \frac{1}{3^k}. \tag{5}$$

From equations (4) and (5), we get $T_{ds}(x.y) = T_{ds}(x).T_{ds}(y)$ for all integers x and y that are component independent dyadic rational numbers.

Case 2. When x and y are component dependent dyadic rational numbers, then there will be terms with coefficient $2(\alpha_j \beta_{k-j})$ in the multiplied form of the equation (3). So we can not apply the dyadic $2S3$ transformation as the dyadic additive components are not distinct.
When x and y are additive distinct dyadic fractions, the terms in the expansion of x are different from that of y. Then the result (iv) holds. This completes the proof. $\square$

Proposition 2. *For all dyadic rational x in (0,1),* $0 < T_{ds}(x) < \frac{1}{2}$.

Proof. For a fixed positive integer n, the maximum possible value of the function T_{ds} is $\dfrac{1}{3^n} + \dfrac{3}{3^n} + \dfrac{9}{3^n} + \cdots + \dfrac{3^{n-1}}{3^n}$. But we have $2(1+3+3^2+ \ \ldots \ +3^{n-1}) < 3^n$ and hence the result. □

Theorem 4. *Let x and y be two dyadic rational numbers in (0,1). Let $T^z = \{(x,y) : x \neq y, T_{ds}(x+y) = T_{ds}(x) + T_{ds}(y)\}$ and $N(T^z) = |T^z|$. Then $N(T^z) = \dfrac{n2^n - n^2 - n}{2} + \delta_n$, where $\delta_n = \dfrac{1}{2}\sum\limits_{r=2}^{n-2}\left[\binom{n}{r}\sum\limits_{k=2}^{n-2}\binom{n-r}{k}\right]$ and $T_{ds}(x)$ is a dyadic $2S3$ transformation.*

Proof. We have $T_{ds}(x+y) = T_{ds}(x) + T_{ds}(y)$ only when x and y are additive distinct dyadic fractions. This will happen when $d(v_x, v_y)$ is either 3 or 4, or $x, y \in \mathscr{D}_n$ in the case of the graph $SMF(\sum_n)$. The number of unordered pairs (x,y) in which $d(v_x, v_y) = 3$ is $n(2^{n-1} - n)$ and that in which $d(v_x, v_y) = 4$ is δ_n as given in the theorem. Hence proved. □

Theorem 5. *Let x and y be two dyadic rational numbers in (0,1). Let $\varpi(T_{ds}) = \{(x,y) : x \neq y, T_{ds}(xy) \neq T_{ds}(x).T_{ds}(y)\}$. Then $|\varpi(T_{ds})| < \binom{2^n-n-1}{2}$.*

Proof. Let the graph $G = SMF(\sum_n)$ be the SM dyadic sum graph for $n \geq 2$. Let $d_s(v_x, v_y)$ denote the number of unordered pairs of vertices for which $d(v_x, v_y) = s$. Here $SM(\sum_n)$ and $SMF(\sum_n)$ are isomorphic for each n. By Lemma 1, when $d(v_x, v_y) = 1$ or 3, we can see that x and y are component independent dyadic rational numbers. In these cases, $T_{ds}(xy) = T_{ds}(x).T_{ds}(y)$.
When $d(v_x, v_y) = 2$, then two cases arise.

Case 1. When $x, y \in \mathscr{D}_n$. In this situation, it is clear that x and y are component independent dyadic rational numbers. Therefore, $T_{ds}(xy) = T_{ds}(x).T_{ds}(y)$.

Case 2. When $x, y \notin \mathscr{D}_n$, then x and y may or may not be component independent dyadic rational numbers. So $T_{ds}(xy)$ may or may not be equal to $T_{ds}(x).T_{ds}(y)$ depending on the Definition 12. The number of unordered pairs for which $x, y \notin \mathscr{D}_n$, x and y have at least one common dyadic additive component is $\left[\binom{2^n-n-1}{2} - \delta_n\right]$. Also, when $d(v_x, v_y) = 4$, there may be some cases in which x and y are component dependent dyadic rational numbers. Since G is a simple graph, $|\varpi(T_{ds}| < d_2(v_x, v_y) + \delta_n - \dfrac{n(n-1)}{2} = \binom{2^n-n-1}{2}$, where $v_x v_y$ is an edge of G and, x and y are distinct. □

Theorem 6. *For each $n \geq 2$, $SMF(\sum_n)$ and $SMF(B_n^+)$ are isomorphic.*

Proof. Let $G_1 = SMF(\sum_n)$ and $G_2 = SMF(B_n^+)$. We make use of the dyadic $2S3$ transformation to show that there exists an isomorphism between the graphs G_1 and G_2.

Let $T_{ds} : N_d \to (0,1)$ such that $T_{ds}(x) = \sum_1^n x_i^*$, where N_d is the set of all dyadic rational numbers in the open interval $(0,1)$ and each x_i^* is obtained by changing the base $\frac{1}{2}$ of x_i to base $\frac{1}{3}$. Let S be the range of T_{ds}. We consider the edge between $y = T_{ds}(x)$ and each x_i^*. Let set of these edges be E' and the graph induced by the edge set E' be G_3 which is then congruent to G_2. Since T_{ds} is one to one, $|V(G_1)| = |V(G_3)|$. Also, the degrees of corresponding vertices are the same. We can clearly observe that T_{ds} preserves the adjacency. Therefore $G_1 \cong G_2$. This completes the proof. □

There exists a homomorphism between $SMF(\sum_n)$ and $SMF(B_n)$.

This homomorphism maps the vertex set of $SMF(\sum_n)$ which is the set of dyadic fractions to the set of vertices of $SMF(B_n)$. In fact, there is an injective correspondence between the vertex set V of $SMF(\sum_n)$ and the set of dyadic fractions in $(0,1)$. By considering the addition and subtraction operations of elements of V, the structure of the additive abelian group is conceived. In view of the Pontryagin duality (the one which explains the general properties of the Fourier transform on locally compact abelian groups), the dual group of the additive dyadic fractions can also be taken as topological group called the dyadic solenoid.

5. Conclusion

In this paper, we extended the $2S3$ transformation for the dyadic rational numbers in (0,1). Two new graphs $SMF(\sum_n)$ and $SMF(B_n)$ were introduced which are similar to the SM sum graphs and SM balancing graphs. And also an isomorphism between $SMF(\sum_n)$ and $SMF(B_n^+)$ was established. The properties of dyadic $2S3$ transformation were examined and found that some properties are similar to the $2S3$ transformation which was defined for all positive integers. The newly introduced transformation T_{ds} was explored through a graph-theoretical way. It has been observed that $0 < T_{ds}(x) < \frac{1}{2}$. Some of the combinatorics and relationship between the parameters of these two graphs are analyzed through this dyadic $2S3$ transformation.

Conflict of Interest

The authors hereby declare that there is no potential conflict of interest.

Acknowledgement

The authors would like to thank the anonymous referee for their helpful comments which greatly improved the paper. This research has been promoted/supported by the University of Kerala, India.

References

[1] J. Białas and Y. Nakamura, *Dyadic numbers and T4 topological spaces*, Formalized Mathematics **5** (1996), no. 3, 361–366.

[2] J. Chung and M.A. Hasan, *Low-weight polynomial form integers for efficient modular multiplication*, IEEE Trans. Comput. **56** (2006), no. 1, 44–57.

[3] A. Genevois and O. Varghese, *Conjugating automorphisms of graph products: Kazhdan's property (T) and SQ-universality*, Bull. Aust. Math. Soc. **101** (2020), no. 2, 272–282.

[4] H. Gundersen and Y. Berg, *A novel balanced ternary adder using recharged semi-floating gate devices*, 36th International Symposium on Multiple-Valued Logic (ISMVL'06), IEEE, 2006, pp. 18–18.

[5] J. Kok and J.V. Kureethara, *Stirling number of the fourth kind and Lucky partitions of a finite set*, Commun. Comb. Optim. **6** (2021), no. 2, 211–219.

[6] S.M. Mirafzal, *The automorphism group of the bipartite Kneser graph*, Proceedings-Mathematical Sciences **129** (2019), no. 3, 1–8.

[7] S. Naduvath and M.T. Ellumkalayil, *A note on δ ˆ(k)-colouring of the Cartesian product of some graphs*, Commun. Comb. Optim. **7** (2022), no. 1, 113–120.

[8] K.G. Sreekumar, *Two-S-Three transformation functions and its properties*, Int. J. Math. Archive **9** (2018), no. 4, 83–88.

[9] K.G. Sreekumar and K. Manilal, *Hosoya polynomial and Harary index of SM family of graphs*, J. Inf. Optim. Sci. **39** (2018), no. 2, 581–590.

[10] ______, *Automorphism groups of some families of bipartite graphs*, Electron. J. Graph Theory Appl. **9** (2021), no. 1, 65–75.

[11] J.-M. Wu, *Null sets for doubling and dyadic doubling measures*, Ann. Acad. Sci. Fenn. Ser. AI Math **18** (1993), no. 1, 77–91.

[12] R. Zou, *Hausdorff dimension of the maximal run-length in dyadic expansion*, Czechoslovak Math. J. **61** (2011), no. 4, 881–888.