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Research Article

Sombor index of some graph transformations

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Abstract: The Sombor index of the graph G is a degree based topological index, defined as $SO = \sum_{uv \in \mathbf{E}(G)} \sqrt{d_G(u)^2 + d_G(v)^2}$, where $d_G(u)$ is the degree of the vertex u of G and $\mathbf{E}(G)$ is the edge set of G. In this paper we calculate SO of some graph transformations.

Keywords: Sombor index, degree (of vertex), graph transformation

AMS Subject classification: 05C07, 05C09, 05C92

1. Introduction

Short time ago, a new of vertex–degree–based graph invariant has been conceived [9], named "*Sombor index*". It was obtained using a geometric approach to (molecular) graphs, applying Euclidean metrics. The index soon attracted the attention of numerous other scholars, and became subject of various mathematical [1, 5–8, 11, 18] and chemical [2, 3, 14] studies, to quote just a few. Continuing research along these lines, we have now determined the ways in which the Sombor index is changed by several (familiar) graph transformations.

In this paper, we consider simple, connected graph G having n vertices and m edges. The vertex and edge sets of the graph G are denoted by $\mathbf{V}(G)$ and $\mathbf{E}(G)$, respectively. An edge joining the vertices u and v is denoted by uv. The number of edges connected

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to the vertex v is known as the degree of this vertex, and will be denoted by $d_G(v)$. For additional graph-theoretical details see in textbooks [10, 17].
The Sombor index of graph G is defined as [9]

$$SO = SO(G) = \sum_{uv \in \mathbf{E}(G)} \sqrt{d_G(u)^2 + d_G(v)^2}\,.$$

In this paper we reports results for the Sombor index of the following graph transformations.
Definition 1: The subdivision graph $S(G)$ is the graph obtained from G by inserting a new vertex into each edge of G.

Definition 2: The line graph $L(G)$ of the graph G is the graph whose vertices are in a one-to-one correspondence with the edges of G, and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

We say that the vertices and edges of the graph G are its elements.

Definition 3: The total graph of G, denoted by $T(G)$, is the graph with vertex set $\mathbf{V}(T(G)) = \mathbf{V}(G) \cup \mathbf{E}(G)$, and two vertices in $T(G)$ are adjacent if and only if they are adjacent elements or they are incident elements in G.

Definition 4 [15]: The semi-total point graph of G, denoted by $T_1(G)$, is the graph with vertex set $\mathbf{V}(T_1(G)) = \mathbf{V}(G) \cup \mathbf{E}(G)$, and two vertices in $T_1(G)$ are adjacent if they are adjacent vertices in G or one is vertex and other is an edge, incident to it.

Definition 5: The semi-total line graph of G, denoted by $T_2(G)$, is a graph with vertex set $\mathbf{V}(T_2(G)) = \mathbf{V}(G) \cup \mathbf{E}(G)$, and two vertices in $T_2(G)$ are adjacent if they are adjacent edges in G or one is a vertex and the other is an edge, incident to it.

Definition 6: The double graph $D(G)$ is obtained by taking two copies of G and joining each vertex in one copy with the neighbors of the corresponding vertex in the other copy. The k^{th} iterated double graph of G for $k = 0, 1, 2, \ldots$ is defined as $D^k(G) = D(D^{k-1}(G))$, where $D^0(G) \equiv G$ and $D^1(G) \equiv D(G)$. If G is a graph of order n, then the order of $D^k(G)$, $k \geq 1$, is $2^k\,n$. More results on double graphs can be found in [12, 16]

Definition 7: The strong double graph $SD(G)$ is obtained by taking two copies of the graph G and joining each vertex v in one copy with the closed neighborhood of the corresponding vertex in another copy. The k^{th} iterated double graph of G for $k = 0, 1, 2, \ldots$ is defined as $SD^k(G) = SD(SD^{k-1}(G))$, where $SD^0(G) \equiv G$ and $SD^1(G) \equiv SD(G)$. If G is a graph of order n, then the order of $SD^k(G)$, $k \geq 1$, is $2^k\,n$.

Definition 8: The generalized transformation graph G^{xy} is a graph whose vertex set is $\mathbf{V}(G) \cup \mathbf{E}(G)$, and $\alpha, \beta \in \mathbf{V}(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:
(a) $\alpha, \beta \in \mathbf{V}(G)$, α, β are adjacent in G if $x = +$, whereas α, β are not adjacent in G if $x = -$.
(b) $\alpha \in \mathbf{V}(G)$ and $\beta \in \mathbf{E}(G)$, α, β are incident in G if $y = +$ whereas α, β are not incident in G if $y = -$.
An example, illustrating Definition 8 is provided in Fig. 1.

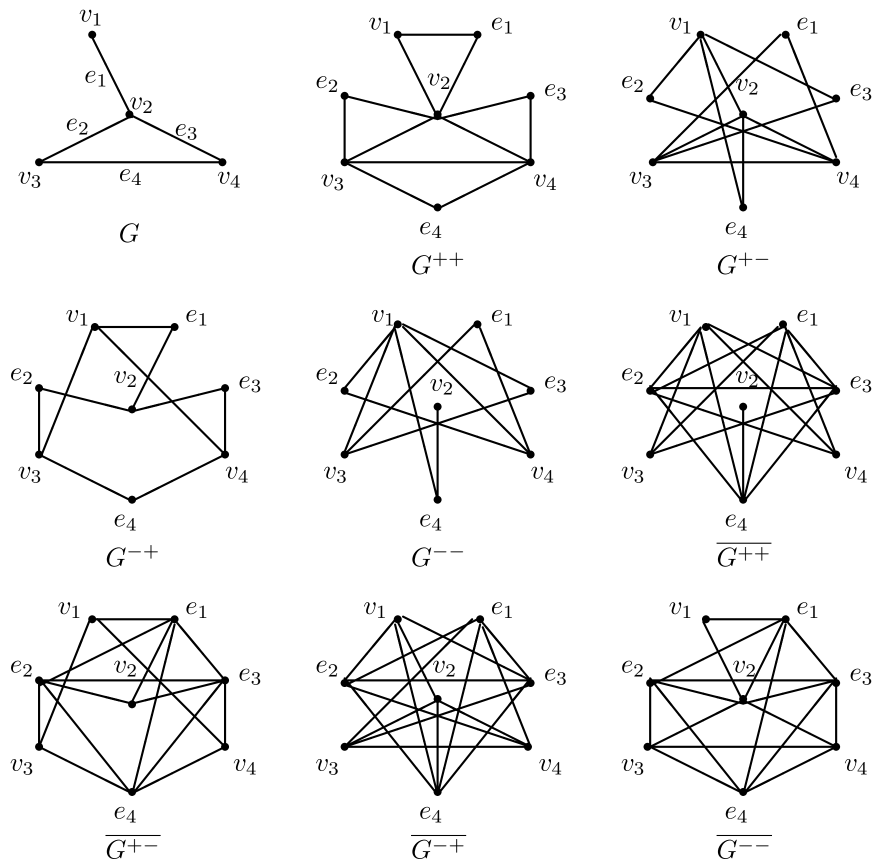

Figure 1: A graph G, its generalized transformations G^{xy} and their complements

2. Sombor index of graph transformations

Theorem 1. *Let G be a graph with n vertices and m edges. Then*

$$SO(S(G)) = \sum_{u \in \mathbf{V}(G)} d_G(u)\sqrt{d_G(u)^2 + 4},$$

where $S(G)$ is the subdivision graph of G.

Proof. By the definition of Sombor index, we have

$$
\begin{aligned}
SO(S(G)) &= \sum_{uv\in\mathbf{E}(S(G))} \sqrt{d_{S(G)}(u)^2 + d_{S(G)}(v)^2} \\
&= \sum_{uv\in\mathbf{E}(G)} \sqrt{d(u)^2+4} = \sum_{u\in\mathbf{V}(G)} d_G(u)\sqrt{d_G(u)^2+4}\,.
\end{aligned}
$$

□

Theorem 2. *Let G be a graph with n vertices and m edges and let $\mathbf{P} = \{uvw$ is a path of length 2 in G, where v is the middle vertex of $uvw\}$. Then,*

$$
SO(L(G)) = \sum_{uvw\in\mathbf{P}} \sqrt{d_G(u)^2+d_G(w)^2-4d_G(u)-4d_G(w)+8+2d_G(v)[d_G(u)+d_G(v)+d_G(w)-4]}\,.
$$

Proof. We know that $d_{L(G)}(e) = d(u)+d(v)-2$, where e is an edge in G. Taking summation over the set $\mathbf{P}$, the Sombor index of the line graph is,

$$
\begin{aligned}
SO(L(G)) &= \sum_{e'e''\in\mathbf{E}(L(G))} \sqrt{d_{L(G)}(e')^2 + d_{L(G)}(e'')^2} \\
&= \sum_{uvw\in\mathbf{V}(G)} \sqrt{[d_G(u)+d_G(v)-2]^2 + [d_G(v)+d_G(w)-2]^2} \\
&= \sum_{uvw\in\mathbf{P}} \sqrt{d_G(u)^2+d_G(w)^2-4d_G(u)-4d_G(w)+2d_G(v)[d_G(u)+d_G(v)+d_G(w)-4]+8}\,.
\end{aligned}
$$

□

Theorem 3. *Let G be a graph and let $\mathbf{P} = \{uvw$ is a path of length 2 in G for each $v \in G\}$. Then the Sombor index of the total graph of G is*

$$
\begin{aligned}
SO(T(G)) &= 2SO(G) + \sum_{uvw\in\mathbf{P}} \sqrt{d_G(u)^2 + d_G(w)^2 + 2d_G(v)[d_G(u)+d_G(v)+d_G(w)]} \\
&\quad + \sum_{u\in\mathbf{V}(G)} \sum_{v\sim u\in\mathbf{V}(G)} d_G(u)\sqrt{5d_G(u)^2 + d_G(v)^2 + 2d_G(u)\,d_G(v)}\,.
\end{aligned}
$$

Proof. By definition we have $d_{T(G)}(u) = 2d_G(u)$, $d_{T(G)}(e) = d_G(u) + d_G(v)$. Partition the edge set of $T(G)$ into three sets, $\mathbf{E}_1 = \{uv \mid uv \in \mathbf{E}(G)\}$, $\mathbf{E}_2 = \{ee' \mid e \sim e'$ in G, $e, e' \in \mathbf{E}(G)\}$, and $\mathbf{E}_3 = \{ue \mid u \sim e$ in G, $u \in \mathbf{V}(G), e \in \mathbf{E}(G)\}$. Then

$$\begin{aligned}
SO(T(G)) &= \sum_{uv\in \mathbf{E}(T(G))} \sqrt{d_{T(G)}(u)^2 + d_{T(G)}(v)^2} \\
&= \sum_{uv\in \mathbf{E}_1} \sqrt{d_{T(G)}(u)^2 + d_{T(G)}(v)^2} + \sum_{uv\in \mathbf{E}_2} \sqrt{d_{T(G)}(u)^2 + d_{T(G)}(v)^2} \\
&+ \sum_{uv\in \mathbf{E}_3} \sqrt{d_{T(G)}(u)^2 + d_{T(G)}(v)^2} \\
&= \sum_{uv\in \mathbf{E}(G)} \sqrt{(2d_G(u))^2 + (2d_G(v))^2} + \sum_{uvw\in \mathbf{P}} \sqrt{[d_G(u)+d_G(v)]^2 + [d_G(v)+d_G(w)]^2} \\
&+ \sum_{uv\in \mathbf{E}(G)} [\sqrt{(2d_G(u))^2 + [d_G(u)+d_G(v)]^2} + \sqrt{(2d_G(v))^2 + [d_G(u)+d_G(v)]^2}] \\
&= 2SO(G) + \sum_{uvw\in \mathbf{P}} \sqrt{[d_G(u)+d_G(w)]^2 + 2d_G(v)[d_G(u)+d_G(v)+d_G(w)]} \\
&+ \sum_{u\in \mathbf{V}(G)} \sum_{v\sim u\in \mathbf{V}(G)} d_G(u)\sqrt{5d_G(u)^2 + d_G(v)^2 + 2d_G(u)\, d_G(v)}\,.
\end{aligned}$$

□

Theorem 4. *Let G be a graph. The Somber index of the semitotal point graph of G is*

$$SO(T_1(G)) = 2SO(G) + 2\sum_{u\in \mathbf{V}(G)} d_G(u)\sqrt{1+d_G(u)^2}\,.$$

Proof. By definition we have $d_{T_1(G)}(u) = 2d_G(u)$ and $d_{T_1(G)}(e) = 2$. The edge set of $T_1(G)$ can be partitioned into two sets, $\mathbf{E}_1 = \{uv \mid uv \in \mathbf{E}(G)\}$ and $\mathbf{E}_2 = \{ue \mid u \sim e \text{ in G}, u \in \mathbf{V}(G), e \in \mathbf{E}(G)\}$. Then

$$\begin{aligned}
SO(T_1(G)) &= \sum_{uv\in \mathbf{E}(T_1(G))} \sqrt{d_{T_1(G)}(u)^2 + d_{T_1(G)}(v)^2} \\
&= \sum_{uv\in \mathbf{E}_1} \sqrt{d_{T_1(G)}(u)^2 + d_{T_1(G)}(v)^2} + \sum_{uv\in \mathbf{E}_2} \sqrt{d_{T_1(G)}(u)^2 + d_{T_1(G)}(v)^2} \\
&= \sum_{uv\in \mathbf{E}(G)} \sqrt{(2d_G(u))^2 + (2d_G(v))^2} + \sum_{uv\in \mathbf{E}(G)} [\sqrt{(2d_G(u))^2 + 4} + \sqrt{(2d_G(v))^2 + 4}] \\
&= 2SO(G) + 2\sum_{u\in \mathbf{V}(G)} d_G(u)\sqrt{1+d_G(u)^2}\,.
\end{aligned}$$

□

Theorem 5. *Let G be a graph. Let* **P** *be same as in Theorem 3. The Sombor index of the semitotal line graph of G is*

$$\begin{aligned}
SO(T_2(G)) &= \sum_{uvw\in \mathbf{P}} \sqrt{d_G(u)^2 + d_G(w)^2 + 2d_G(v)[d_G(u)+d_G(v)+d_G(w)]} \\
&+ \sum_{u\in \mathbf{V}(G)} \sum_{v\sim u\in \mathbf{V}(G)} d_G(u)\sqrt{2d_G(u)^2 + d_G(v)^2 + 2d_G(u)\, d_G(v)}\,.
\end{aligned}$$

Proof. Partition the edge set of the semitotal line graph into two sets, $\mathbf{E}_1 = \{ue \mid u \sim e \text{ in G}, u \in V(G), e \in E(G)\}$ and $\mathbf{E}_2 = \{ee' \mid e \sim e' \text{ in G}, e, e' \in E(G)\}$. Note, in addition, that $d_{T_2(G)}(u) = d_G(u)$, $d_{T_2(G)}(e) = d_G(u) + d_G(v)$. Then

$$
\begin{aligned}
SO(T_2(G)) &= \sum_{uv \in \mathbf{E}(T_2(G))} \sqrt{d_{T_2(G)}(u)^2 + d_{T_2(G)}(v)^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{d_{T_2(G)}(u)^2 + d_{T_2(G)}(v)^2} + \sum_{uv \in \mathbf{E}_2} \sqrt{d_{T_2(G)}(u)^2 + d_{T_2(G)}(v)^2} \\
&= \sum_{uv \in \mathbf{E}(G)} \left[\sqrt{d_G(u)^2 + [d_G(u) + d_G(v)]^2} + \sqrt{d_G(v)^2 + [d_G(u) + d_G(v)]^2} \right] \\
&\quad + \sum_{uvw \in \mathbf{P}} \sqrt{[d_G(u) + d_G(v)]^2 + [d_G(v) + d_G(w)]^2} \\
&= \sum_{u \in \mathbf{V}(G)} \sum_{v \sim u \in \mathbf{V}(G)} d_G(u) \sqrt{2d_G(u)^2 + d_G(v)^2 + 2d_G(u)\, d_G(v)} \\
&+ \sum_{uvw \in \mathbf{P}} \sqrt{d_G(u)^2 + d_G(w)^2 + 2d_G(v)[d_G(u) + d_G(v) + d_G(w)]}\,.
\end{aligned}
$$

□

3. Sombor index of double graph and strong double graph

Theorem 6. *Let G be a graph and $D^k(G)$ be its iterated double graph. Then*

$$SO(D^k(G)) = 2^{3k}\, SO(G)\,.$$

Proof. We know that $d_{D^k(G)}(u) = 2^k\, d_G(u)$. Partition the edge set $\mathbf{E}(D^k(G))$ into two sets $\mathbf{E}_1$ and $\mathbf{E}_2$, where $\mathbf{E}_1 = \{uv \mid uv \in \mathbf{E}(G)\}$ $\mathbf{E}_2 = \{uv \mid uv \notin \mathbf{E}(G)\}$. It is easy to check that $|\mathbf{E}_1| = 2^k\, m$ and $|\mathbf{E}_2| = 2^k\, m(2^k - 1)$. Therefore

$$
\begin{aligned}
SO(D^k(G)) &= \sum_{uv \in \mathbf{E}(D^k(G))} \sqrt{d_{D^k(G)}(u)^2 + d_{D^k(G)}(v)^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{d_{D^k(G)}(u)^2 + d_{D^k(G)}(v)^2} + \sum_{uv \in \mathbf{E}_2} \sqrt{d_{D^k(G)}(u)^2 + d_{D^k(G)}(v)^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{(2^k d_G(u))^2 + (2^k d_G(v))^2} + \sum_{uv \in \mathbf{E}_2} \sqrt{(2^k d_G(u))^2 + (2^k d_G(v))^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{2^{2k} d_G(u)^2 + 2^{2k} d_G(v)^2} + \sum_{uv \in \mathbf{E}_2} \sqrt{2^{2k} d_G(u)^2 + 2^{2k} d_G(v)^2} \\
&= 2^{2k} \sum_{uv \in \mathbf{E}(G)} \sqrt{d_G(u)^2 + d_G(v)^2} + 2^{2k}(2^k - 1) \sum_{uv \in \mathbf{E}(G)} \sqrt{d_G(u)^2 + d_G(v)^2} \\
&= 2^{3k} SO(G)\,.
\end{aligned}
$$

□

Corollary 1. *Let G be any graph with n vertices. Then $SO(D(G)) = 8\, SO(G)$.*

Theorem 7. *Let G be a graph with n vertices and m edges. Then the Sombor index of its strong double graph is*

$$SO(SD(G)) = 4 \sum_{uv \in \mathbf{E}(G)} \sqrt{4[d_G(u)^2 + d_G(v)^2] + 4[d_G(u) + d_G(v)] + 2} + \sqrt{2}(4m + n)\,.$$

Proof. We know that $d_{SD(G)}(u) = 2d_G(u) + 1$. Partition the edge set into three sets $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, where $\mathbf{E}_1 = \{uv \mid uv \in \mathbf{E}(G)\}$, $\mathbf{E}_2 = \{uv \mid uv \notin \mathbf{E}(G)\}$ and $\mathbf{E}_3 = \{uu \mid u \in \mathbf{V}(G)\}$. It is easy to check that $|\mathbf{E}_1| = 2m$, $|\mathbf{E}_2| = 2m$, and $|\mathbf{E}_3| = n$. Therefore by definition, we have

$$\begin{aligned}
SO(SD(G)) &= \sum_{uv \in \mathbf{E}(SD(G))} \sqrt{d_{SD(G)}(u)^2 + d_{SD(G)}(v)^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{[2d_G(u) + 1]^2 + [2d_G(v) + 1]^2} + \sum_{uv \in \mathbf{E}_2} \sqrt{[2d_G(u) + 1]^2 + [2d_G(v) + 1]^2} \\
&+ \sum_{uv \in \mathbf{E}_3} \sqrt{2[2d_G(u) + 1]^2} \\
&= 2 \sum_{uv \in E(G)} \sqrt{[2d_G(u) + 1]^2 + [2d_G(v) + 1]^2} \\
&+ 2 \sum_{uv \in \mathbf{E}(G)} \sqrt{[2d_G(u) + 1]^2 + [2d_G(v) + 1]^2} + \sqrt{2}(4m + n) \\
&= 4 \sum_{uv \in \mathbf{E}(G)} \sqrt{4[d_G(u)^2 + d_G(v)^2] + 4[d_G(u) + d_G(v)] + 2} + \sqrt{2}(4m + n)\,.
\end{aligned}$$

□

Theorem 8. *Let G be a graph and $SD^k(G)$ be its iterated strong double graph. Then, for $k \geq 2$*

$$\begin{aligned}
SO(SD^k(G)) &= 2^k \sum_{uv \in \mathbf{E}(G)} \sqrt{2^{2k}[d_G(u)^2 + d_G(v)^2] + 2^{k+1}(2^k - 1)[d_G(u) + d_G(v)] + 2(2^k - 1)^2} \\
&+ \sum_{uv \in \mathbf{E}_2} \sqrt{2^{2k}[d_G(u)^2 + d_G(v)^2] + 2^{k+1}(2^k - 1)[d_G(u) + d_G(v)] + 2(2^k - 1)^2} \\
&+ 2^{k-\frac{1}{2}}(k + 1)[2^{k+1}m + (2^k - 1)n]\,.
\end{aligned}$$

Proof. We know that $d_{SD^k(G)}(u) = 2^k[d_G(u) + 1] - 1$. Partition the edge set $\mathbf{E}(SD^k(G))$ same as in the proof of Theorem 7. It is easy to check that $|\mathbf{E}_1| = 2^k\, m$, $|\mathbf{E}_2| = 2^k m(2^k - 1) + 2^{k-1}n(2^k - k - 2)$, and $|\mathbf{E}_3| = n2^{k-1}(k + 1)$. Therefore, by definition, we have

$$\begin{aligned}
SO(SD^k(G)) &= \sum_{uv\in\mathbf{E}(SD^k(G))} \sqrt{d_{SD^k(G)}(u)^2 + d_{SD^k(G)}(v)^2} \\
&= \sum_{uv\in\mathbf{E}_1} \sqrt{d_{SD^k(G)}(u)^2 + d_{SD^k(G)}(v)^2} + \sum_{uv\in\mathbf{E}_2} \sqrt{d_{SD^k(G)}(u)^2 + d_{SD^k(G)}(v)^2} \\
&+ \sum_{uv\in\mathbf{E}_3} \sqrt{d_{SD^k(G)}(u)^2 + d_{SD^k(G)}(v)^2} \\
&= \sum_{uv\in\mathbf{E}_1} \sqrt{[2^k(d_G(u)+1)-1]^2 + [2^k(d_G(v)+1)-1]^2} \\
&+ \sum_{uv\in\mathbf{E}_2} \sqrt{[2^k(d_G(u)+1)-1]^2 + [2^k(d_G(v)+1)-1]^2} \\
&+ \sum_{uv\in\mathbf{E}_3} \sqrt{2[2^k(d_G(u)+1)-1]^2} \\
&= \sum_{uv\in\mathbf{E}_1} \sqrt{2^{2k}[d_G(u)^2+d_G(v)^2] + 2^{k+1}(2^k-1)[d_G(u)+d_G(v)] + 2(2^k-1)^2} \\
&+ \sum_{uv\in\mathbf{E}_2} \sqrt{2^{2k}[d_G(u)^2+d_G(v)^2] + 2^{k+1}(2^k-1)[d_G(u)+d_G(v)] + 2(2^k-1)^2} \\
&+ \sqrt{2} \sum_{u\in\mathbf{V}(G)} \left[2^k(d_G(u)+1)-1\right] \\
&= 2^k \sum_{uv\in\mathbf{E}(G)} \sqrt{2^{2k}[d_G(u)^2+d_G(v)^2] + 2^{k+1}(2^k-1)[d_G(u)+d_G(v)] + 2(2^k-1)^2} \\
&+ \sum_{uv\in\mathbf{E}_2} \sqrt{2^{2k}[d_G(u)^2+d_G(v)^2] + 2^{k+1}(2^k-1)[d_G(u)+d_G(v)] + 2(2^k-1)^2} \\
&+ 2^{k-\frac{1}{2}}(k+1)[2^{k+1}m + (2^k-1)n]\,.
\end{aligned}$$

□

In particular, for $k = 2$,

$$SO(SD^2(G)) = 16\sqrt{2} \sum_{uv\in\mathbf{E}(G)} \sqrt{8[d_G(u)^2+d_G(v)^2]+12[d_G(u)+d_G(v)]+9} + 6\sqrt{2}(8+3n)\,.$$

4. Sombor index of generalized graph transformations

Proposition 1. [4] *Let G be a graph with n vertices and m edges. Let $u \in \mathbf{V}(G)$ and $e \in \mathbf{E}(G)$. Then the degrees of point and line vertices in G^{xy} are*
(i) $d_{G^{++}}(u) = 2d_G(u)$ and $d_{G^{++}}(e) = 2$.
(ii) $d_{G^{+-}}(u) = m$ and $d_{G^{+-}}(e) = n-2$.
(iii) $d_{G^{-+}}(u) = n-1$ and $d_{G^{-+}}(e) = 2$.
(iv) $d_{G^{--}}(u) = n+m-1-2d_G(u)$ and $d_{G^{--}}(e) = n-2$.

Theorem 9. *Let G be a graph with n vertices and m edges. Then*

$$SO(G^{++}) = 2SO(G) + 2 \sum_{u \in \mathbf{V}(G)} d_G(u)\sqrt{d_G(u)^2 + 1}\,.$$

Proof. Partition the edge set $\mathbf{E}(G^{++})$ into two sets $\mathbf{E}_1$ and $\mathbf{E}_2$, where $\mathbf{E}_1 = \{uv \mid uv \in E(G)\}$ and $\mathbf{E}_2 = \{ue \mid$ the vertex u is incident to the edge e in $G\}$. It is easy to check that $|\mathbf{E}_1| = m$, and $|\mathbf{E}_2| = 2m$. By Proposition 1, if $u \in \mathbf{V}(G)$, then $d_{G^{++}}(u) = 2d_G(u)$ and if $e \in \mathbf{E}(G)$, then $d_{G^{++}}(e) = 2$. Therefore,

$$\begin{aligned}
SO(G^{++}) &= \sum_{uv \in \mathbf{E}(G^{++})} \sqrt{d_{G^{++}}(u)^2 + d_{G^{++}}(v)^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{d_{G^{++}}(u)^2 + d_{G^{++}}(v)^2} + \sum_{ue \in \mathbf{E}_2} \sqrt{d_{G^{++}}(u)^2 + d_{G^{++}}(e)^2} \\
&= \sum_{uv \in \mathbf{E}(G)} \sqrt{4d_G(u)^2 + 4d_G(v)^2} + \sum_{ue \in \mathbf{E}_2} \sqrt{4d_G(u)^2 + 4} \\
&= 2 \sum_{uv \in \mathbf{E}(G)} \sqrt{d_G(u)^2 + d_G(v)^2} + 2 \sum_{ue \in \mathbf{E}_2} \sqrt{d_G(u)^2 + 1} \\
&= 2 \sum_{uv \in \mathbf{E}(G)} \sqrt{d_G(u)^2 + d_G(v)^2} + 2 \sum_{uv \in \mathbf{E}(G)} \left(\sqrt{d_G(u)^2 + 1} + \sqrt{d_G(v)^2 + 1}\right) \\
&= 2SO(G) + 2 \sum_{u \in V(G)} d_G(u)\sqrt{d_G(u)^2 + 1}\,.
\end{aligned}$$

□

In a fully analogous manner, taking into account that $|\mathbf{E}_1| = m$ and $|\mathbf{E}_2| = m(n-2)$ for G^{+-}, $|\mathbf{E}_1| = \binom{n}{2} - m$ and $|\mathbf{E}_2| = 2m$ for G^{-+}, whereas $|\mathbf{E}_1| = \binom{n}{2} - m$ and $|\mathbf{E}_2| = m(n-2)$ for G^{--}, we prove:

Theorem 10. *Let G be a graph with n vertices and m edges. Then*

$$SO(G^{+-}) = \sqrt{2}m^2 + m(n-2)\sqrt{m^2 + (n-2)^2}$$

$$SO(G^{-+}) = \sqrt{2}(n-1)\left[\binom{n}{2} - m\right] + 2m\sqrt{(n-1)^2 + 4}\,,$$

and

$$\begin{aligned}
SO(G^{--}) &= \sum_{uv \notin \mathbf{E}(G)} \sqrt{2(n+m-1)^2 + 4(d_G(u)^2 + d_G(v)^2) - 4(n+m-1)(d_G(u) + d_G(v))} \\
&+ \sum_{u \in \mathbf{V}(G)} (m - d_G(u))\sqrt{2n^2 + (m-1)^2 + 2mn - 6n + 4d_G(u)^2 - 4d_G(u)(n+m-1)}\,.
\end{aligned}$$

Corollary 2. *Let G_1 and G_2 be two different graphs, having same number of vertices and same number of edges. Then*
(i) $SO(G_1^{+-}) = SO(G_2^{+-})$
(ii) $SO(G_1^{-+}) = SO(G_2^{-+})$.

5. Sombor index of $\overline{G^{xy}}$

Let $\overline{G}$ denote the complement of the graph G. Examples illustrating the graphs $\overline{G^{xy}}$ can be found in Fig. 1.

Proposition 2. [13] *Let G be a graph with n vertices and m edges. Let $u \in \mathbf{V}(G)$ and $e \in \mathbf{E}(G)$. Then the degrees of point and line vertices in $\overline{G^{xy}}$ are*
(i) $d_{\overline{G^{++}}}(u) = n + m - 1 - 2d_G(u)$ and $d_{\overline{G^{++}}}(e) = n + m - 3$.
(ii) $d_{\overline{G^{+-}}}(u) = n - 1$ and $d_{\overline{G^{+-}}}(e) = m + 1$.
(iii) $d_{\overline{G^{-+}}}(u) = m$ and $d_{\overline{G^{-+}}}(e) = n + m - 3$.
(iv) $d_{\overline{G^{--}}}(u) = 2d_G(u)$ and $d_{\overline{G^{--}}}(e) = m + 1$.

Theorem 11. *Let G be a graph with n vertices and m edges. Then*

$$SO\left(\overline{G^{-+}}\right) = \sqrt{2}m^2 + m(n-2)\sqrt{m^2 + (n+m-3)^2} + \frac{m(m-1)}{\sqrt{2}}(n+m-3)\,.$$

Proof. Partition the edge set $\mathbf{E}(\overline{G^{-+}})$ into three sets $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, where $\mathbf{E}_1 = \{uv \mid uv \in \mathbf{E}(G)\}$, $\mathbf{E}_2 = \{ue \mid \text{the vertex } u \text{ is not incident to the edge } e \text{ in } G\}$ and $\mathbf{E}_3 = \{ef \mid e, f \in \mathbf{E}(G)\}$. It is easy to check that $|\mathbf{E}_1| = m$, $|\mathbf{E}_2| = m(n-2)$ and $\mathbf{E}_3 = \binom{m}{2}$. By Proposition 2, if $u \in \mathbf{V}(G)$, then $d_{\overline{G^{-+}}}(u) = m$, whereas if $e \in \mathbf{E}(G)$, then $d_{\overline{G^{-+}}}(e) = n + m - 3$. Therefore,

$$\begin{aligned}
SO\left(\overline{G^{-+}}\right) &= \sum_{uv \in \mathbf{E}(\overline{G^{-+}})} \sqrt{d_{\overline{G^{-+}}}(u)^2 + d_{\overline{G^{-+}}}(v)^2} \\
&= \sum_{uv \in \mathbf{E}_1} \sqrt{d_{\overline{G^{-+}}}(u)^2 + d_{\overline{G^{-+}}}(v)^2} + \sum_{ue \in \mathbf{E}_2} \sqrt{d_{\overline{G^{-+}}}(u)^2 + d_{\overline{G^{-+}}}(e)^2} \\
&+ \sum_{ef \in \mathbf{E}_3} \sqrt{d_{\overline{G^{-+}}}(e)^2 + d_{\overline{G^{-+}}}(f)^2} \\
&= \sum_{uv \in \mathbf{E}(G)} \sqrt{m^2 + m^2} + \sum_{ue \in E_2} \sqrt{m^2 + (n+m-3)^2} \\
&+ \sum_{ef \in \mathbf{E}_3} \sqrt{(n+m-3)^2 + (n+m-3)^2} \\
&= \sum_{uv \in \mathbf{E}(G)} \sqrt{2}m + \sum_{ue \in E_2} \sqrt{m^2 + (n+m-3)^2} + \sum_{ef \in \mathbf{E}_3} \sqrt{2}(n+m-3) \\
&= \sqrt{2}m^2 + m(n-2)\sqrt{m^2 + (n+m-3)^2} + \frac{m(m-1)}{\sqrt{2}}(n+m-3)\,.
\end{aligned}$$

□

Analogously, noting that in the case of the graph $\overline{G^{++}}$, $|\mathbf{E}_1| = \binom{n}{2} - m$, $|\mathbf{E}_2| = m(n-2)$, and $\mathbf{E}_3 = \binom{m}{2}$, we arrive at:

Theorem 12. *Let G be a graph with n vertices and m edges. Then*

$$\begin{aligned} SO\left(\overline{G^{++}}\right) &= \sum_{uv\notin \mathbf{E}(G)} \sqrt{[n+m-1-2d_G(u)]^2+[n+m-1-2d_G(v)]^2} \\ &+ \sum_{u\in \mathbf{V}(G)} [m-d_G(u)]\sqrt{[n+m-2-d_G(u)]^2+(n+m-3)^2} \\ &+ \frac{m(m-1)(n+m-3)}{\sqrt{2}}\,. \end{aligned}$$

Theorem 13. *Let G be a graph with n vertices and m edges. Then*

$$SO\left(\overline{G^{+-}}\right) = \sqrt{2}(n-1)\left[\binom{n}{2}-m\right] + 2m\sqrt{(n-1)^2+(m+1)^2} + \frac{m(m-1)(m+1)}{\sqrt{2}}\,.$$

Proof. Partition the edge set $\mathbf{E}(\overline{G^{+-}})$ into three sets $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$, where

$$\begin{aligned} \mathbf{E}_1 &= \{uv \mid uv\notin \mathbf{E}(G)\}, \\ \mathbf{E}_2 &= \{ue \mid \text{the vertex } u \text{ is incident to the edge } e \text{ in } G\}, \\ \mathbf{E}_3 &= \{ef \mid e,f\in \mathbf{E}(G)\}. \end{aligned}$$

Then, $|\mathbf{E}_1| = \binom{n}{2}-m$, $|\mathbf{E}_2| = 2m$, and $\mathbf{E}_3 = \binom{m}{2}$. By Proposition 2, if $u\in \mathbf{V}(G)$, then $d_{\overline{G^{+-}}}(u) = n-1$, whereas if $e\in \mathbf{E}(G)$, then $d_{\overline{G^{+-}}}(e) = m+1$. Therefore,

$$\begin{aligned} SO\left(\overline{G^{+-}}\right) &= \sum_{uv\in \mathbf{E}(\overline{G^{+-}})} \sqrt{d_{\overline{G^{+-}}}(u)^2+d_{\overline{G^{+-}}}(v)^2} \\ &= \sum_{uv\in \mathbf{E}_1} \sqrt{d_{\overline{G^{+-}}}(u)^2+d_{\overline{G^{+-}}}(v)^2} + \sum_{ue\in \mathbf{E}_2} \sqrt{d_{\overline{G^{+-}}}(u)^2+d_{\overline{G^{+-}}}(e)^2} \\ &+ \sum_{ef\in \mathbf{E}_3} \sqrt{d_{\overline{G^{+-}}}(e)^2+d_{\overline{G^{+-}}}(f)^2} \\ &= \sum_{uv\notin \mathbf{E}(G)} \sqrt{(n-1)^2+(n-1)^2} + \sum_{ue\in \mathbf{E}_2} \sqrt{(n-1)^2+(m+1)^2} \\ &+ \sum_{ef\in \mathbf{E}_3} \sqrt{(m+1)^2+(m+1)^2} \\ &= \sum_{uv\notin \mathbf{E}(G)} \sqrt{2}(n-1) + \sum_{ue\in E_2} \sqrt{(n-1)^2+(m+1)^2} + \sum_{ef\in \mathbf{E}_3} \sqrt{2}(m+1) \\ &= \sqrt{2}(n-1)\left[\binom{n}{2}-m\right] + 2m\sqrt{(n-1)^2+(m+1)^2} + \frac{m(m-1)(m+1)}{\sqrt{2}}\,. \end{aligned}$$

□

Using the same notation as in Theorem 13, and noting that in the case of the graph $\overline{G^{--}}$, $|\mathbf{E}_1| = m$, $|\mathbf{E}_2| = 2m$, and $\mathbf{E}_3 = \binom{m}{2}$, we obtain:

Theorem 14. *Let G be a graph with n vertices and m edges. Then*

$$SO\left(\overline{G^{--}}\right) = 2SO(G) + \sum_{u \in V(G)} d_G(u)\sqrt{(4d_G(u))^2 + (m+1)^2} + \frac{m(m-1)(m+1)}{\sqrt{2}}.$$

By this, the Sombor indices of all graphs G^{xy} and $\overline{G^{xy}}$ have been calculated.

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