Communications in Combinatorics and Optimization
Vol. 8, No. 1 (2023), pp. 261-270
DOI: 10.22049/CCO.2022.27641.1305

CCO
Commun. Comb. Optim.

Research Article

A new upper bound on the independent 2-rainbow domination number in trees

Elham Gholami[1,†], **Nader Jafari Rad**[2,*], **Abolfazl Tehranian**[1,‡], **Hamid Rasouli**[1,§]

[1] Department of Mathematics, Science and Research branch, Islamic Azad University, Tehran, Iran
[†] gholamiel1363@gmail.com
[‡] tehranian@srbiau.ac.ir
[§] hrasouli@srbiau.ac.ir

[2] Department of Mathematics, Shahed University, Tehran, Iran
n.jafarirad@gmail.com

Received: 18 November 2021; Accepted: 4 February 2022
Published Online: 6 February 2022

Abstract: A 2-rainbow dominating function on a graph G is a function g that assigns to each vertex a set of colors chosen from the subsets of $\{1,2\}$ so that for each vertex with $g(v) = \emptyset$ we have $\bigcup_{u \in N(v)} g(u) = \{1,2\}$. The weight of a 2-rainbow dominating function g is the value $w(g) = \sum_{v \in V(G)} |f(v)|$. A 2-rainbow dominating function g is an independent 2-rainbow dominating function if no pair of vertices assigned nonempty sets are adjacent. The 2-rainbow domination number $\gamma_{r2}(G)$ (respectively, the independent 2-rainbow domination number $i_{r2}(G)$) is the minimum weight of a 2-rainbow dominating function (respectively, independent 2-rainbow dominating function) on G. We prove that for any tree T of order $n \geq 3$, with l leaves and s support vertices, $i_{r2}(T) \leq (14n + \ell + s)/20$, thus improving the bound given in [Independent 2-rainbow domination in trees, Asian-Eur. J. Math. 8 (2015) 1550035] under certain conditions.

Keywords: Rainbow domination, Independent rainbow domination, Tree

AMS Subject classification: 05C69

1. Introduction

In this paper, we continue the study of a variant of 2-rainbow dominating functions, namely, independent 2-rainbow dominating function. We first present some necessary definitions and notations. For notation and graph theory terminology not given here,

* *Corresponding Author*

we follow [12]. We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices of a graph G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and the *degree* of v, denoted by $\deg_G(v)$, is the cardinality of its open neighborhood. A *leaf* of a tree T is a vertex of degree one, while a *support vertex* of T is a vertex adjacent to a leaf. A *strong support vertex* is a support vertex adjacent to at least two leaves, while *weak support vertex* is a support vertex adjacent to precisely one leaf. In this paper, we denote the set of all support vertices of T by $S(T)$ and the set of leaves by $L(T)$. We denote $\ell(T) = |L(T)|$ and $s(T) = |S(T)|$. We also denote by $L(x)$ the set of leaves adjacent to a support vertex x, and denote $\ell_x = |L(x)|$. A *star* is the graph $K_{1,k}$, where $k \geq 1$. For a star with $k > 1$ leaves, the central vertex is the unique vertex of degree greater than one. For $r, s \geq 1$, the *double star* $S(r, s)$ is the tree with exactly two vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors. We denote a path on n vertices by P_n. A *rooted tree* T distinguishes one vertex r called the root. For each vertex $v \neq r$ of T, the parent of v is the neighbor of v on the unique (r, v)-path, while a child of v is any other neighbor of v. The set of children of v is denoted by $C(v)$. A descendant of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v, while an ancestor of v is a vertex $u \neq v$ that belongs to the (r, v)-path in T. The maximal subtree of T rooted at v is denoted by T_v. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices of G is the *diameter* of G, denoted by $\text{diam}(G)$.

A 2-*rainbow dominating function* (*2RDF*) of a graph G is a function g that assigns to each vertex a set of colors chosen from the subsets of $\{1, 2\}$ so that for each vertex v with $g(v) = \emptyset$ we have $\bigcup_{u \in N(v)} g(u) = \{1, 2\}$. The weight of a 2-rainbow dominating function g is the value $w(g) = \sum_{v \in V(G)} |f(v)|$. The 2-*rainbow domination number* $\gamma_{r2}(G)$ is the minimum weight of a 2-rainbow dominating function on G. The concept of 2-rainbow domination was introduced by Brešar, Henning, and Rall [7] and has been studied by several authors (see for example [1–3, 8, 11, 13, 14]).

A 2-rainbow dominating function g is an *independent* 2-*rainbow dominating function* (*I2RDF*) if no two vertices assigned nonempty sets are adjacent. The weight of a 2-rainbow dominating function g is the value $w(g) = \sum_{v \in V(G)} |f(v)|$. The *independent* 2-*rainbow domination number* $i_{r2}(G)$ is the minimum weight of an independent 2-rainbow dominating function on G. We refer to an independent 2-rainbow dominating function on G of minimum weight as an i_{r2}-function. The independent 2-rainbow domination number was investigated in [4, 6, 9, 10].

Chellali et al. [10] posed the following problem: Find a sharp bound for $i_{r2}(T)$ in terms of the order of a tree T. Amjadi et al. [5] answered the above problem and proved the following bound for the independent 2-rainbow domination number of a tree.

Theorem 1 (Amjadi et al. [5]). *If T is a tree of order $n \geq 3$, then $i_{r2}(T) \leq \frac{3n}{4}$. Furthermore, this bound is sharp.*

In this paper we present a new bound for the independent 2-rainbow domination number of a tree of order $n \geq 3$ with l leaves and s support vertices. Our bound improves the bound given in Theorem 1 for trees when $\ell + s < n$.

2. Main Result

Theorem 2. *For any tree T of order $n \geq 3$, with l leaves and s support vertices, $i_{r2}(T) \leq (14n + \ell + s)/20$.*

Proof. We use induction on the order $n = n(T)$ of a tree T. According to [4] for stars and double stars, the base step is correct for $n \leq 4$. Assume that for any tree T' of order $n' < n$, with ℓ' leaves and s' support vertices, $i_{r2}(T') \leq (14n' + \ell' + s')/20$. Now consider the tree T of order $n \geq 5$, with l leaves and s support vertices. If T is a star, then the function that assigns $\{1, 2\}$ to the central vertex and $\emptyset$ to every leaf of the star is an *I2RDF* of T of weight 2, and so $i_{r2}(T) = 2 < (14n + \ell + s)/20$. Hence, we may assume that $\mathrm{diam}(T) \geq 3$. Suppose that $\mathrm{diam}(T) = 3$, and so T is a double star $T \cong S(r, k)$, where $r \geq k \geq 1$. Let u and v be the two vertices of T that are not leaves, where u has r leaf neighbors and v has k leaf neighbors. The function that assigns $\{1, 2\}$ to u, $\{1\}$ to the leaf neighbors of v, and $\emptyset$ to the remaining vertices of T is a an *I2RDF* of T of weight $2 + k$, and so $i_{r2}(T) \leq 2 + k \leq (14n + \ell + s)/20$. Hence, we may assume that $\mathrm{diam}(T) \geq 4$, for otherwise the desired result follows.
We root T at a leaf x_0 of a diametrical path $x_0x_1 \ldots x_d$ from x_0 to a leaf x_d farthest from x_0 such that $\deg(x_{d-1}) = \max\{\deg(u) : d(x_0, u) = d - 1\}$. The remainder of the proof proceeds by establishing eight claims and then deducing from those claims that the statement of the theorem is true.

Claim 1. If u and v are two strong support vertices of T such that $N(u) \cap (V(T) - L(T)) = \{v\}$, then $i_{r2}(T) < (14n + \ell + s)/20$.

Proof of Claim 1. Let u' be a leaf neighbor of u and v' be a leaf neighbor of v. Let $T' = T - \{u', v'\}$. Then $n(T') = n' = n - 2$, $\ell(T') = \ell' = \ell - 2$ and $s(T') = s' = s$. Among all i_{r2}-functions on T', let f' be chosen so that the weight assigned to leaves is as small as possible. We first assume that $f'(u) = \emptyset$. If $f'(v) = \emptyset$, then $\deg(u) \geq 3$. Then re-assigning $\{1, 2\}$ to u and $\emptyset$ to all leaf neighbors of u produces a new *I2RDF* g' of T' such that $w(g') \leq w(f')$ and the sum of the values assigned to all leaves under g' is less than the sum of the values assigned to all leaves under f', a contradiction. Hence we may assume that $f'(v) \neq \emptyset$. Since, v is a support vertex, we can assume that $f'(v) = \{1, 2\}$. Then we can extend f' to a *I2RDF* f of T by assigning $\emptyset$ to v' and $\{1\}$ to u', and so by the inductive hypothesis,

$$i_{r2}(T) \leq i_{r2}(T') + 1 \leq (14(n - 2) + (\ell - 2) + s)/20 + 1 < (14n + \ell + s)/20.$$

Next assume that $f'(u) \neq \emptyset$. Since u is a support vertex, we have $f'(u) = \{1,2\}$. Also, $f'(v) = \emptyset$, since f' is a i_{r2}-function. Then we can extend f' to a *I2RDF* f of T by assigning $\emptyset$ to u' and $\{1\}$ to v'. As above, we get that

$$i_{r2}(T) \leq i_{r2}(T') + 1 < (14n + \ell + s)/20. \quad \blacklozenge$$

If $\deg(x_{d-1}) \geq 3$, then by Claim 1, we may assume that x_{d-2} is not a strong support vertex, for otherwise the desired result follows.

Claim 2. If u is a strong support vertex of T such that $N(u) \cap (V(T) - L(T)) = \{v\}$ and there exists at least one weak support vertex of degree two in $N(v)$, then $i_{r2}(T) < (14n + \ell + s)/20$.

Proof of Claim 2. Let u' be the leaf neighbor of u and x be the weak support vertex adjacent to v with leaf neighbor y. Let $T' = T - \{u', x, y\}$. Then $n' = n - 3$, $\ell' = \ell - 2$ and $s' = s - 1$. Among all i_{r2}-functions on T', let f' be chosen so that the weight assigned to leaves is as small as possible.
First assume that $f'(u) = \emptyset$. If $f'(v) = \emptyset$, then $\deg(u) \geq 3$, and we can extend f' to a *I2RDF* f of T by re-assigning $\{1,2\}$ to u and x, and $\emptyset$ to u', y and all leaf neighbors of u in T'. Thus

$$i_{r2}(T) \leq i_{r2}(T') + 2 \leq (14(n-3) + (l-2) + (s-1))/20 + 2 < (14n + l + s)/20.$$

Now assume that $f'(v) \neq \emptyset$. Without loss of generality, we can assume that $2 \in f(v)$. Then we can extend f' to a *I2RDF* f of T by assigning $\emptyset$ to x and $\{1\}$ to y, u', and so

$$i_{r2}(T) \leq i_{r2}(T') + 2 < (14n + \ell + s)/20.$$

Next assume that $f'(u) \neq \emptyset$. Since u is a support vertex in tree T', we have $f'(u) = \{1,2\}$. Also, $f'(v) = \emptyset$, since f' is a i_{r2}-function. Then we can extend f' to a *I2RDF* f of T by assigning $\emptyset$ to y, u' and $\{1,2\}$ to x. As above,

$$i_{r2}(T) \leq i_{r2}(T') + 2 < (14n + \ell + s)/20. \quad \blacklozenge$$

If $\deg(x_{d-1}) \geq 3$, then by Claim 2, we may assume that there is no weak support vertex of degree two as a child of vertex x_{d-2}, for otherwise the desired result follows.

Claim 3. If $\deg(x_{d-1}) \geq 3$, then $i_{r2}(T) < (14n + \ell + s)/20$.

Proof of Claim 3. By Claim 1, we may assume that x_{d-2} is not a strong support vertex and by Claim 2 every child support vertex of x_{d-2} in tree T has degree at least three. Let r be the number of children of x_{d-2} that are leaves, and let k be the number of children support vertex of x_{d-2}. Claim 1, implies that $r \leq 1$. Further, since x_{d-1} has degree at least three, we note that $k \geq 1$. Let $T' = T - T_{x_{d-2}}$. Then $n' = n - \sum_{u \in N(x_{d-2})-(L(T)\cup\{x_{d-3}\})} \deg(u) - r - 1 \leq n - 3k - r - 1$, $s' \leq s - k - r + 1$ and $\ell' \leq \ell - 2k - r + 1$. Assume that f' is an i_{r2}-function of T'. Then we can extend f' to a *I2RDF* f of T by assigning $\{1,2\}$ to all child support vertices of x_{d-2}, $\{1\}$ to leaf neighbors of x_{d-2} in T', if any, and $\emptyset$ to the remaining vertices in tree $T_{x_{d-2}}$. Hence,

$$\begin{aligned} i_{r2}(T) &\leq i_{r2}(T') + 2k + r \\ &\leq (14n' + \ell' + s')/20 + 2k + r \\ &= (14(n - 3k - r - 1) + (\ell - 2k - r + 1) + (s - k - r + 1))/20 + 2k + r \\ &< (14n + \ell + s)/20. \quad \blacklozenge \end{aligned}$$

By Claim 3, we may assume that $\deg(x_{d-1}) = 2$, for otherwise the desired result follows.

Claim 4. If $\deg(x_{d-2}) = 2$ and $\deg(x_{d-3}) \geq 3$, then $i_{r2}(T) < (14n + l + s)/20$.

Proof of Claim 4. Let $T' = T - \{x_d, x_{d-1}, x_{d-2}\}$. Then $n' = n - 3$, $\ell' = \ell - 1$ and $s' = s - 1$. Assume that f' is a i_{r2}-function. Then we can extend f' to a *I2RDF* f of T by assigning $\{1,2\}$ to x_{d-1} and $\emptyset$ to x_d and x_{d-2}. Hence,

$$\begin{aligned} i_{r2}(T) &\leq i_{r2}(T') + 2 \\ &\leq (14n' + l' + s')/20 + 2 \\ &= (14(n-3) + (\ell - 1) + (s - 1))/20 + 2 \\ &< (14n + l + s)/20. \quad \blacklozenge \end{aligned}$$

Claim 5. If $\deg(x_{d-2}) = 2$ and $\deg(x_{d-3}) = 2$, then $i_{r2}(T) < (14n + \ell + s)/20$.

Proof of Claim 5. Let $T' = T - \{x_d, x_{d-1}\}$. Then $n' = n - 2$, $\ell' = \ell$ and $s' = s$. Assume that f' is an i_{r2}-function. We first assume that $f'(x_{d-2}) \neq \emptyset$. Without loss of generality, we assume that $2 \in f'(x_{d-2})$. Then we can extend f' to a *I2RDF* f of T by assigning $\{1\}$ to x_d and $\emptyset$ to x_{d-1}, and so we deduce that

$$\begin{aligned} i_{r2}(T) &\leq i_{r2}(T') + 1 \\ &\leq (14n' + l' + s')/20 + 2 \\ &= (14(n-2) + \ell + s)/20 + 1 \\ &< (14n + \ell + s)/20. \end{aligned}$$

Next assume that $f'(x_{d-2}) = \emptyset$. Then $f'(x_{d-3}) = \{1,2\}$. If for every vertex $u \in N(x_{d-4})$ other than x_{d-3}, $f'(u) = \emptyset$, then re-assigning $\{1\}$ to x_{d-4} and $\{2\}$ to x_{d-2} the set, produces a new *I2RDF* g' of T' such that $g'(x_{d-2}) \neq \emptyset$ and so as before the desired result follows. Thus we may assume that there exists a vertex $w \in N(x_{d-4}) - \{x_{d-3}\}$ such that $f'(w) \neq \emptyset$. Without loss of generality, we may assume that $2 \in f'(w)$. Then we can extend f' to a *I2RDF* f of T by re-assigning $\{1\}$ to x_{d-3}, $\{1, 2\}$ to x_{d-1}, and $\emptyset$ to x_d. Hence,

$$\begin{aligned} i_{r2}(T) &\leq i_{r2}(T') + 1 \\ &\leq (14n' + l' + s')/20 + 2 \\ &= (14(n-2) + \ell + s)/20 + 1 \\ &< (14n + \ell + s)/20. \quad \blacklozenge \end{aligned}$$

By Claims 4 and 5, we can assume that $\deg(x_{d-2}) \geq 3$, for otherwise the desired result follows.

Claim 6. If $\deg(x_{d-3}) = 2$, then $i_{r2}(T) < (14n + \ell + s)/20$.

Proof of Claim 6. Let $T' = T - T_{x_{d-3}}$. Then we can assume that every children of x_{d-2} is a leaf or a weak support vertex. Let r be the number of children of x_{d-2} of degree 2 and k be the number of leaf neighbors of x_{d-2}. Then $r + k \geq 2$, $n' = n - 2r - k - 2$, $\ell' \leq \ell - r - k + 1$ and $s' \leq s - r - k' + 1$, where $k' = 1$ if $k \neq 0$ and $k' = 0$ otherwise. Assume that f' is a $i_{r2}(T')$-function. Then we can extend f' to a *I2RDF* f of T by assigning $\{1, 2\}$ to x_{d-2}, $\emptyset$ to every vertex in $N(x_{d-2})$ and $\{1\}$ to the remaining vertices of T. Hence,

$$\begin{aligned} i_{r2}(T) &\leq i_{r2}(T') + r + 2 \\ &\leq (14n' + \ell' + s')/20 + r + 2 \\ &= (14(n - 2r - k - 2) + (\ell - r - k + 1) + (s - r - k' + 1))/20 + r + 2 \\ &< (14n + \ell + s)/20. \quad \blacklozenge \end{aligned}$$

By Claim 6, we can assume that $\deg(x_{d-3}) \geq 3$, for otherwise the desired result follows.

Claim 7. If x_{d-3} is a strong support vertex, then $i_{r2}(T) < (14n + l + s)/20$.

Proof of Claim 7. Let $u \in L(x_{d-3})$ and $T' = T - \{x_d, x_{d-1}, u\}$. Then, $n' = n - 3$, $\ell' = \ell - 2$ and $s' = s - 1$. Among all $i_{r2}(T')$-functions, let f' be chosen so that the weight assigned to leaves is as small as possible. We first assume that $f'(x_{d-2}) \neq \emptyset$. Without loss of generality, we assume that $2 \in f'(x_{d-2})$. Then we can extend f' to a

I2RDF f of T by assigning $\{1\}$ to x_d and u and assigning $\emptyset$ to x_{d-1}.So,

$$\begin{aligned} i_{r2}(T) &\le i_{r2}(T') + 2 \\ &\le (14n' + \ell' + s')/20 + 2 \\ &= (14(n-3) + (\ell - 2) + (s-1))/20 + 2 \\ &< (14n + \ell + s)/20. \end{aligned}$$

Next assume that $f'(x_{d-2}) = \emptyset$. We know that every child of x_{d-2} in tree T' is a leaf or a weak support vertex of degree two. Let r be the number of leaf neighbors of x_{d-2}. Since $f'(x_{d-2}) = \emptyset$, for every leaf $v \in L(x_{d-2})$ we have $|f'(v)| \ge 1$ and for every child support vertex z with leaf neighbors z' we have $|f'(z)| + |f'(z')| \ge 2$. Assume that $f'(x_{d-3}) = \emptyset$. If $r \ne 0$. Then the function f defined by $f(x_{d-2}) = \{1,2\}$, $f(w) = \emptyset$ for $w \in N(x_{d-2})$, $f(w) = \{2\}$ if w is a leaf in $T_{x_{d-2}}$ at distance 2 from x_{d-2}, $f(u) = \{1\}$ and $g'(w) = f'(w)$ otherwise, is a *I2RDF* for T with $w(f) \le w(f') + 2$. Also, if $r = 0$, then the function f defined by $f(x_{d-2}) = \{1\}$, $f(w) = \emptyset$ for $w \in N(x_{d-2})$, $f(w) = \{2\}$ if w is a leaf in $T_{x_{d-2}}$ at distance 2 from x_{d-2}, $f(u) = \{1\}$ and $g'(w) = f'(w)$ otherwise, is a *I2RDF* for tree T with $w(f) \le w(f') + 2$. Hence,

$$\begin{aligned} i_{r2}(T) &\le i_{r2}(T') + 2 \\ &\le (14n' + \ell' + s')/20 + 2 \\ &= (14(n-3) + (\ell - 2) + (s-1))/20 + 2 \\ &< (14n + \ell + s)/20. \end{aligned}$$

Thus we may assume that $f'(x_{d-3}) \ne \emptyset$. Since x_{d-3} is a support vertex in the tree T', we can assume that $f(x_{d-3}) = \{1,2\}$. Then we can extend f' to a *I2RDF* f of T by assigning $\{1,2\}$ to x_{d-1} and assigning $\emptyset$ to $\{u, x_d\}$. Hence,

$$\begin{aligned} i_{r2}(T) &\le i_{r2}(T') + 2 \\ &\le (14n' + \ell' + s')/20 + 2 \\ &= (14(n-3) + (\ell - 2) + (s-1))/20 + 2 \\ &< (14n + \ell + s)/20. \quad \blacklozenge \end{aligned}$$

By Claim 7, we can assume that x_{d-3} is not a strong support vertex.

Claim 8. If x_{d-3} has a child which is a weak support vertex of degree two, then $i_{r2}(T) < (14n + \ell + s)/20$.

Proof of Claim 8. Assume that x_{d-3} has a child u that is a weak support vertex of degree two. Let $L(u) = \{v\}$ and $T' = T - \{x_d, x_{d-1}, u, v\}$. Then $n' = n - 4$, $\ell' = \ell - 2$ and $s' = s - 2$. Among all $i_{r2}(T')$-functions, let f' be chosen so that the weight assigned to leaves is as small as possible. We first assume that $f'(x_{d-2}) \ne \emptyset$.

Without loss of generality, we assume that $2 \in f'(x_{d-2})$. Then we can extend f' to a *I2RDF* f of T by assigning $\{1\}$ to x_d, $\{1,2\}$ to u and $\emptyset$ to x_{d-1} and v. So

$$\begin{aligned} i_{r2}(T) &\le i_{r2}(T') + 3 \\ &\le (14n' + l' + s')/20 + 3 \\ &= (14(n-4) + (\ell-2) + (s-2))/20 + 3 \\ &= (14n + \ell + s)/20. \end{aligned}$$

Next assume that $f'(x_{d-2}) = \emptyset$. If $f'(x_{d-3}) \neq \emptyset$, then we may assume that $2 \in f'(x_{d-3})$. Then we can extend f' to a *I2RDF* f of T by assigning $\{1\}$ to v, $\{1,2\}$ to x_{d-1} and $\emptyset$ to x_d and u. Then

$$\begin{aligned} i_{r2}(T) &\le i_{r2}(T') + 3 \\ &\le (14n' + l' + s')/20 + 3 \\ &= (14(n-4) + (\ell-2) + (s-2))/20 + 3 \\ &= (14n + \ell + s)/20. \end{aligned}$$

Now assume that $f'(x_{d-3}) = \emptyset$. Let $r = \ell_{x_{d-2}}$. Since $f'(x_{d-2}) = \emptyset$, for every leaf $v \in L(x_{d-2})$, $|f'(v)| = 1$, and for every child support vertex z with leaf neighbors z', $|f'(z)| + |f'(z')| = 2$. If $r \neq 0$, then the function f defined by $f(x_{d-2}) = \{1,2\}$, $f(w) = \emptyset$ for $w \in N(x_{d-2})$, $f(w) = \{2\}$ if w is a leaf in $T_{x_{d-2}}$ at distance 2 from x_{d-2}, $f(u) = \{1,2\}$, $f(v) = \emptyset$ and $g'(w) = f'(w)$ otherwise, is a *I2RDF* for T with $w(f) \le w(f') + 3$. Also, if $r = 0$, then the function f defined by $f(x_{d-2}) = \{1\}$, $f(w) = \emptyset$ for $w \in N(x_{d-2})$, $f(w) = \{2\}$ if w is a leaf in $T_{x_{d-2}}$ at distance 2 from x_{d-2}, $f(u) = \{1,2\}$, $f(v) = \emptyset$ and $g'(w) = f'(w)$ otherwise, is a *I2RDF* for T with $w(f) \le w(f') + 3$. Hence,

$$\begin{aligned} i_{r2}(T) &\le i_{r2}(T') + 3 \\ &\le (14n' + l' + s')/20 + 3 \\ &= (14(n-4) + (\ell-2) + (s-2))/20 + 2 \\ &= (14n + \ell + s)/20. \quad \blacklozenge \end{aligned}$$

Thus we may assume that x_{d-3} has no weak support vertex of degree two as a child. Let R be the set of all support vertices $u \in N(x_{d-3}) \cap V(T_{x_{d-3}})$ such that $N(u) - (L(T) \cup \{x_{d-3}\}) \neq \emptyset$, $R_0 = L(R)$, $K = (S(T) \cap V(T_{x_{d-3}})) - N(x_{d-3})$, $K_0 = L(K)$, P be the set of all strong support vertices $u \in N(x_{d-3}) \cap V(T_{x_{d-3}})$ such that $N(u) - (L(T) \cup \{x_{d-3}\}) = \emptyset$, $P_0 = L(P)$ and $B = V(T_{x_{d-3}}) - (S(T_{x_{d-3}}) \cup L(T_{x_{d-3}}))$. Also, let $|R| = r$, $|R_0| = r_0$, $|K| = k$, $|K_0| = k_0$, $|P| = p$, $|P_0| = p_0$ and $|B| = b$. It is easy to see that $r_0 \ge r$, $k = k_0$, $p_0 \ge 2p$ and $k \ge r + 2b$.

We first assume that x_{d-3} is a support vertex and so as before, $\ell_{x_{d-3}} = 1$. Let $T' = T - T_{x_{d-3}}$. Then $n' = n - r - r_0 - k - k_0 - p - p_0 - 2$, $\ell' \le \ell - r_0 - k_0 - p_0$,

$s' \leq s - r - k - p$. Among all $i_{r2}(T')$-functions, let f' be chosen so that the weight assigned to leaves is as small as possible. Then we can extend f' to a *I2RDF* f of T by assigning $\{1,2\}$ to vertices of $R \cup P$, $\emptyset$ to vertices of $R_0 \cup P_0 \cup \{x_3\}$, $\{2\}$ to vertices of B, $\{1\}$ to the vertices of $L(x_{d-3}) \cup K_0$ and $\emptyset$ to the remaining vertices of $T_{x_{d-3}}$. Hence

$$\begin{aligned}
i_{r2}(T) &\leq i_{r2}(T') + 2r + 2p + b + k_0 + 1 \\
&\leq (14n' + l' + s')/20 + 2r + 2p + b + k_0 + 1 \\
&= (14(n - r - r_0 - k - k_0 - p - p_0 - b - 2) \\
&+ (\ell - r_0 - k_0 - p_0) + (s - r - k - p))/20 + 2r + 2p + b + k_0 + 1 \\
&\leq (14n + \ell + s)/20 + (25r - 15r_0 - 15k + 15k_0 + 25p - 15p_0 + 6b - 8)/20 \\
&\leq (14n + \ell + s)/20 + (10r - 10k - 5p_0 + 6b - 8)/20 \\
&\leq (14n + \ell + s)/20 + (-14b - 5p_0 - 8)/20 \\
&< (14n + \ell + s)/20.
\end{aligned}$$

Next assume that x_{d-3} is not a support vertex. Let $T' = T - T_{x_{d-3}}$. Then $n' = n - r - r_0 - k - k_0 - p - p_0 - 1$, $\ell' \leq \ell - r_0 - k_0 - p_0 + 1$, $s' \leq s - r - k - p + 1$. Among all $i_{r2}(T')$-functions, let f' be chosen so that the weight assigned to leaves is as small as possible. Then we can extend f' to a *I2RDF* f of T by assigning $\{1,2\}$ to the vertices of $R \cup P$, $\emptyset$ to the vertices of $R_0 \cup P_0 \cup \{x_{d-3}\}$, $\{2\}$ to the vertices set B, $\{1\}$ to the vertices of K_0 and $\emptyset$ to the remaining vertices of $T_{x_{d-3}}$. Hence

$$\begin{aligned}
i_{r2}(T) &\leq i_{r2}(T') + 2r + 2p + b + k_0 \\
&\leq (14n' + l' + s')/20 + 2r + 2p + b + k_0 \\
&= (14(n - r - r_0 - k - k_0 - p - p_0 - b - 1) + (\ell - r_0 - k_0 - p_0 + 1) \\
&+ (s - r - k - p + 1))/20 + 2r + 2p + b + k_0 \\
&\leq (14n + \ell + s)/20 + (25r - 15r_0 - 15k + 5k_0 + 25p - 15p_0 + 6b - 12)/20 \\
&\leq (14n + \ell + s)/20 + (10r - 10k - 5p_0 + 6b - 12)/20 \\
&\leq (14n + \ell + s)/20 + (-14b - 5p_0 - 12)/20 \\
&< (14n + \ell + s)/20.
\end{aligned}$$

□

Acknowledgements

The authors would like to thank both referees for their careful review and helpful comments. They also would like to thank Dr Hadi Rahbani for his assistance on the paper.

References

[1] H. Abdollahzadeh Ahangar, J. Amjadi, M. Chellali, S. Nazari-Moghaddam, and S.M. Sheikholeslami, *Total 2-rainbow domination numbers of trees*, Discuss. Math. Graph Theory **41** (2021), no. 2, 345–360.

[2] H. Abdollahzadeh Ahangar, J. Amjadi, N. Jafari Rad, and V. Samodivkin, *Total k-rainbow domination numbers in graphs*, Commun. Comb. Optim. **3** (2018), no. 1, 37–50.

[3] H. Abdollahzadeh Ahangar, M. Khaibari, N. Jafari Rad, and S.M. Sheikholeslami, *Graphs with large total 2-rainbow domination number*, Iran. J. Sci. Technol. Trans. A Sci. **42** (2018), no. 2, 841–846.

[4] J. Amjadi, M. Chellali, M. Falahat, and S.M. Sheikholeslami, *Unicyclic graphs with strong equality between the 2-rainbow domination and independent 2-rainbow domination numbers*, Trans. Comb. **4** (2015), no. 2, 1–11.

[5] J. Amjadi, N. Dehgardi, N. Mohammadi, S.M. Sheikholeslami, and L. Volkmann, *Independent 2-rainbow domination in trees*, Asian-Eur. J. Math. **8** (2015), no. 2, Article ID: 1550035.

[6] J. Amjadi, M. Falahat, S.M. Sheikholeslami, and N. Jafari Rad, *Strong equality between the 2-rainbow domination and independent 2-rainbow domination numbers in trees*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 1, 205–218.

[7] B. Brešar, M.A. Henning, and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008), no. 1, 213–225.

[8] B. Brešar and T.K. Šumenjak, *On the 2-rainbow domination in graphs*, Discrete Appl. Math. **155** (2007), no. 17, 2394–2400.

[9] S. Brezovnik and T.K. Šumenjak, *Complexity of k-rainbow independent domination and some results on the lexicographic product of graphs*, Appl. Math. Comput. **349** (2019), 214–220.

[10] M. Chellali and N. Jafari Rad, *Independent 2-rainbow domination in graphs*, J. Combin. Math. Combin. Comput. **94** (2015), 133–148.

[11] N. Dehgardi, *On the outer independent 2-rainbow domination number of Cartesian products of paths and cycles*, Commun. Comb. Optim. **6** (2021), no. 2, 315–324.

[12] T.W. Haynes, S. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc. New York, 1998.

[13] Z. Shao, Z. Li, A. Peperko, J. Wan, and J. Žerovnik, *Independent rainbow domination of graphs*, Bull. Malays. Math. Sci. Soc. **42** (2019), no. 2, 417–435.

[14] T.K. Šumenjak, D.F. Rall, and A. Tepeh, *On k-rainbow independent domination in graphs*, Appl. Math. Comput. **333** (2018), 353–361.