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Research Article

A study on graph topology

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Abstract: The concept of topology defined on a set can be extended to the field of graph theory by defining the notion of graph topologies on graphs where we consider a collection of subgraphs of a graph G in such a way that this collection satisfies the three conditions stated similarly to that of the three axioms of point-set topology. This paper discusses an introduction and basic concepts to the graph topology. A subgraph of G is said to be open if it is in the graph topology $\mathcal{T}_G$. The paper also introduces the concept of the closed graph and the closure of graph topology in graph topological space using the ideas of decomposition-complement and neighborhood-complement.

Keywords: Graph topology, graph topological space, $\mathcal{T}$-interior, $\mathcal{T}$-neighbourhood, d-closure, n-closure

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1. Introduction

Topology is the area of mathematics that deals with space and deformation. Being a prolific research area in mathematics with wide range of applications, intense research has been done in topology. For the terminology in topology, we refer [6, 9, 10].
A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ a family of subset of X satisfying the three following properties:

(i) $\emptyset, X \in \mathcal{T}$;

(ii) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$;

(iii) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

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The family $\mathcal{T}$ is called a *topology* on the set X. Members of $\mathcal{T}$ are called *open* in X or open subset of X.

Defining topologies on well-known discrete structures such as graphs have become challenging problems for many researchers came across. The first known attempt in this direction can be found in [1]. In [7], authors proposed that the collection of all subgraphs of a graph G forms a topology (in the usual sense) of a graph under the operations graph union and graph intersection. Also, the notion of topology on a graph consisting of its spanning subgraphs has been introduced and its properties have been studied in [8].

Motivated by the above mentioned studies, in this paper, we discuss some graph topologies of graphs and their characteristics. For terms and definitions in graph theory we refer [2, 3, 5, 12]. We also investigate the idea of closed graphs and the closure of a subgraph in a graph topological space.

2. Graph Topology of a Graph

Throughout this paper, by a *null graph*, we mean a graph whose vertex set and edge set are empty (see [12]). We denote the null graph by K_0. Generally, the null graph K_0 is considered to be a subgraph of all graphs. Also, note that an empty graph is an edge-less graph. Now, the term graph topology of a given graph is defined as given below:

Definition 1 (Graph topology). Let $G = (V, E)$ be a graph. A *graph topology*, $\mathcal{T}_G$, on G is a collection of subgraphs of G which satisfies the following three properties:

(i) $K_0, G \in \mathcal{T}_G$;

(ii) Any union of members of $\mathcal{T}_G$ is in $\mathcal{T}_G$;

(ii) Finite intersection of members of $\mathcal{T}_G$ is in $\mathcal{T}_G$

We call the pair $(G, \mathcal{T}_G)$ a *graph topological space*. If the context is clear, we can write $\mathcal{T}$ instead of $\mathcal{T}_G$. The conditions that define graph topology are called axioms of graph topology. A subgraph H of a graph G is said to be *open* in G if $H \in \mathcal{T}$.

Analogous to the terminology in topology, we have the following definitions.

(i) For a non-empty graph G, the collection $\mathcal{T} = \{K_0, G\}$ is a graph topology on G, which is called the *indiscrete graph topology* on G.

(ii) For a non-empty graph G, the collection of all subgraphs of G,denoted by $\mathcal{P}$ is also a graph topology on G called the *discrete graph topology* on G.

(ii) A graph topology $\mathcal{T}$ on a non-empty graph G is said to be a *co-finite graph topology* on G if $\mathcal{T} = \{H : G - H \text{ is finite}\}$.

(iv) A graph topology $\mathcal{T}_1$ is *coarser* or *weaker* than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_2$ is finer or stronger than $\mathcal{T}_1$ if $\mathcal{T}_2 \supset \mathcal{T}_1$.

Note that the indiscrete graph topology is the smallest graph topology on G whereas the discrete graph topology is the largest graph topology on G.

Theorem 1. *Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots$ be graph topologies defined on a non-empty graph G. Then $\mathcal{T} = \bigcap\limits_{i \in I} \mathcal{T}_i$ is a graph topology on G and is weaker than each $\mathcal{T}_i$. Consider a graph topology $\mathcal{T}^{'}$ which is weaker than each $\mathcal{T}_i$ then $\mathcal{T}$ is stronger than $\mathcal{T}^{'}$.*

Proof. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots$ be graph topologies on G and let $\mathcal{T} = \bigcap\limits_{i \in I} \mathcal{T}_i$. We need to prove that $\mathcal{T}$ is a graph topology on G. Since $\mathcal{T}$ is the intersection of topologies the empty set and the full graph is in $\mathcal{T}$ as it belongs to each $\mathcal{T}_i$. Now we need to prove that $\mathcal{T}$ is closed under finite intersection. For this consider the subgraphs $H_1, H_2, \dots, H_n$ be in $\mathcal{T}$ and let $H = \bigcap\limits_{j \in_n I} H_j$. We need to show that $H \in \mathcal{T}$. Since $\mathcal{T} = \bigcap\limits_{i \in I} \mathcal{T}_i$, each subgraph H_j belongs to each $\mathcal{T}_i$. Since $\mathcal{T}_i$ is a graph topology, it is closed under finite intersection. So, $H \in \mathcal{T}_i$ for each $i \in I$. Then $H \in \mathcal{T}$ as $\mathcal{T} = \bigcap\limits_{i \in I} \mathcal{T}_i$. Hence, $\mathcal{T}$ is closed under finite intersection. The property that $\mathcal{T}$ is closed under arbitrary union can be established in a similar way.
Since $\mathcal{T} \subset \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots$, it is weaker than each $\mathcal{T}_i$. Also, since $\mathcal{T}^{'}$ is weaker than each $\mathcal{T}_i$, $\mathcal{T}^{'} \subset \mathcal{T}$, that is, $\mathcal{T}$ is stronger than $\mathcal{T}^{'}$. □

Let $(G, \mathcal{T})$ be a graph topological space. A subfamily $\mathcal{G}$ of $\mathcal{T}$ is said to be a *base* or a *basis* of $\mathcal{T}$ if every member of $\mathcal{T}$ can be expressed as a union of some members of $\mathcal{G}$. The subcollection $\mathcal{G}$ is a base of a graph topology $\mathcal{T}$ if $\mathcal{G}$ generates the graphs in $\mathcal{T}$ with the union of members of $\mathcal{G}$. It is the smallest collection of open subgraph that generates $\mathcal{T}$ with their union. The base containing all the single vertex subgraphs of a graph generates a graph topology containing spanning empty subgraphs. The following theorem characterises the base of a graph topology.

Theorem 2. *Let $(G, \mathcal{T})$ be a graph topological space and let $\mathcal{G} \subset \mathcal{T}$. Then, $\mathcal{G}$ is a base for the topological space if and only if,*

(i) *for each $v \in V(G)$, there exists $G_i \in \mathcal{G}$ such that $v \in V(G_i) \subseteq V(G)$.*

(ii) *for each $e \in E(G)$, there exist $G_j \in \mathcal{G}$ such that $e \in E(G_j) \subseteq E(G)$.*

Proof. Given a topological space $(G, \mathcal{T})$ and let $\mathcal{G}$ be a subcollection of $\mathcal{T}$. Suppose that $\mathcal{G}$ is a base for the topological space. Let $v \in V(G)$ and let H be an open subgraph of G containing v. We need to prove that there exist a $G_i \in \mathcal{G}$ containing the vertex v. Since H is an open subgraph of G, by the definition of base of graph topology, H can be expressed as the union of members of $\mathcal{G}$, the base for the graph topology $\mathcal{T}$. That is,

$$H = \bigcup_{i \in I} G_i$$

where each $G_i \in \mathcal{G}$. Since v is a vertex of H, it is clear that $v \in V(G_i) \subseteq V(H) \subseteq V(G)$ for some i. The edge analogue of this condition can also be proved in the similar manner.

Conversely, suppose that the two conditions are satisfied for every vertices and edges of graph G. We need to prove that the collection of subgraphs $\mathcal{G}$ is a base for the graph topology $\mathcal{T}$. That is, we need to prove that any open subgraph of G can be expressed as the union of elements of $\mathcal{G}$. Consider a subgraph H of graph G which is open. By Condition (i) of the statement, for each vertex $v \in V(H)$, there exist a G_i in the collection $\mathcal{G}$ such that $v \in V(G_i) \subseteq V(G)$. In a similar way, for each $e \in E(H)$, there exist $G_j \in \mathcal{G}$ such that $e \in E(G_j) \subseteq E(G)$. Therefore, we have

$$H = \bigcup_{v \in V(G_i)} G_i.$$

That is, H can be expressed as the union of members of $\mathcal{G}$. Hence, $\mathcal{G}$ is a base for the graph topological space $(G, \mathcal{T})$. □

Theorem 3. *Let $(G, \mathcal{T})$ be a graph topological space and let $H_1, H_2 \in \mathcal{G}$. Then,*

(i) *for each $v \in V(H_1 \cap H_2)$, there exists $H_3 \in \mathcal{G}$ such that $v \in V(H_3) \subseteq V(H_1 \cap H_2)$.*

(ii) *for each $e \in E(H_1 \cap H_2)$, there exists $H_3 \in \mathcal{G}$ such that $e \in E(H_3) \subseteq E(H_1 \cap H_2)$.*

Proof. Let $(G, \mathcal{T})$ be a graph topological space and let $H_1, H_2 \in \mathcal{G}$. Since each basis element is itself an element of $\mathcal{T}$ and since $\mathcal{T}$ is closed under intersection, $H_1 \cap H_2 \in \mathcal{T}$. Then for each $v \in V(H_1 \cap H_2)$, by Theorem 2 the condition holds. Similarly, the conditions for edges can be proved. □

A collection $\mathcal{K}$ of subgraphs of a graph G is called a *subgraph cover* of the graph G whenever, $\bigcup_{H \in \mathcal{K}} E(H) = E(G)$ (see [11]). In the current discussion, by the term cover, we mean the concept of cover in graph theory.

Theorem 4. *The subgraph cover of any graph is a base.*

Proof. The subgraph cover of a graph G contains the subgraphs of G such that the union of the edge sets of those subgraphs is the edge set of G. Hence, we can construct a graph topology by taking union of the subgraphs from the subgraph cover. □

Let G be a graph and $(G, \mathcal{T})$ be a graph topological space. The $\mathcal{T}$-*neighbourhood* of a subgraph H of G in $(G, \mathcal{T})$ is a subgraph K of G such that there exist a graph $H_i \in \mathcal{T}$ such that $H \subset H_i \subset K$.
We can say that any subgraph K of G containing the open supergraphs of H is a $\mathcal{T}$-neighbourhood of H. Following definitions depict the neighbourhood of vertex and edges of a graph.

Let G be a graph and $(G, \mathcal{T})$ be a graph topological space. Then,

(i) the $\mathcal{T}$-*neighbourhood* of a vertex v in $(G, \mathcal{T})$ is a subgraph H of G such that there exist a graph $G_i \in \mathcal{T}$ such that $v \in V(G_i) \subset V(H)$.

(ii) the $\mathcal{T}$-*neighbourhood* of an edge e in $(G, \mathcal{T})$ is a subgraph H of G such that there exist a graph $G_i \in \mathcal{T}$ such that $e \in E(G_i) \subset E(H)$.

A vertex v of a graph G is called a $\mathcal{T}$-*isolated vertex* of the graph topological space $(G, \mathcal{T})$ if the induced subgraph $G[v] \in \mathcal{T}$. Similarly, an edge e of a graph G is said to be a $\mathcal{T}$-*isolated edge* of the graph topological space $(G, \mathcal{T})$ if the induced subgraph $G[e] \in \mathcal{T}$.

Theorem 5. *Let H be a subgraph of G. Then, H is said to be*

(i) *a $\mathcal{T}$-neighbourhood of $v \in V(H)$ if and only if v is a $\mathcal{T}$-isolated vertex of G.*

(ii) *a $\mathcal{T}$-neighbourhood of $e \in E(H)$ if and only if e is a $\mathcal{T}$-isolated edge of G.*

Proof. Assume that every subgraph H of a graph G containing v is a neighbourhood of v. Then, there exist open subgraph H_i such that $v \in H_i \subset H$. By the third axiom of graph topology, we have $\bigcap_{i \in I} H_i = G[v] \in \mathcal{T}$. That is, v is $\mathcal{T}$-isolated vertex.
Conversely, assume that the vertex v is $\mathcal{T}$-isolated vertex. Then, $G[v]$ is an open subgraph for every subgraph of G containing the vertex v. Therefore, every subgraph H of G containing v is a $\mathcal{T}$-neighbourhood of v.
In a similar way, we can prove the conditions required for edges also. This completes the proof. □

As a consequence of the above result, we can see that if $\mathcal{T}$ contains single vertex subgraph of G, then the collection of all subgraphs of G containing vertex v forms a *neighbourhood system* for that vertex. Similarly, if $\mathcal{T}$ contains a subgraph of G induced by a single edge, then the collection of all subgraphs of G containing that edge forms a $\mathcal{T}$-*neighbourhood system* for that edge.
We can extend the concept of the $\mathcal{T}$-neighbourhood of vertices and edges of a graph to the notion of the $\mathcal{T}$-neighbourhood of a subgraph of G as follows: Let H be a subgraph H of G, which need not necessarily be open in G. Then, a subgraph G_i of G is said to be a $\mathcal{T}$-neighbourhood of H if there exists an open subgraph H_i of G such that $H \subseteq H_i \subseteq G_i$. Then, as immediate consequence of Theorem 5, we have the following result.

Proposition 1. *Every subgraph H_i of a graph G containing a subgraph H of G is a $\mathcal{T}$-neighbourhood of H if and only if $H \in \mathcal{T}$.*

For a graph G, a vertex v of a subgraph H of G is said to be a $\mathcal{T}$-*interior vertex* or a $\mathcal{T}$-*interior point* of H if there exist an open subgraph H_i such that $v \in H_i \subseteq H$. Note that any vertex of a graph G is a $\mathcal{T}$-interior point if we can find an open subgraph of G containing that vertex. In a similar way, an edge e of H is said to be a $\mathcal{T}$-*interior edge* of H if there exist an open subgraph H_i such that $e \in H_i \subseteq H$. The $\mathcal{T}$-*interior* of a graph $H \subset G$ is the union of all subgraphs of H that are in $\mathcal{T}$. The vertices and edges of the $\mathcal{T}$-interior of the graph H is called $\mathcal{T}$-interior vertices and $\mathcal{T}$-interior edges of H.

A subgraph H of a graph G is said to be a $\mathcal{T}$-*interior subgraph* of G if

(i) every vertex of H is a $\mathcal{T}$-interior vertex of H, and

(ii) every edge of H is a $\mathcal{T}$-interior edge of H.

The following theorem establishes a necessary and sufficient condition for a subgraph H of G to be a $\mathcal{T}$-interior subgraph of G.

Theorem 6. *Let $(G, \mathcal{T})$ be a graph topological space and H be a subgraph of G. Then, H is a $\mathcal{T}$-interior subgraph of G if and only if $H = \bigcup_i H_i$, where each H_i is in $\mathcal{T}$.*

Proof. Consider a topological space $(G, \mathcal{T})$. Let H be a $\mathcal{T}$-interior subgraph of G. Then, we have every element (vertices and edges) are $\mathcal{T}$-interior elements of H. Since every vertex is a $\mathcal{T}$-interior vertex of H, for each vertex $v_i \in V(H)$, there exist an open subgraph H_i of G such that $v_i \in V(H_i) \subseteq V(H)$. Therefore, $H = \bigcup\limits_{v_i \in H} H_i$.
Conversely, suppose H is a union of open proper subgraphs of G. That is, $H = \bigcup H_i$, where each H_i is an open subgraph of G. Therefore, each vertex v of H should be an element of some H_i. That is, $v \in V(H_i) \subseteq V(H)$ for each $v \in V(H)$. Thus, every vertex v of H is a $\mathcal{T}$-interior vertex of H.
In a similar way, we can establish the condition for edges of H also. This completes the proof. □

The smallest topology of a non-empty set X, which is neither trivial nor discrete, is the Sierpinski topology of X. Invoking the notion of the Sierpinski topology of a set X, we define two types of topologies on a general graph G as follows:

$$\begin{aligned} \mathcal{T}_v &= \{G, K_0, G[v]\} \\ \mathcal{T}_e &= \{G, K_0, G[e]\} \end{aligned}$$

where v and e respectively are an arbitrary vertex and an arbitrary edge of G. The above-mentioned topologies are the smallest graph topologies which are neither trivial nor discrete.

Theorem 7. *Let G be a graph and let $\mathcal{T}_v = \{G, K_0, G[v]\}$ and $\mathcal{T}_e = \{G, K_0, G[e]\}$ be two graph topologies on G. Then, their union is a graph topology on G if and only if v is an end vertex of the edge e in G.*

Proof. Let v be an end vertex of the edge e. In this case, $\mathcal{T}_v \cup \mathcal{T}_e = \{G, K_0, G[v], G[e]\}$. Note that $G[v] \cup G[e] = G[e]$ and $G[v] \cap G[e] = G[v]$. Hence, the union of $\mathcal{T}_v$ and $\mathcal{T}_e$ are graph topology of G.
If possible, let v be not an end vertex of the edge e. Then, $\mathcal{T}_v \cup \mathcal{T}_e = \{G, K_0, G[v], G[e]\}$. Note that $G[v] \cup G[e] = G[v, e]$, where $G[v, e]$ is the disjoint union of $G[v]$ and $G[e]$. Since $G[v, e]$ is not element of $\mathcal{T}_v \cup \mathcal{T}_e$, it is not a graph topology on G. Hence the theorem is proved. □

Note that $\mathcal{T}_v \cap \mathcal{T}_e = \{G, K_0\}$, which is the indiscrete graph topology of G.

3. Closed Graphs in Graph Topological Space

In order to define the closed subgraph in the graph G or in graph topological space $(G, \mathcal{T}_G)$, we define two types of complements - decomposition-complement and the neighbourhood-complement of graphs.

3.1. Decomposition-Closed Subgraphs

The notion of decomposition-complement of a graph G is defined as below:

Definition 2. Let G be a graph and let $H = (V_H, E_H)$ be a subgraph of the graph $G = (V, E)$. The complement of the subgraph H with respect to the graph G is the graph $H^* = (V^*, E^*)$ where $E^* = E - E_H$ and the vertex set of H^* is the set of all vertices incident to the edges in $E(H^*)$ is called the *decomposition-complement* of H.

In other words, the decomposition-complement H^* of the the subgraph H of G is the subgraph of G induced by the edge set $E - E_H$. That is, $H^* = \langle E - E_H \rangle$.

Now, we define the notion of a closed subgraph in a graph topological space with regard to the decomposition-complement of a subgraph of G.

Definition 3. A subgraph H in a topological space is *decomposition closed* or *d-closed* if its decomposition-complement H^* is open in the topological space. A subgraph which is both open and d-closed are called *d-clopen subgraph*. The null graph K_0 and graph G are both open and d-closed in the subgraph topological space.

Proposition 2. *The null graph K_0 and the graph G in a graph topological space is decomposition-closed.*

Proof. Let $(G, \mathcal{T}_G)$ be a graph topological space. We need to show that the null graph K_0 and the graph G are decomposition-closed. That is, we need to show that the decomposition-complement of K_0 and G are open.
Considering the null graph K_0 we have,

$$\begin{aligned} E(K_0^*) &= E(G) - E(K_0) \\ &= E(G) \end{aligned}$$

The subgraph induced by the $E(G)$ is the graph G which is open in $\mathcal{T}$. Hence, K_0 is decomposition-closed. Now, consider the graph G. We have,

$$\begin{aligned} E(G^*) &= E(G) - E(G) \\ &= \emptyset \end{aligned}$$

By Definition 2, the vertex set of G^* is the set of all vertices incident to the edges in $E(G^*)$ which is empty. Now, the subgraph obtained by the empty vertex set and edge set is the graph K_0 which is open. Hence the graph G is decomposition-closed. □

The following result discusses decomposition-closed subgraphs of empty graphs.

Proposition 3. *The set of all empty subgraphs $N_i; 1 \leq i \leq n$ of graph G which are not open in $\mathcal{T}_G$ are always decomposition-closed.*

Proof. Let $(G, \mathcal{T}_G)$ be a topological space. Let N_i be a subgraph of G. Then,

$$\begin{aligned} E({N_i}^*) &= E(G) - E(N_i) \\ &= E(G) \end{aligned}$$

Since the decomposition-complement of the set of all isolated vertices and empty subgraphs N_i of graph G are always open, the subgraphs of G, which are not open, are always decomposition-closed. □

It is also noted that every member in discrete and indiscrete graph topology, is both open and decomposition-closed. The following theorem characterises the decomposition-closed subgraph of a subgraph topology.

Theorem 8. *Let $(G, \mathcal{T}_G)$ be a subgraph topological space and $\mathcal{C}$ be the collection of all decomposition-closed graphs in the subgraph topological space $\mathcal{T}_G$. Then,*

(i) $K_0, G \in \mathcal{C}$

(ii) *Any union of d-closed subgraphs is d-closed in $\mathcal{C}$.*

(iii) *Arbitrary intersection of d-closed subgraphs is d-closed in $\mathcal{C}$.*

Conversely, for any graph G, there exists a subgraph topology on G which coincides with the family of all decomposition-closed subgraphs satisfying the above properties.

Proof. The proof for the first one is trivial as the decomposition-complement of the null graph and graph G are open.
Let $(G, \mathcal{T}_G)$ be a graph topological space and let

$$H_1 = (V_1, E_1), H_2 = (V_2, E_2), \dots, H_n = (V_n, E_n)$$

be d-closed subgraphs in the topological space. By Definition 3, we have the decomposition-complement of each of these graphs are open. That is,

$$H_1^* = (V_1^*, E_1^*), H_2^* = (V_2^*, E_2^*), \dots, H_n^* = (V_n^*, E_n^*)$$

are open in the topological space where $E_i^* = E - E_i$.
From the second axiom of graph topology we have, $\bigcup_{i\in I} H_i^*$ is open. That is,

$$\begin{aligned}\bigcup_{i\in I} H_i^* &= \bigcup_{i\in I}(V_i^*, E_i^*)\\ &= \bigcup_{i\in I}(V_i^*, E - E_i)\\ &= (V^{'}, \bigcup_{i\in I}(E - E_i))\\ &= (V^{'}, E - \bigcup_{i\in I} E_i)\end{aligned}$$

where $V^{'}$ is the set of vertices which are the end vertices of the edges in $E - \bigcup_{i\in I} E_i$. This implies that the graph obtained by the union of edges of H_i is d-closed. That is, arbitrary union of d-closed graphs is d-closed. Similarly, we can show that intersection of d-closed graphs is d-closed. Consider the d-closed graphs $H_1 = (V_1, E_1), H_2 = (V_2, E_2), \dots, H_n = (V_n, E_n)$ with the decomposition-complements $H_1^* = (V_1^*, E_1^*), H_2^* = (V_2^*, E_2^*), \dots, H_n^* = (V_n^*, E_n^*)$. The definition of graph topology we have $\bigcap_{i\in I} H_i^*$ is open. That is,

$$\begin{aligned}\bigcap_{i\in I} H_i^* &= \bigcap_{i\in I}(V_i^*, E_i^*)\\ &= \bigcap_{i\in I}(V_i^*, E - E_i)\\ &= (V^{''}, \bigcap_{i\in I}(E - E_i))\\ &= (V^{''}, E - \bigcap_{i\in I} E_i)\end{aligned}$$

where $V^{''}$ is the set of vertices which are the end vertices of the edges in $E - \bigcap_{i\in I} E_i$. The converse part is trivial as the collection of subgraphs satisfying the three axioms is a graph topology on graph G. □

3.2. Neighbourhood-Closed Graphs

In this section we consider the neighbourhood-complement of a graph to define the closed subgraph in a graph topological space.

Definition 4. [4] Let A be a subset of vertices of a graph G and $N(A)$ denote the vertices of $V(G) \setminus A$ which has a neighbour in A. Then complement of A, denoted by $\overline{A}$, is the complement of $N(A) \cup A$ in $V(G)$.

Definition 5. Let G be a graph and let H be a subgraph of G with vertex set $V(H)$. Let $N(V(H))$ denote the vertices of $V(G) \setminus V(H)$ which has a neighbour in $V(H)$. Then,

$$V(H)^{\varsigma} = (V(H) \cup N(V(H)))^{\varsigma}$$

The subgraph induced by the vertex set $V(H)^{\varsigma}$ is called *neighbourhood-complement* of the graph H.

Definition 6. A subgraph H of G in a graph topological space $(G, \mathcal{T}_G)$ is called *neighbourhood-closed* or *n-closed* if its neighbourhood-complement graph is open.

Proposition 4. *In a graph topological space $(G, \mathcal{T}_G)$, the null graph K_0 and the graph G are always neighbourhood-closed.*

Proof. Let G be a graph and $(G, \mathcal{T})$ be a graph topological space. In order to prove that G and K_0 are d-closed, we need to show that the neighbourhood-complement graph is open.
Let us first consider G with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and the neighbourhood set $N(V(G)) = \emptyset$ with respect to the vertex set of the graph G. Then, we have

$$\begin{aligned} V(G)^{\varsigma} &= (\{v_1, v_2, ..., v_n\} \cup \emptyset)^{\varsigma} \\ &= \{v_1, v_2, \ldots, v_n\}^{\varsigma} \\ &= \emptyset \end{aligned}$$

The induced subgraph with the empty vertex set is the null graph K_0 which is open. Hence, the graph G is n-closed. Similarly, let us consider the null graph K_0 with the vertex set $V(K_0) = \emptyset$ and $N(V(K_0)) = \emptyset$.

$$\begin{aligned} V(K_0)^{\varsigma} &= (\emptyset \cup \emptyset)^{\varsigma} \\ &= \emptyset^{\varsigma} \\ &= V(G) \end{aligned}$$

The induced subgraph with the vertex set $V(G)$ is the graph G which is open. □

We now characterise a n-closed graph in a graph topological space defined with respect to the neighbourhood-complement.

Theorem 9. *Let $\mathcal{Z}$ be the collection of all neighbourhood-closed graphs in the subgraph topological space $(G, \mathcal{T}_G)$. Then,*

(i) $K_0, G \in \mathcal{Z}$

(ii) *Any union of n-closed subgraphs is n-closed in $\mathcal{Z}$.*

(iii) *Arbitrary intersection of n-closed subgraphs is n-closed in $\mathcal{Z}$.*

Conversely, the collection of neighbourhood-closed subgraphs satisfying the above three properties is a graph topology on G.

Proof. Part-(i) is immediate from Proposition 4. That is, $K_0, G \in \mathcal{Z}$.
Now we have to prove that any (finite or infinite) union of n-closed subgraphs is n-closed in $\mathcal{Z}$. Let $H_1, H_2, ..., H_n$ be n-closed subgraphs in a graph topological space. That is, $H_1, H_2, \ldots, H_n \in \mathcal{Z}$. To prove that $\bigcup_{i \in I} H_i$ is n-closed. It is enough to prove that the subgraph induced by the vertex set $(\bigcup_{i \in I} V(H_i)^{\varsigma})$ is open.
By Definition 6, the neighbourhood-complement of these subgraphs $H_1, H_2, \ldots, H_n$ are open. That is, the subgraph induced by the neighbourhood-complement of each of these graphs is open. More precisely, the subgraph induced by the vertex sets $v(H_1)^{\varsigma}, v(H_2)^{\varsigma}, v(H_3)^{\varsigma}, \ldots, v(H_n)^{\varsigma}$, are open. By the definition of graph topology, we know that the union of these subgraphs is open. Let those graphs be $H_1^{\varsigma}, H_2^{\varsigma}, \ldots, H_n^{\varsigma}$ which are open in $\mathcal{T}_G$. By axiom of graph topology, the union $\bigcup_{i \in I} H_i^{\varsigma}$ is open. This is nothing but the subgraph induced by the set $(\bigcup_{i \in I} V(H_i)^{\varsigma})$.
Now, we need to prove that the arbitrary intersection of n-closed subgraphs is always n-closed. For this, consider the n-closed graphs $H_1, H_2, H_3,$
$\ldots, H_n \in \mathcal{Z}$. In order to prove that $\bigcap_{i \in I} H_i$ is n-closed, we have to show that the subgraph induced by the vertex set $(\bigcap_{i \in I} V(H_i)^{\varsigma})$ is open. Since the neighbourhood-complements of each if these graphs $H_1^{\varsigma}, H_2^{\varsigma}, \ldots, H_n^{\varsigma}$ are open in $\mathcal{T}_G$, their intersection $\bigcap_{i \in I} H_i^{\varsigma}$ is also open by the definition of graph topology, which is the subgraph induced by $(\bigcup_{i \in I} V(H_i)^{\varsigma})$.
Suppose that we have a collection of n-closed subgraphs satisfying the three properties. These properties coincide with the axioms of graph topology and hence the collection of these n-closed subgraph forms a graph topology on G. □

4. Closure of a Subgraph

The notion of the closure of a subgraph with regard to a given graph topology is introduced as follows:

Definition 7. Let $(G, \mathcal{T}_G)$ be a topological space. The $\mathcal{T}_G$*-closure* of a subgraph H of G, denoted by $Cl_{\mathcal{T}}(H)$ or $\widetilde{H}$, is the intersection of all closed subgraph containing H. That is,

$$Cl_{\mathcal{T}}(H) = \{\cap H_i \text{ where } H_i \text{ is closed and } H \subset H_i\}$$

Following are a few properties of the $\mathcal{T}_G$-closure of a subgraph.

Proposition 5. *Let H be a subgraph of a graph G in the graph topological space $(G, \mathcal{T}_G)$. Then,*

(i) *$\tilde{H}$ is closed subgraph of G and is the smallest closed subgraph containing H.*

(ii) *Closure of a null graph is null graph.*

(iii) *The subgraph H is closed if and only if the closure of H is H itself.*

(iv) *$\tilde{\tilde{H}} = \tilde{H}$ if H is not closed and $\tilde{\tilde{H}} = \tilde{H} = H$ if H is closed.*

Proof. **(i)** Since arbitrary intersection of closed graph is closed we have closure of the subgraph H is closed.. Suppose that $\mathcal{H}$ be the closure of H, that is the smallest closed graph containing H. Let K be a closed subgraph of G containing H. Then $\mathcal{H} \subseteq K$.

(ii) The proof is trivial.

(iii) Suppose that H is a closed subgraph. Then H is the smallest closed subgraph containing itself. Clearly the $\mathcal{T}_G$-closure of H is H. Conversely, suppose $\mathcal{T}_G$-closure of H is H. Then H is closed by condition (i).

(iv) Applying property (iii) to $\tilde{H}$, we have $\tilde{\tilde{H}} = \tilde{H}$ if H is not closed and $\tilde{\tilde{H}} = \tilde{H} = H$ if H is closed.

□

Proposition 6. *Let $(G, \mathcal{T}_G)$ be a graph topological space and H be a subgraph of G. Let $\tilde{H}$ be the closure of the graph H. Then, every element (a vertex or an edge) is in $\tilde{H}$ if and only if every open subgraph containing it intersects with H.*

Proof. Let $(G, \mathcal{T}_G)$ be a graph topological space and H be a subgraph of G. Let $\tilde{H}$ be the $\mathcal{T}_G$-closure of H and X be an open subgraph in $\mathcal{T}_G$. We will prove this using contrapositive method. Suppose that vertex, say v in $V(G)$, is not a vertex of $\tilde{H}$. Then, $E(X) = E(G) - E(\tilde{H})$ induces an open subgraph containing v such that $E(X) \cap E(\tilde{H}) = \emptyset$. The same argument is applicable for any edge e in $E(G)$ also, as required.
Conversely, let X be an open graph such that $E(X) \cap E(\tilde{H}) = \emptyset$. Then, $E(G) - E(X)$ is a closed subgraph containing H. Therefore by Definition 7, $E(\tilde{H}) \subseteq E(G) - E(X)$ and hence $\tilde{H}$ does not contain the vertex v. This completes the proof. □

5. Conclusion

In this article, we have initiated a study on the graph theoretical version of topology and extended some concepts and results of topology on a set to that area. We can see that when graph G is an empty graph, then its graph topology, as defined in this paper, coincides with the definition of a topology on a set $X = V(G)$. Hence, intensive studies are possible in this area. We also discussed the idea of closed graphs using the notions of two graph complements- decomposition complement and neighbourhood complement in this paper. We have also studied the closure of a subgraph and the properties of the same has been explored. Other topological properties such as countability, relationship between topological connectedness and usual connectedness of graphs and compactness can also be studied. All these points seem to be much promising for further intensive research.

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