

*Short Note*

## Remarks on the restrained Italian domination number in graphs

Lutz Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University, 52056 Aachen, Germany  
volkm@math2.rwth-aachen.de

*Received: 2 October 2021; Accepted: 10 December 2021*

*Published Online: 13 December 2021*

**Abstract:** Let  $G$  be a graph with vertex set  $V(G)$ . An Italian dominating function (IDF) is a function  $f : V(G) \rightarrow \{0, 1, 2\}$  having the property that  $f(N(u)) \geq 2$  for every vertex  $u \in V(G)$  with  $f(u) = 0$ , where  $N(u)$  is the neighborhood of  $u$ . If  $f$  is an IDF on  $G$ , then let  $V_0 = \{v \in V(G) : f(v) = 0\}$ . A restrained Italian dominating function (RIDF) is an Italian dominating function  $f$  having the property that the subgraph induced by  $V_0$  does not have an isolated vertex. The weight of an RIDF  $f$  is the sum  $\sum_{v \in V(G)} f(v)$ , and the minimum weight of an RIDF on a graph  $G$  is the restrained Italian domination number. We present sharp bounds for the restrained Italian domination number, and we determine the restrained Italian domination number for some families of graphs.

**Keywords:** Italian domination, restrained Italian domination, restrained domination

**AMS Subject classification:** 05C69

### 1. Introduction

For definitions and notations not given here we refer to [14]. We consider simple graphs  $G$  with vertex set  $V = V(G)$  and edge set  $E = E(G)$ . The *order* of  $G$  is  $n = n(G) = |V|$ . The *open neighborhood* of a vertex  $v$  is the set  $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$  and its *closed neighborhood* is the set  $N[v] = N_G[v] = N(v) \cup \{v\}$ . The *degree* of vertex  $v \in V$  is  $d(v) = d_G(v) = |N(v)|$ . The *maximum degree* and *minimum degree* of  $G$  are denoted by  $\Delta = \Delta(G)$  and  $\delta = \delta(G)$ , respectively. The *complement* of a graph  $G$  is denoted by  $\overline{G}$ . For a subset  $D$  of vertices in a graph  $G$ , we denote by  $G[D]$  the subgraph of  $G$  induced by  $D$ . A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. A *strong support vertex* is a support vertex adjacent to more than one leaf. A set  $S \subseteq V(G)$  is called a *dominating set* if every vertex is either an element of  $S$  or is adjacent to an element of  $S$ . The *domination number*  $\gamma(G)$  of a graph  $G$  is the minimum cardinality of a dominating set of  $G$ . A

*restrained dominating set* is a set  $S \subseteq V(G)$  where every vertex in  $V(G) \setminus S$  is adjacent to a vertex in  $S$  as well as to another vertex in  $V(G) \setminus S$ . The *restrained domination number* of  $G$ , denoted by  $\gamma_r(G)$ , is the smallest cardinality of a restrained dominating set of  $G$ . Restrained domination was formally defined by Domke, Hattingh, S.T. Hedetniemi, Laskar and Markus in their 1999 paper [12]. For more information on this parameter we refer the reader to the survey paper [13]. We write  $P_n$  for the path of order  $n$ ,  $C_n$  for the cycle of length  $n$  and  $K_n$  for the complete graph of order  $n$ . Also, let  $K_{n_1, n_2, \dots, n_p}$  denote the complete  $p$ -partite graph with vertex set  $S_1 \cup S_2 \cup \dots \cup S_p$  where  $|S_i| = n_i$  for  $1 \leq i \leq p$ . For  $n \geq 2$ , the *star*  $K_{1, n-1}$  has one vertex of degree  $n-1$  and  $n-1$  leaves. A *subdivision* of an edge  $uv$  is obtained by introducing a new vertex  $w$  and replacing the edge  $uv$  with edges  $uw$  and  $wv$ . A *wounded spider* is a tree obtained from  $K_{1, r}$ ,  $r \geq 1$ , by subdividing at most  $r-1$  of its edges. By  $S_{p, q}$  we denote the *double star*, where one center vertex is adjacent to  $p$  leaves and the other one to  $q$  leaves.

Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [11] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (see [7–10]). A *Roman dominating function* (RDF) on a graph  $G$  is a function  $f : V(G) \rightarrow \{0, 1, 2\}$  satisfying the condition that each vertex  $u$  with  $f(u) = 0$  has a neighbor  $v$  with  $f(v) = 2$ . The weight of an RDF  $f$  is the sum  $w(f) = \sum_{v \in V(G)} f(v)$ . The *Roman domination number*  $\gamma_R(G)$  equals the minimum weight of a Roman dominating function on  $G$ . An RDF of  $G$  with weight  $\gamma_R(G)$  is called a  $\gamma_R(G)$ -function. For an RDF  $f$ , one can denote  $f = (V_0, V_1, V_2)$ , where  $V_i = \{v \in V(G) : f(v) = i\}$  for  $i = 0, 1, 2$ .

In 2015, Roushini Leely Pushpam and Padmapriya [17] defined the *restrained Roman dominating function* (RRDF) as a Roman dominating function  $f$  with the property that  $G[V_0]$  does not have an isolated vertex. The weight of an RRDF  $f$  is the sum  $w(f) = \sum_{v \in V(G)} f(v)$ . The *restrained Roman domination number*  $\gamma_{rR}(G)$  equals the minimum weight of a restrained Roman dominating function on  $G$ . An RRDF of  $G$  with weight  $\gamma_{rR}(G)$  is called a  $\gamma_{rR}(G)$ -function. The restrained Roman domination has been studied by several authors (see [3, 19]).

In 2016, Chellali, Haynes, S.T. Hedetniemi and MacRae [6] defined a new variant of Roman dominating functions, the so called Italian dominating functions. An *Italian dominating function* (IDF) on a graph  $G$  is a function  $f : V \rightarrow \{0, 1, 2\}$  having the property that  $f(N(u)) \geq 2$  for each vertex  $u$  with  $f(u) = 0$ . The weight of an IDF  $f$  is the sum  $w(f) = \sum_{v \in V(G)} f(v)$ , and the minimum weight of an IDF in a graph  $G$  is the *Italian domination number*, denoted by  $\gamma_I(G)$ . In [1, 2, 4, 5, 15, 20], the authors consider variants of Italian domination.

In 2021, Samadi, Alishahi, Masoumi and Mojdeh [18] defined the *restrained Italian dominating function* (RIDF) as an IDF  $f$  having the property that the subgraph induced by  $V_0$  does not have an isolated vertex. The weight of an RIDF  $f$  is the sum  $\sum_{v \in V(G)} f(v)$ , and the minimum weight of an RIDF on a graph  $G$  is the *restrained Italian domination number*, denoted by  $\gamma_{rI}(G)$ . An RIDF of  $G$  with weight  $\gamma_{rI}(G)$  is called a  $\gamma_{rI}(G)$ -function. For an RIDF  $f$ , one can denote  $f = (V_0, V_1, V_2)$ , where  $V_i = \{v \in V(G) : f(v) = i\}$  for  $i = 0, 1, 2$ . Clearly,  $\gamma_{rI}(G) \leq \gamma_{rR}(G)$ .

In this paper, we present further bounds and Nordhaus-Gaddum type results for the restrained Italian domination number. In addition, we determine the restrained Italian domination number for some families of graphs.

We make use of the following results.

**Proposition 1.** [12] *If  $n \geq 2$  is an integer, then  $\gamma_r(K_{1,n-1}) = n$ .*

**Proposition 2.** [12] *If  $T$  is a tree of order  $n \geq 3$ , then  $\gamma_r(T) \geq \Delta(T)$ . Furthermore,  $\gamma_r(T) = \Delta(T)$  if and only if  $T$  is a wounded spider which is not a star.*

**Proposition 3.** [17] *Let  $P_n$  be a path of order  $n \geq 4$ . Then  $\gamma_{rR}(P_n) = \frac{2n+3+r}{3}$ , where  $n \equiv r \pmod{3}$  for  $r \in \{1, 2, 3\}$ .*

**Proposition 4.** [17] *Let  $C_n$  be a cycle of order  $n \geq 3$ . Then  $\gamma_{rR}(C_n) = \frac{2n+3+r}{3}$ , when  $n \equiv r \pmod{3}$  for  $r \in \{1, 2\}$  and  $\gamma_{rR}(C_n) = \frac{2n}{3}$ , when  $n \equiv 0 \pmod{3}$ .*

**Proposition 5.** *If  $n \geq 1$ , then  $\gamma_{rR}(P_n) = \gamma_{rI}(P_n)$  and  $\gamma_{rR}(C_n) = \gamma_{rI}(C_n)$  for  $n \geq 3$ .*

*Proof.* Let  $G \in \{P_n, C_n\}$ . If  $f$  is an RIDF, then  $\Delta(G) \leq 2$  implies that every vertex  $u$  with  $f(u) = 0$  has a neighbor  $v$  with  $f(v) = 0$  and a neighbor  $w$  with  $f(w) = 2$ . Therefore  $f$  is also an RRDF and thus  $\gamma_{rR}(G) \leq \gamma_{rI}(G)$ . Because of  $\gamma_{rR}(G) \geq \gamma_{rI}(G)$ , we obtain  $\gamma_{rR}(G) = \gamma_{rI}(G)$ .  $\square$

The following inequality chain is obviously.

**Proposition 6.** *If  $G$  is a graph, then  $\gamma_r(G) \leq \gamma_{rI}(G) \leq \gamma_{rR}(G) \leq 2\gamma_r(G)$ .*

Propositions 1 and 6 lead to the next observation immediately.

**Proposition 7.** *If  $n \geq 2$ , then  $\gamma_{rI}(K_{1,n-1}) = n$ .*

**Proposition 8.** [18] *If  $G$  is a connected graph of order  $n \geq 2$ , then  $\gamma_{rI}(G) \leq n$  with equality if and only if  $G$  is star or  $G \in \{C_4, C_5, P_4, P_5, P_6\}$ .*

Let  $C_{5,5}$  be the graph of order 9 consisting of two cycles of length five with one vertex in common. Let  $R_6$  be the graph of order 6 consisting of a cycle  $C_5 = v_1v_2v_3v_4v_5v_1$  with a further vertex  $y$  and two further edges  $yv_1$  and  $yv_3$ . It is straightforward to verify that  $\gamma_{rI}(C_{5,5}) = 8$  and  $\gamma_{rI}(R_6) = 5$ .

**Proposition 9.** [18] *Let  $G$  be a connected graph of order  $n$  with  $\delta(G) \geq 2$ . If  $G \notin \{C_3, C_4, C_5, C_7, C_8, K_{2,3}, R_6, C_{5,5}\}$ , then  $\gamma_{rI}(G) \leq n - 2$ .*

**Proposition 10.** [18] *If  $G$  is a graph of order  $n \geq 2$ , then  $\gamma_{rI}(G) \geq 2$ , with equality if and only if  $\Delta(G) = n - 1$  and  $G$  contains a vertex  $w$  of maximum degree such that  $\delta(G[N_G(w)]) \geq 1$  or  $G$  contains two vertices  $u$  and  $v$  such that the remaining  $n - 2$  vertices are adjacent to  $u$  and  $v$  and  $G[V(G) \setminus \{u, v\}]$  has no isolated vertex.*

The proof of the next observation is easy and therefore omitted.

**Proposition 11.** (i)  $\gamma_{rI}(K_n) = 2$  for  $n \geq 2$ ,

(ii)  $\gamma_{rI}(K_{m,n}) = 4$  for  $m, n \geq 2$

(iii) Let  $K_{n_1, n_2, \dots, n_p}$  be the complete  $p$ -partite graph such that  $p \geq 3$  and  $n_1 \leq n_2 \leq \dots \leq n_p$ . Then  $\gamma_{rI}(K_{1, n_2, \dots, n_p}) = \gamma_{rI}(K_{2, n_2, \dots, n_p}) = 2$  and  $\gamma_{rI}(K_{n_1, n_2, \dots, n_p}) = 3$  for  $n_1 \geq 3$ .

Let  $p \geq 1$  and  $0 \leq r \leq 2$  be integers, and let  $G_{3p+r}$  be the graph obtained from a cycle  $C_{3p+r} = v_1 v_2 \dots v_{3p+r} v_1$  by adding two leaves  $a_i$  and  $b_i$  to each vertex  $v_i$  for  $1 \leq i \leq 3p+r$ . Clearly,  $\gamma_{rI}(G_{3p+r}) = 6p + 2r$ . Now let  $f$  be a  $\gamma_{rR}(G_{3p+r})$ -function. Then we observe that

$$\begin{aligned} f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(a_i) + f(a_{i+1}) + f(a_{i+2}) \\ + f(b_i) + f(b_{i+1}) + f(b_{i+2}) \geq 7. \end{aligned}$$

This leads to  $\gamma_{rR}(G_{3p+r}) \geq 7p + 2r$ . Hence we observe

**Proposition 12.** *There exist graphs  $G$  for which  $\gamma_{rR}(G) - \gamma_{rI}(G)$  can be made arbitrarily large.*

## 2. Bounds

The *clique number*  $c(G)$  of a graph  $G$  is the maximum order among the complete subgraphs of  $G$ .

**Observation 1.** Let  $G$  be a graph of order  $n$ .

(i) If  $\delta(G) \geq 3$ , then  $\gamma_{rI}(G) \leq n + 1 - \delta(G)$ .

(ii) If  $c(G) \geq 3$ , then  $\gamma_{rI}(G) \leq n + 2 - c(G)$ .

*Proof.* (i) Let  $\delta = \delta(G) \geq 3$ , and let  $z$  be a vertex of minimum degree with the neighbors  $v_1, v_2, \dots, v_\delta$ . Define the function  $f$  by  $f(z) = f(v_1) = f(v_2) = \dots = f(v_{\delta-2}) = 0$  and  $f(x) = 1$  otherwise. Then  $G[V_0]$  is connected of order at least two, and every vertex of  $V_0$  has at least two neighbors of weight one. Therefore  $f$  is an RIDF on  $G$  of weigh  $n + 1 - \delta$  and thus  $\gamma_{rI}(G) \leq n + 1 - \delta(G)$ .

(ii) Let  $c = c(G) \geq 3$ , and let  $u_1, u_2, \dots, u_c$  be the vertices of a clique of  $G$ . Define the function  $f$  by  $f(u_1) = f(u_2) = \dots = f(u_{c-1}) = 0$ ,  $f(u_c) = 2$  and  $f(x) = 1$  otherwise. Then it is easy to see that  $f$  is an RIDF on  $G$  of weigh  $n + 2 - c$  and thus  $\gamma_{rI}(G) \leq n + 2 - c(G)$ .  $\square$

The complete graphs demonstrate that Observation 1 (i) and (ii) are sharp. In addition, let  $H$  be the graph obtained from a complete graph  $K_{n-1}$  ( $n \geq 4$ ) by adding a leaf  $w$ . Then it is straightforward to verify that  $\gamma_{rI}(H) = n + 2 - c(H)$ . This is a further example which shows that Observation 1 (ii) is sharp.

**Theorem 2.** [18] *If  $T$  is a tree of order  $n \geq 3$  different from the double star  $S_{2,2}$ , then  $\gamma_{rI}(T) \geq \frac{n+3}{2}$ .*

If  $\Delta(T) \geq \frac{n(T)+2}{2}$ , then the next lower bound on  $\gamma_{rI}(T)$  is better than this one in Theorem 2.

**Theorem 3.** *If  $T$  is a tree, then  $\gamma_{rI}(T) \geq \Delta(T) + 1$ .*

*Proof.* Let  $n$  be the order of  $T$ . If  $1 \leq n \leq 3$ , then  $\gamma_{rI}(T) = n = \Delta(T) + 1$ . Let now  $n \geq 4$ . If  $T$  is a star, then Proposition 7 implies  $\gamma_{rI}(T) = n = \Delta(T) + 1$ . If  $T$  is a wounded spider, which is not a star, then it is easy to verify that  $\gamma_{rI}(T) \geq n - 1 \geq \Delta(T) + 1$ . If  $T$  is not a wounded spider, then it follows from Propositions 2 and 6 that  $\gamma_{rI}(T) \geq \gamma_r(T) \geq \Delta(T) + 1$ .  $\square$

Let  $S_{2,q}$  be the double star with  $q \geq 1$ . Then  $\gamma_{rI}(S_{2,q}) = \Delta(S_{2,q}) + 1$ . These double stars and the stars demonstrate that Theorem 3 is sharp.

**Theorem 4.** *Let  $L$  be the set of leaves of a connected graph  $G$ . If  $|L| \geq 1$ , then  $\gamma_{rI}(G) \geq |L|$  with equality if and only if  $G$  is not a star and each vertex  $v \in V(G) \setminus L$  is a strong support vertex.*

*Proof.* Let  $f$  be an RIDF on  $G$ . Then  $f(u) \geq 1$  for each  $u \in L$  and so  $\gamma_{rI}(G) \geq |L|$ . Now let  $G$  be not a star, and let each vertex  $v \in V(G) \setminus L$  be a strong support vertex. Define the function  $f$  by  $f(x) = 1$  for  $x \in L$  and  $f(x) = 0$  for  $x \in V(G) \setminus L$ . Then  $f(N(x)) \geq 2$  for each  $x \in V(G) \setminus L$ . Since  $G$  is connected and not a star,  $G - L$  is connected and  $|V(G) \setminus L| \geq 2$ . Thus  $f$  is an RIDF on  $G$  and therefore  $\gamma_{rI}(G) = |L|$ .

Conversely, assume that  $\gamma_{rI}(G) = |L|$ . Then  $G$  is not a star. Let  $f$  be a  $\gamma_{rI}(G)$ -function. Since  $f(u) \geq 1$  for each  $u \in L$ , we note that  $f(u) = 1$  for each  $u \in L$  and  $f(x) = 0$  for each  $x \in V(G) \setminus L$ . Assume first that there exists a vertex  $w \in V(G) \setminus L$  which is not a support vertex. Then  $N[w] \subseteq V(G) \setminus L$  and  $f(N[w]) \geq 1$ . This leads to the contradiction  $\gamma_{rI}(G) \geq |L| + 1$ . Hence each vertex  $x \in V(G) \setminus L$  is a support vertex. Assume that there exists a vertex  $u \in V(G) \setminus L$  with exactly one leaf neighbor  $v$ . It follows that  $N(u) \setminus \{v\} \subseteq V(G) \setminus L$ . If  $f(u) = 0$ , then  $f(v) = 2$  or  $f(x) = 1$  for at least one vertex  $y \in V(G) \setminus L$ . In both cases we arrive at the contradiction

$\gamma_{rI}(G) \geq |L| + 1$ . If  $f(u) \geq 1$ , then we also obtain the contradiction  $\gamma_{rI}(G) \geq |L| + 1$ . Consequently, each vertex  $v \in V(G) \setminus L$  is a strong support vertex.  $\square$

### 3. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [16], Nordhaus and Gaddum discussed this problem for the chromatic number. We present such inequalities for the restrained Italian domination number

**Theorem 5.** *If  $G$  is a graph of order  $n \geq 3$ , then  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \geq 5$ .*

*Proof.* Assume, without loss of generality, that  $\gamma_{rI}(G) \leq \gamma_{rI}(\overline{G})$ . According to Proposition 10 we only need to show that if  $\gamma_{rI}(G) = 2$ , then  $\gamma_{rI}(\overline{G}) \geq 3$ . Assume that  $\gamma_{rI}(G) = 2$ . It follows from Proposition 10 that  $\Delta(G) = n - 1$  and  $G$  contains a vertex  $w$  of maximum degree such that  $\delta(G[N_G(w)]) \geq 1$  or  $G$  contains two vertices  $u$  and  $v$  such that the remaining  $n - 2$  vertices are adjacent to  $u$  and  $v$  and  $G[V(G) \setminus \{u, v\}]$  has no isolated vertex. In the first case,  $\overline{G} = H \cup \{w\}$ , where  $w$  is an isolated vertex of  $\overline{G}$ . Since  $n(H) \geq 2$ , Proposition 10 leads to  $\gamma_{rI}(\overline{G}) \geq \gamma_{rI}(H) + 1 \geq 3$ . In the second case, we can assume, without loss of generality, that  $u$  and  $v$  are not adjacent in  $G$ . Thus  $\overline{G} = H \cup \{uv\}$ , where  $uv$  is an isolated edge of  $\overline{G}$ . Since  $n(H) \geq 1$ , we deduce that  $\gamma_{rI}(\overline{G}) \geq \gamma_{rI}(H) + 2 \geq 3$ . This completes the proof.  $\square$

**Example 1.** Let  $Wd(2, p)$  be the *windmill graph* consisting of a center vertex  $z$  which is adjacent to the vertices of  $p$  copies of the complete graph  $K_2$ . If  $p \geq 3$ , then  $\overline{Wd(2, p)}$  consists of an isolated vertex  $z$  and a complete  $p$ -partite graph  $K_{n_1, n_2, \dots, n_p}$  such that  $n_1 = n_2 = \dots = n_p = 2$ . Hence it follows from Proposition 11 (iii) that  $\gamma_{rI}(\overline{Wd(2, p)}) = 3$ . Thus we obtain  $\gamma_{rI}(Wd(2, p)) + \gamma_{rI}(\overline{Wd(2, p)}) = 5$ .

Example 1 shows that Theorem 5 is sharp.

**Theorem 6.** *If  $G$  is a graph  $G$  of order  $n \geq 6$ , then  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 2$ .*

*Proof.* If  $G$  or  $\overline{G}$  is neither a star nor the path  $P_6$ , then it follows from Proposition 8 that  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq (n - 1) + (n - 1) = 2n - 2$ . Assume next, without loss of generality, that  $G$  is a star. Then  $\overline{G}$  is the union of an isolated vertex and a complete graph  $K_{n-1}$ . Thus Propositions 7 and 11 (i) imply  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n + 3 \leq 2n - 2$ . Finally, if, without loss of generality,  $G = P_6$ , then it is easy to see that  $\gamma_{rI}(\overline{P_6}) = 3$  and so  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + 3 = 9 \leq 2n - 2$ .  $\square$

If  $G \in \{P_1, P_2, P_3, P_4, C_5\}$ , then we observe that  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) = 2n$ . Therefore the condition  $n \geq 6$  in Theorem 6 is necessary.

**Theorem 7.** *If  $G$  is a graph of order  $n \geq 7$ , then  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$ .*

*Proof.* Assume, without loss of generality, that  $\delta(G) \leq \delta(\overline{G})$ . We distinguish four cases.

**Case 1.** Assume that  $\delta(G) = 0$ . Let  $u$  be a vertex such that  $d_G(u) = 0$ . Let  $x_1, x_2, \dots, x_{n-1}$  be the vertices of  $G - u$  such that  $d_{\overline{G}}(x_1), d_{\overline{G}}(x_2), \dots, d_{\overline{G}}(x_k) \geq 2$  and  $d_{\overline{G}}(x_{k+1}) = d_{\overline{G}}(x_{k+2}) = \dots = d_{\overline{G}}(x_{n-1}) = 1$ . If  $k \geq 2$ , then the function  $f$  with  $f(u) = 2$ ,  $f(x_1) = f(x_2) = \dots = f(x_k) = 0$  and  $f(x_{k+1}) = f(x_{k+2}) = \dots = f(x_{n-1}) = 1$  is an RIDF on  $\overline{G}$  of weight  $n - k + 1$ . If  $k \geq 5$ , then it follows that  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n + (n - k + 1) \leq 2n - 4$ . If  $2 \leq k \leq 4$ , then  $\gamma_{rI}(G - u) = 2$  according to Proposition 10. This leads to  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 3 + n - k + 1 \leq 2n - 4$ . If  $d_{\overline{G}}(x_1) = d_{\overline{G}}(x_2) = \dots = d_{\overline{G}}(x_{n-1}) = 1$ , then  $G - u$  is the complete graph, and we deduce from Proposition 11 (i) that  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 3 + n \leq 2n - 4$ .

**Case 2** Assume that  $\delta(G) = 1$ . Let  $u$  be a vertex such that  $d_G(u) = 1$ , and let  $v$  be adjacent to  $u$  in  $G$ . Let  $H = G - v$ , and let  $x_1, x_2, \dots, x_{n-2}$  be the vertices of  $G - \{u, v\}$  such that  $d_{\overline{H}}(x_1), d_{\overline{H}}(x_2), \dots, d_{\overline{H}}(x_k) \geq 2$  and  $d_{\overline{H}}(x_{k+1}) = d_{\overline{H}}(x_{k+2}) = \dots = d_{\overline{H}}(x_{n-1}) = 1$ . If  $k \geq 2$ , then the function  $f$  with  $f(u) = 2$ ,  $f(x_1) = f(x_2) = \dots = f(x_k) = 0$  and  $f(v) = f(x_{k+1}) = f(x_{k+2}) = \dots = f(x_{n-1}) = 1$  is an RIDF on  $\overline{G}$  of weight  $n - k + 1$ . If  $k \geq 5$ , then it follows that  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n + (n - k + 1) \leq 2n - 4$ . If  $2 \leq k \leq 4$  and  $n \geq 8$ , then  $\gamma_{rI}(G) = 4$  according to Proposition 10. This leads to  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + n - k + 1 \leq 2n - 4$ .

Let next  $k = 4$  and  $n = 7$ . If  $v$  is adjacent to  $x_5$  in  $G$ , then the function  $f$  with  $f(u) = 2$ ,  $f(v) = f(x_5) = 0$  and  $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 1$  is an RIDF of  $G$  and therefore  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + n - k + 1 = n + 3 = 10 = 2n - 4$ . If  $v$  is adjacent to  $x_i$  in  $G$  for one  $i \in \{1, 2, 3, 4\}$ , say to  $x_4$ , then the function  $f$  with  $f(v) = 2$ ,  $f(x_4) = f(x_5) = 0$  and  $f(x_1) = f(x_2) = f(x_3) = f(u) = 1$  is an RIDF of  $G$  and so  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + n - k + 1 = n + 3 = 10 = 2n - 4$ . It remains the case that  $v$  is adjacent to  $x_1, x_2, x_3, x_4$  and  $x_5$  in  $\overline{G}$ . Then the function  $f$  with  $f(u) = f(v) = f(x_5) = 1$  and  $f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$  is an RIDF of  $\overline{G}$  and so  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 7 + 3 = 10 = 2n - 4$ . Let next  $2 \leq k \leq 3$  and  $n = 7$ . Then it is easy to see that  $\gamma_{rI}(G) \leq 4$  and hence  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + n - k + 1 \leq 10 = 2n - 4$ . If  $d_{\overline{H}}(x_1) = d_{\overline{H}}(x_2) = \dots = d_{\overline{H}}(x_{n-2}) = 1$ , then  $G - \{u, v\}$  is the complete graph. If  $n \geq 8$ , then we have  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + n \leq 2n - 4$  immediately. Let now  $n = 7$ . If  $v$  has at least two neighbors in  $G - u$ , say  $x_4$  and  $x_5$ , then the function  $f$  with  $f(x_5) = 2$ ,  $f(u) = 1$  and  $f(v) = f(x_1) = f(x_2) = f(x_3) = f(x_4) = 0$  is an RIDF of  $G$  and thus  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 3 + 7 = 10 = 2n - 4$ . It remains the case that  $v$  has at least 4 neighbors in  $\overline{G} - u$ , say  $x_1, x_2, x_3$  and  $x_4$  are neighbors of  $v$  in  $\overline{G} - u$ . Then the function  $f$  with  $f(u) = f(x_1) = 2$ ,  $f(x_5) = 1$  and  $f(v) = f(x_2) = f(x_3) = f(x_4) = 0$  is an RIDF of  $\overline{G}$ , and we obtain  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 4 + 5 \leq 2n - 4$ .

**Case 3** Assume that  $\delta(G) \geq 2$ . Then the assumption  $\delta(G) \leq \delta(\overline{G})$  leads to  $\delta(\overline{G}) \geq 2$ . Assume first that  $G$  is not connected. Let  $H_1$  be a component of  $G$  and let  $H_2 = G - H_1$ . Since  $\delta(G) \geq 2$ , we note that  $|H_1|, |H_2| \geq 3$ . If  $u \in V(H_1)$  and  $v \in V(H_2)$ , then the function  $f$  with  $f(u) = f(v) = 2$  and  $f(x) = 0$  otherwise is an RIDF of

weight 4 on  $\overline{G}$  and therefore  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq n + 4 \leq 2n - 4$  when  $n \geq 8$ . If  $n = 7$ , then let, without loss of generality,  $|H_1| = 3$ . Then  $\gamma_{rI}(H_1) = 2$  and so  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 6 + 4 = 10 = 2n - 4$ . The same holds when  $\overline{G}$  is not connected. Assume next that  $G$  and  $\overline{G}$  are connected. If  $G, \overline{G} \notin \{C_7, C_8, C_{5,5}\}$ , then Proposition 9 leads to  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$ . If  $G \in \{C_7, C_8, C_{5,5}\}$  or  $\overline{G} \in \{C_7, C_8, C_{5,5}\}$ , then it is straightforward to verify that  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$ .

**Case 4** Assume that  $\delta(G) \geq 3$ . Then  $\delta(\overline{G}) \geq 3$ . Now Observation 1 (i) leads to  $\gamma_{rI}(G) + \gamma_{rI}(\overline{G}) \leq 2n - 4$ .  $\square$

By Proposition 7, we have  $\gamma_{rI}(K_{1,5}) = 6$ . Hence it follows that  $\gamma_{rI}(K_{1,5}) + \gamma_{rI}(\overline{K_{1,5}}) = 6 + 3 = 2n - 3$  for  $n = 6$ . In addition, we note that  $\gamma_{rI}(R_6) + \gamma_{rI}(\overline{R_6}) = 6 + 3 = 2n - 3$  for  $n = 6$ . Consequently, the condition  $n \geq 7$  in Theorem 7 is necessary.

## References

- [1] H. Abdollahzadeh Ahangar, M. Chellali, M. Hajjari, and S.M. Sheikholeslami, *Further progress on the total Roman  $\{2\}$ -domination number of graphs*, Bull. Iranian Math. Soc. (In press).
- [2] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami, and J.C. Valenzuela-Tripodoro, *Total Roman 2-dominating functions in graphs*, Discuss. Math. Graph Theory (In press).
- [3] H. Abdollahzadeh Ahangar and S.R. Mirmehdipour, *Bounds on the restrained Roman domination number of a graph*, Commun. Comb. Optim. **1** (2016), no. 1, 75–82.
- [4] F. Azvin and N. Jafari Rad, *Bounds on the double Italian domination number of a graph*, Discuss. Math. Graph Theory (In press).
- [5] F. Azvin, N. Jafari Rad, and L. Volkmann, *Bounds on the outer-independent double Italian domination number*, Commun. Comb. Optim. **6** (2021), no. 1, 123–136.
- [6] M. Chellali, T.W. Haynes, S.T. Hedetniemi, and A. MacRae, *Roman  $\{2\}$ -domination*, Discrete Appl. Math. **204** (2016), 22–28.
- [7] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2020, pp. 365–409.
- [8] ———, *A survey on Roman domination parameters in directed graphs*, J. Combin. Math. Comb. Comput. **151** (2020), 141–171.
- [9] ———, *Varieties of Roman domination II*, AKCE Int. J. Graphs Comb. **17** (2020), no. 3, 966–984.
- [10] ———, *Varieties of Roman domination*, Structures of Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, Berlin/Heidelberg, 2021, pp. 273–307.



- 
- [11] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.
  - [12] G.S. Domke, J.H. Hattingh, S.T. Hedetniemi, R.C. Laskar, and L.R. Markus, *Restrained domination in graphs*, Discrete Math. **203** (1999), no. 1-3, 61–69.
  - [13] J.H. Hattingh and E.F. Joubert, *Restrained and total restrained domination in graphs*, Topics in Domination in Graphs (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer, 2020, pp. 129–150.
  - [14] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.
  - [15] D.A. Mojdeh and L. Volkmann, *Roman  $\{3\}$ -domination (double Italian domination)*, Discrete Appl. Math. **283** (2020), 555–564.
  - [16] E.A. Nordhaus and J.W. Gaddum, *On complementary graphs*, Amer. Math. Monthly **63** (1956), no. 3, 175–177.
  - [17] R.L. Pushpam and S. Padmapriya, *Restrained Roman domination in graphs*, Trans. Comb. **4** (2015), no. 1, 1–17.
  - [18] B. Samadi, M. Alishahi, I. Masoumi, and D.A. Mojdeh, *Restrained Italian domination in graphs*, RAIRO-Oper. Res. **55** (2021), no. 2, 319–332.
  - [19] F. Siahpour, H. Abdollahzadeh Ahangar, and S.M. Sheikholeslami, *Some progress on the restrained Roman domination*, Bull. Malays. Math. Sci. Soc. **44** (2021), no. 2, 733–751.
  - [20] L. Volkmann, *Restrained double Italian domination in graphs*, Commun. Comb. Optim. (In press).