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Research Article

Bounds for fuzzy Zagreb Estrada index

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Abstract: Let $G(V, \sigma, \mu)$ be a fuzzy graph of order n, where $\sigma(u)$ is the vertex membership, $\mu(u, v)$ is membership value of an edge and $\mu(u)$ is the strength of vertex. The first fuzzy Zagreb index is the sum $\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)$ where $u_iu_j \in \mu$ and the corresponding fuzzy Zagreb matrix is the square matrix of order n whose $(i, j)^{th}$ entry whenever $i \neq j$, is $\sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j)$ and zero otherwise. In this paper, we introduce the Zagreb Estrada index of fuzzy graphs and establish some bounds for it.

Keywords: Fuzzy graphs, fuzzy Zagreb indices, fuzzy Zagreb Matrices, fuzzy Zagreb energies, fuzzy Zagreb Estrada index

AMS Subject classification: 05C07, 05C72, 15B15

1. Introduction

Topological indices are studied and applied in various fields by engineers, pharmacist, graph theorist and mathematicians. Gutman [5] in 1972, introduced the first Zagreb index and is oldest among the topological indices. Gutman, Eliasi, Kulli, KC Das and many other experts have contributed significantly in the developments of different Zagreb indices of simple graphs.

In case of classical graphs, both the vertices and edges have membership value one, but in case of fuzzy graphs both vertices and edges are equally important along with their fuzzy membership values. If the description of objects or their relationships or both are vague in nature, then we design a Fuzzy Graph model. In 1965, Zadeh [11] introduced the concept of fuzzy sets and fuzzy relations. Further Rosenfeld

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[19], Zimmerman [22], Thomson [21] and many experts in [4, 12, 14–16, 20] have contributed significantly in the developments of fuzzy graphs. In [1], Anjali and Mathew introduced energy of fuzzy graphs and in [18] authors introduced Laplacian energy of fuzzy graphs. In [2, 6], authors discussed Weiner index of fuzzy graph and found relationships between connectivity index and Wiener index of a fuzzy graph. Recently, Kale in [10] introduced concepts of first, second and hyper Zagreb indices of fuzzy graphs. The authors also studied the Zagreb matrices and the associated Zagreb energies of fuzzy graphs along with bounds for the energies. In [7, 8], authors discussed few applications of first Zagreb index on fuzzy graphs.
The paper is organized as follows: In section 2, we discuss the preliminary definitions required for the development of the content and in section 3, provides some bounds for the fuzzy Zagreb Estrada index.

2. Preliminaries

In this section, we recall some notions of fuzzy graphs and the definitions of fuzzy Zagreb indices and fuzzy Zagreb matrix which will play an important role in the subsequent sections of the paper. Basics of Zagreb indices can be referred in [3, 9] and basics of fuzzy graphs can be referred in [13, 17, 19]. Fuzzy Zagreb indices, Zagreb matrices and Zagreb energies can be refered in [10].

Definition 1. A fuzzy graph $G(V, \sigma, \mu)$ is a graph with vertex-membership function $\sigma : V \to [0, 1]$ and edge-membership function $\mu : V \times V \to [0, 1]$ such that for each $x, y \in V$, $\mu(x, y) \le \min\{\sigma(x), \sigma(y)\}$.
Also, we denote strength of vertex u by $\mu(u)$, it represents the minimum of membership values of edges incident to the vertex u, that is

$$\mu(u) = \bigwedge_{uv_i \in \mu} \mu(u, v_i).$$

Definition 2. The fuzzy Zagreb first index of $G(V, \sigma, \mu)$ is defined as

$$FM_1(G) = \sum_{uv\in\mu} \big[\sigma(u)\mu(u) + \sigma(v)\mu(v)\big].$$

Definition 3. The fuzzy hyper Zagreb index of $G(V, \sigma, \mu)$ is defined as

$$FHM(G) = \sum_{uv\in\mu} \big[\sigma(u)\mu(u) + \sigma(v)\mu(v)\big]^2.$$

Definition 4. Let $G(V, \sigma, \mu)$ be a fuzzy graph with $V = \{u_1, u_2, \dots, u_n\}$. The first fuzzy Zagreb matrix is defined as $FZ = (fz)_{i,j}$, where

$$(fz)_{i,j} = \begin{cases} \sigma(u_i)\mu(u_i) + \sigma(u_j)\mu(u_j), & \text{if } \ i \neq j \text{ and } \ u_i.u_j \in \mu \\ 0, & \text{if } \ u_i.u_j \notin \mu \\ 0, & \text{if } \ i = j. \end{cases}$$

Definition 5. Let $G(V, \sigma, \mu)$ be a fuzzy graph with $V = \{u_1, u_2, \ldots, u_n\}$ and let *FZ* be the first fuzzy Zagreb matrix with its eigen values $\xi_1, \xi_2, \ldots, \xi_n$. The first fuzzy Zagreb energy is defined as

$$FZE = \sum_{i=1}^{n} |\, \xi_i|.$$

Lemma 1. *([10]) If $G(V, \sigma, \mu)$ with $V = \{u_1, u_2, \ldots, u_n\}$ is a fuzzy graph and FZ is its fuzzy Zagreb matrix then*
(1) $N_1 = tr(FZ) = 0$
(2) $N_2 = tr(FZ^2) = 2.FHM.$

Lemma 2. *If $a_1, a_2, \ldots, a_n$ are positive numbers then*

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \leq \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Lemma 3. *If $a_1, a_2, \ldots, a_n$ are non-negative numbers and $k \geq 2$ then*

$$\sum_{i=1}^{n} {a_i}^k \leq \left(\sum_{i=1}^{n} {a_i}^2 \right)^{k/2}.$$

3. Fuzzy Zagreb Estrada index

In this section, we introduce the definition of fuzzy Zagreb Estrada index and present some lower and upper bounds for it.
Fuzzy Zagreb Estrada index of $G(V, \sigma, \mu)$ is defined as

$$FZEE = \sum_{i=1}^{n} e^{\xi_i},$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of fuzzy Zagreb matrix *FZ*. It follows that,

$$FZEE = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{{\xi_i}^k}{k!} = \sum_{k=0}^{\infty} \frac{N_k}{k!}$$

where $N_k = \sum\limits_{i=1}^{n} (\xi_i)^k = tr(FZ^k)$.

Theorem 1. *Let $G(V, \sigma, \mu)$ be a fuzzy graph with $V = \{u_1, u_2, \ldots, u_n\}$, FZ be the corresponding fuzzy Zagreb matrix with eigenvalues $\xi_1, \xi_2, \ldots, \xi_n$ and FHM be its fuzzy hyper Zagreb index. Then*

$$FZEE < n - 1 + e^{\sqrt{2FHM - 1}}.$$

Proof. Let $G(V,\sigma,\mu)$ be a fuzzy graph and $\xi_1 \geq \xi_2 \geq \cdots \geq \xi_n$ be eigenvalues of fuzzy Zagreb matrix *FZ*. Assume that n_+ be the number of positive eigenvalues. As the function $f(x) = e^x$ is monotonically increasing, we get

$$\begin{aligned}
FZEE = \sum_{i=1}^{n} e^{\xi_i} &< (n-n_+) + \sum_{i=1}^{n_+} e^{\xi_i} \\
&= (n-n_+) + \sum_{i=1}^{n_+}\sum_{k=0}^{\infty} \frac{{\xi_i}^k}{k!} \\
&= (n-n_+) + \sum_{i=1}^{n_+} 1 + \sum_{i=1}^{n_+}\sum_{k=1}^{\infty} \frac{{\xi_i}^k}{k!} \\
&= n + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{n_+} {\xi_i}^k \\
&\leq n + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n_+} {\xi_i}^2 \right)^{k/2} && \text{(using Lemma 3)} \\
&= n + \sum_{k=1}^{\infty} \frac{1}{k!} (2FHM-1)^{k/2} && \text{(using Lemma 1)} \\
&= n - 1 + e^{\sqrt{2FHM-1}}.
\end{aligned}$$

□

Theorem 2. *If $G(V,\sigma,\mu)$ is a fuzzy graph and $V = \{u_1, u_2, \ldots, u_n\}$, FZ is the corresponding fuzzy Zagreb matrix with eigen values $\xi_1, \xi_2, \ldots, \xi_n$ and FHM is its fuzzy hyper Zagreb index then*

$$\sqrt{n^2 + 4FHM} \leq FZEE \leq n - 1 + e^{\sqrt{2FHM}}.$$

Proof. First we prove the lower bound. Using the definition of fuzzy Zagreb Estrada index, we get

$$(FZEE)^2 = \sum_{i=1}^{n} e^{2\xi_i} + 2 \sum_{1 \leq i < j \leq n} e^{\xi_i}.e^{\xi_j}. \tag{1}$$

By Lemma 2, we get

$$\begin{aligned}
2\sum_{1\le i<j\le n} e^{\xi_i}.e^{\xi_j} &\ge n(n-1)\left(\prod_{i<j} e^{\xi_i}.e^{\xi_j}\right)^{2/n(n-1)}\\
&= n(n-1)\left[\left(\prod_{i=1}^{n} e^{\xi_i}\right)^{n-1}\right]^{2/n(n-1)}\\
&= n(n-1)\left(e^{\sum_{i=1}^{n}\xi_i}\right)^{2/n}\\
&= n(n-1)
\end{aligned}$$

$$2\sum_{1\le i<j\le n} e^{\xi_i}.e^{\xi_j} = n(n-1), \tag{2}$$

and using Lemma 1, we get

$$\begin{aligned}
\sum_{i=1}^{n} e^{2\xi_i} &= \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{(2\xi_i)^k}{k!}\\
&= n + 4FHM + \sum_{i=1}^{n}\sum_{k=3}^{\infty}\frac{(2\xi_i)^k}{k!}\\
&\ge n + 4FHM + \psi\sum_{i=1}^{n}\sum_{k=3}^{\infty}\frac{(\xi_i)^k}{k!}
\end{aligned}$$

where $\psi \in [0,8]$. Hence, we get

$$\sum_{i=1}^{n} e^{2\xi_i} = n + 4FHM - \psi n - \psi FHM + \psi\sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{(\psi_i)^k}{k!}. \tag{3}$$

Applying equations (2) and (3) in (1), we get

$$FZEE \ge \frac{\psi}{2} + \sqrt{\left(n-\frac{\psi}{2}\right)^2 + (4-\psi)FHM}. \tag{4}$$

Since the function, $f(x) = \frac{x}{2} + \sqrt{\left(n-\frac{x}{2}\right)^2 + (4-x)FHM}$ is monotonically decreasing function in $[0,8]$, hence best lower bound for FZEE is attained at $\psi = 0$. Thus (4) leads to

$$FZEE \ge \sqrt{n^2 + 4FHM}. \tag{5}$$

Now we prove the upper bounds. By definition of *FZEE* we have

$$\begin{aligned} FZEE &= \sum_{i=1}^{n} e^{\xi_i} = \sum_{i=1}^{n}\sum_{k=0}^{\infty} \frac{{\xi_i}^k}{k!} \\ &= n + \sum_{i=1}^{n}\sum_{k=1}^{\infty} \frac{{\xi_i}^k}{k!} \\ &= n + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{n} \left(\xi_i^2\right)^{k/2} \\ &\leq n + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} {\xi_i}^2\right)^{k/2} \\ &= n + \sum_{k=1}^{\infty} \frac{1}{k!} (2FHM)^{k/2} \\ &= n - 1 + \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{2FHM}\right)^{k} \\ &= n - 1 + e^{\sqrt{2FHM}}, \end{aligned}$$

and the proof is complete. □

Theorem 3. *If $G(V, \sigma, \mu)$ is a fuzzy graph with $V = \{u_1, u_2, \dots, u_n\}$, FZ is the corresponding fuzzy Zagreb matrix with eigenvalues $\xi_1, \xi_2, \dots, \xi_n$ and FHM is its fuzzy hyper Zagreb index, then*

$$FZEE \leq e^{\sqrt{2FHM}}.$$

Proof. By definition of *FZEE*, we have

$$\begin{aligned} FZEE &= \sum_{k=0}^{\infty}\sum_{i=1}^{n} \frac{{\xi_i}^k}{k!} \\ &\leq \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} {\xi_i}^2\right)^{k/2} && \text{(using Lemma 3)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (2FHM)^{k/2} && \text{(using Lemma 1)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{2FHM}\right)^{k} \\ &= e^{\sqrt{2FHM}}. \end{aligned}$$

This completes the proof. □

4. Conclusion

Fuzzy Zagreb Estrada Index is studied. Some bounds for fuzzy Zagreb Estrada index are prsented. Further study on these fuzzy Zagreb indices may reveal more analogous results of these kind and will be discussed in the forthcoming papers.

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