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Research Article

Efficient algorithms for independent Roman domination on some classes of graphs

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Abstract: Let $G = (V, E)$ be a given graph of order n. A function $f : V \to \{0, 1, 2\}$ is an independent Roman dominating function (IRDF) on G if for every vertex $v \in V$ with $f(v) = 0$ there is a vertex u adjacent to v with $f(u) = 2$ and $\{v \in V : f(v) > 0\}$ is an independent set. The weight of an IRDF f on G is the value $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an IRDF among all IRDFs on G is called the independent Roman domination number of G. In this paper, we give algorithms for computing the independent Roman domination number of G in $O(|V|)$ time when $G = (V, E)$ is a tree, unicyclic graph or proper interval graph.

Keywords: Independent Roman dominating function, tree, unicyclic graph, proper interval graph

AMS Subject classification: 05C69, 68Q25

1. Introduction

Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. Throughout this paper, all graphs that we consider are finite, undirected, and simple. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. For any $S \subseteq V$ the *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all edges in E that have both endpoints in S. Let $u, v \in V$, let $uv \in E$, and w be a vertex not in V. We denote the graph obtained from G by adding a new edge between u and w by $G + uw$, that is, $G + uw = (V \cup \{w\}, E \cup \{uw\})$ and by removing the edge uv by $G - uv$, that is, $G - uv = (V, E \setminus \{uv\})$. A *unicyclic graph* is a graph obtained from a tree T of order at least three by joining precisely two non-adjacent vertices of T [11]. A graph $G = (V, E)$ is an *interval graph* if there is an interval I_v on the real line one-to-one correspondence with each vertex $v \in V$ such that $uv \in E$ if and only if $I_v \cap I_u \neq \emptyset$. A *proper interval graph* is an interval graph in which no interval properly contains another [1]. The following result is clear.

Proposition 1. *Let $G = (V, E)$ be a proper interval graph. For all $S \subseteq V$, the induced subgraph $G[S]$ is also a proper interval graph.*

For a graph G, an *independent Roman dominating function* (IRDF) of G is a function $f : V \to \{0, 1, 2\}$ such that each vertex $v \in V$ with $f(v) = 0$ is adjacent to a vertex u with $f(u) = 2$ and also $\{u \in V : f(u) > 0\}$ is an independent set. The *weight* of an IRDF f on G is the value $f(V) = \sum_{v \in V} f(v)$, denoted by $w(f)$, and the minimum weight of an IRDF among all IRDFs on G is called the independent Roman domination number of G, denoted by $i_R(G)$.
The independent concept in dominating functions of graphs have been studied extensively in the literature, for example [2, 4, 6–8, 12, 13]. Liu and Chang [9] have shown that the decision problem associated with the independent Roman domination is NP-complete even for bipartite graphs. Furthermore, they have used a linear programming method to give an algorithm for computing the independent Roman domination number of a given strongly chordal graph $G = (V, E)$ with a strong elimination ordering provided in $O(|V| + |E|)$ time. Chakradhar and Venkata Subba Reddy [5] have shown that the decision problem associated with the independent Roman domination is NP-complete even when restricted to star convex bipartite graphs and comb convex bipartite graphs. Furthermore, they have proven that the problem of computing the independent Roman domination number is linear time solvable for bounded tree-width graphs, chain graphs, and threshold graphs.
Let $G = (V, E)$ be a graph. In this paper, we use a dynamic programming approach to compute $i_R(G)$ when G is a tree or unicyclic graph in $O(|V|)$ time. Next, we propose a dynamic programming algorithm for computing $i_R(G)$ for a given proper interval graph G, a subclass of strongly chordal graphs, with a consecutive ordering of vertices in G provided in $O(|V|)$ time. If G is a disconnected graph, then clearly $i_R(G)$ is the sum of the independent Roman domination numbers of its components. So, in the rest of the paper we only consider connected graphs.

2. Trees

In this section we give a linear algorithm (Algorithm 2.1) to compute independent Roman domination number of a given tree. Before we introduce our algorithm, we need to define the following notations. Let $G = (V, E)$ be a graph, let $v \in V$, let w be a vertex not in V and let $a \in \{0, 1, 2\}$.

- $i'_R(G, v) = \min\{w(f) : f$ is an IRDF on $G + vw$ with $f(v) = 0$ and $f(w) = 2\}$.
- $i^a_R(G, v) = \min\{w(f) : f$ is an IRDF on G with $f(v) = a$ (if there exists)$\}$.

To prove that Algorithm 2.1 works correctly we need the following lemma.

Lemma 1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$, let $v \in V_1$ and $u \in V_2$ and let $G = (V = V_1 \cup V_2, E_1 \cup E_2 \cup \{uv\})$. Then,*

Algorithm 2.1: IRDNT($\mathtt{T}$)

Input: A tree $\mathtt{T} = (V, E)$ with $|V| = n$.
Output: The independent Roman domination number of $\mathtt{T}$.

Compute a canonical ordering $(v_1, v_2, \dots, v_n)$ of vertices T, where T is a rooted tree of $\mathtt{T}$.
for $i \leftarrow 1$ **to** n **do**
  $i_R^0(v_i) \leftarrow \infty$;
  $i_R^1(v_i) \leftarrow 1$;
  $i_R^2(v_i) \leftarrow 2$;
  $i'_R(v_i) \leftarrow 2$;
end for
for $i \leftarrow n$ **downto** 2 **do**
  Let v_j be the parent of v_i;
  $i_R^0(v_j) \leftarrow \min\{i_R^0(v_j) + i_R^0(v_i), i_R^0(v_j) + i_R^1(v_i), i'_R(v_j) + i_R^2(v_i) - 2\}$;
  $i_R^1(v_j) \leftarrow i_R^1(v_j) + i_R^0(v_i)$,;
  $i_R^2(v_j) \leftarrow i_R^2(v_j) + i'_R(v_i) - 2$;
  $i'_R(v_j) \leftarrow \min\{i'_R(v_j) + i_R^0(v_i), i'_R(v_j) + i_R^1(v_i), i'_R(v_j) + i_R^2(v_i)\}$;
end for
return $(i_R^0(v_1), i_R^1(v_1), i_R^2(v_1), i'_R(v_1))$;

(i) $i_R^0(G,v) = \min\{i_R^0(G_1,v) + i_R^0(G_2,u), i_R^0(G_1,v) + i_R^1(G_2,u), i'_R(G_1,v) + i_R^2(G_2,u) - 2\}$,

(ii) $i_R^1(G,v) = i_R^1(G_1,v) + i_R^0(G_2,u)$,

(iii) $i_R^2(G,v) = i_R^2(G_1,v) + i'_R(G_2,u) - 2$,

(iv) $i'_R(G,v) = \min\{i'_R(G_1,v) + i_R^0(G_2,u), i'_R(G_1,v) + i_R^1(G_2,u), i'_R(G_1,v) + i_R^2(G_2,u)\}$.

Proof. Let w, w_1, w_2 be vertices not in V. Let f_1 (resp. f_2) be the restriction of f to G_1 (resp. G_2) for a given IRDF f on G. Clearly, $f = f_1 \cup f_2$ and so $w(f) = w(f_1) + w(f_2)$. Also, assume that g_1^a (resp. g_2^a) is an IRDF on G_1 (resp. G_2) with minimum weight and $g_1^a(v) = a$ (resp. $g_2^a(u) = a$) for each $a \in \{0,1,2\}$ and g'_1 (resp. g'_2) is an IRDF on $G_1 + vw_1$ (resp. $G_2 + uw_2$) with minimum weight, $g'_1(v) = 0$ and $g'_1(w_1) = 2$ (resp. $g'_2(u) = 0$ and $g'_2(w_2) = 2$). So, $w(g_1^a) = i_R^a(G_1,v)$, $w(g_2^a) = i_R^a(G_2,u)$, $w(g'_1) = i'_R(G_1,v)$ and $w(g'_2) = i'_R(G_2,u)$ for all $a \in \{0,1,2\}$.
We first prove (i). Let $i_R^0 = \min\{i_R^0(G_1,v) + i_R^0(G_2,u), i_R^0(G_1,v) + i_R^1(G_2,u), i'_R(G_1,v) + i_R^2(G_2,u) - 2\}$ and assume that f is an IRDF on G with minimum weight and $f(v) = 0$. So, $w(f) = i_R^0(G,v)$. Since u is adjacent to v in G, $f(u) \in \{0,1,2\}$. In the following we consider these cases. If $f(u) = 0$, then f_1 is an IRDF on G_1 with $f_1(v) = 0$ and f_2 is an IRDF on G_2 with $f_2(u) = 0$ and so $w(f_1) \le i_R^0(G_1,v)$ and $w(f_2) \le i_R^0(G_2,u)$. Thus, $i_R^0(G,v) = w(f) = w(f_1) + w(f_2) \le i_R^0(G_1,v) + i_R^0(G_2,u)$. If $f(u) = 1$, then f_1 is an IRDF on G_1 with $f_1(v) = 0$ and f_2 is an IRDF on G_2 with $f_2(u) = 1$ and so $w(f_1) \le i_R^0(G_1,v)$ and $w(f_2) \le i_R^1(G_2,u)$. Thus, $i_R^0(G,v) = w(f) = w(f_1) + w(f_2) \le i_R^0(G_1,v) + i_R^1(G_2,u)$. If $f(u) = 2$, then $h = f_1 \cup \{(w,2)\}$ is an IRDF on $G_1 + vw$ with $h(v) = 0$ and $h(w) = 2$ and f_2 is an IRDF on G_2 with $f_2(u) = 2$ and so $w(h) = w(f_1) + 2 \le i'_R(G_1,v)$ and $w(f_2) \le i_R^2(G_2,u)$. Thus, $i_R^0(G,v) = w(f) = w(f_1) + w(f_2) \le i'_R(G_1,v) + i_R^2(G_2,u) - 2$. Therefore,

$$i_R^0(G,v) \le i_R^0. \quad (1)$$

Conversely, $h_1 = g_1^0 \cup g_2^0$ is an IRDF on G with $h_1(v) = 0$ and so $w(h_1) = i_R^0(G_1, v) + i_R^0(G_2, u) \leq i_R^0(G, v)$. Also, $h_2 = g_1^0 \cup g_2^1$ is an IRDF on G with $h_2(v) = 0$ and so $w(h_2) = i_R^0(G_1, v) + i_R^1(G_2, u) \leq i_R^0(G, v)$. Let h be the restriction of $g_1' \cup g_2^2$ to G. We deduce that h is an IRDF on G with $h(v) = 0$ and so $w(h) = i_R'(G_1, v) + i_R^2(G_2, u) - 2 \leq i_R^0(G, v)$. Therefore,

$$i_R^0 \leq i_R^0(G, v). \tag{2}$$

The proof of (i) follows from (1) and (2).
Now, we prove (ii). Assume that f is an IRDF on G with minimum weight and $f(v) = 1$. So, $w(f) = i_R^1(G, v)$. Since $uv \in E$, $f(u) = 0$. We obtain that f_1 is an IRDF on G_1 with $f_1(v) = 1$ and f_2 is an IRDF on G_2 with $f_2(u) = 0$ and so $w(f_1) \leq i_R^1(G_1, v)$ and $w(f_2) \leq i_R^0(G_2, u)$. Thus, $i_R^1(G, v) = w(f) = w(f_1) + w(f_2) \leq i_R^1(G_1, v) + i_R^0(G_2, u)$. Conversely, $h = g_1^1 \cup g_2^0$ is an IRDF on G with $h(v) = 1$ and so $w(h) = i_R^1(G_1, v) + i_R^0(G_2, u) \leq i_R^1(G, v)$. This completes the proof of (ii). Similarly, we can prove (iii) and (iv). □

Let T be a rooted tree of order n. A *canonical ordering* of vertices of T is an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of T with the property that the label of parent of v_i is less than i. Therefore, the label of the root of T is 1. Now, we can prove that Algorithm 2.1 returns the independent Roman domination number of trees in linear time.

Theorem 1. *For a given tree* $\mathtt{T} = (V, E)$ *of order* n*, let* $(v_1, v_2, \ldots, v_n)$ *be a canonical ordering of vertices* T *computed in Algorithm 2.1, where* T *is a rooted tree of* $\mathtt{T}$*. Algorithm 2.1 returns* $(i_R^0(T, v_1), i_R^1(T, v_1), i_R^2(T, v_1), i_R'(T, v_1))$ *in* $O(|V|)$ *time.*

Proof. Clearly, $i_R^0(\mathtt{T}, v_1) = i_R^0(T, v_1)$, $i_R^1(\mathtt{T}, v_1) = i_R^1(T, v_1)$, $i_R^2(\mathtt{T}, v_1) = i_R^2(T, v_1)$ and $i_R'(\mathtt{T}, v_1) = i_R'(T, v_1)$. We can compute T and $(v_1, v_2, \ldots, v_n)$ in $O(n)$ time.
The proof is by induction on n. If $n = 1$, then it is easy to check that the lemma is true. Let $n > 1$ and assume that the lemma holds for any subtree of T. In the last iteration of the for-loop of Algorithm 2.1 we have $i = 2$ and therefore $j = 1$. Let T_i be the subtree of T with the root v_i and let $T_j = T - T_i$, where $T = (V(T_i) \cup V(T_j), E(T_i) \cup E(T_j) \cup \{v_i v_j\})$. By the induction hypothesis, since $v_1, v_2, \ldots, v_n$ is a canonical ordering of vertices of T, the lemma holds for T_i and T_j. By Lemma 1, the lemma holds for T. This proves the lemma. It is easy to see that the running time of Algorithm 2.1 is $O(n)$. This completes the proof of the theorem. □

Let T be a given tree and let v be a vertex of T. We obtain that $i_R(T) = \{i_R^0(T, v), i_R^1(T, v), i_R^2(T, v)\}$. By Theorem 1, we obtain the following result.

Corollary 1. *There is a linear algorithm to compute the independent Roman domination number of a given tree.*

Algorithm 3.1: IRDNUG(T, u, v)

Input: A rooted tree $T = (V, E)$ with root u and $v \in V$.
Output: $(i_R^{00}(T,u,v), i_R^{10}(T,u,v), i_R^{2}(T,u,v))$.

Let $w_0(=v), \ldots, w_k(=u)$ be the shortest path between u and v in T and let T_w be the subtree of T with the root v for each $w \in V$.
$T' \leftarrow T_{w_1} - T_{w_0}$;
$i_R^{00} \leftarrow i_R^{0}(T', w_1) + i_R^{0}(T_{w_0}, w_0)$;
$i_R^{10} \leftarrow i_R^{1}(T', w_1) + i_R^{0}(T_{w_0}, w_0)$;
$i_R^{20} \leftarrow i_R^{2}(T', w_1) + i'_R(T_{w_0}, w_0) - 2$;
$i_R^{00,2} \leftarrow i'_R(T', w_1) + i_R^{0}(T_{w_0}, w_0)$;
$j_R^{00,2} \leftarrow i_R^{0}(T', w_1) + i'_R(T_{w_0}, w_0)$;
$j_R^{10,2} \leftarrow i_R^{1}(T', w_1) + i'_R(T_{w_0}, w_0)$;
$j_R^{20,2} \leftarrow i_R^{2}(T', w_1) + i'_R(T_{w_0}, w_0)$;
$j_R^{00,22} \leftarrow i'_R(T', w_1) + i'_R(T_{w_0}, w_0)$;
for $i \leftarrow 2$ **to** k **do**
  $T' \leftarrow T_{w_i} - T_{w_{i-1}}$;
  $i_R^{00} \leftarrow \min\{i_R^{0}(T', w_i) + i_R^{00}, i_R^{0}(T', w_i) + i_R^{10}, i'_R(T', w_i) + i_R^{20} - 2\}$;
  $i_R^{10} \leftarrow i_R^{1}(T', w_i) + i_R^{00}$;
  $i_R^{20} \leftarrow i_R^{2}(T', w_i) + i_R^{00,2} - 2$;
  $i_R^{00,2} \leftarrow i'_R(T', w_i) + \min\{i_R^{00}, i_R^{10}, i_R^{20}\}$;
  $j_R^{00,2} \leftarrow \min\{i_R^{0}(T', w_i) + j_R^{00,2}, i_R^{0}(T', w_i) + j_R^{10,2}, i'_R(T', w_i) + j_R^{20,2} - 2\}$;
  $j_R^{10,2} \leftarrow i_R^{1}(T', w_i) + j_R^{00,2}$;
  $j_R^{20,2} \leftarrow i_R^{2}(T', w_i) + j_R^{00,22} - 2$;
  $j_R^{00,22} \leftarrow i'_R(T', w_i) + \min\{j_R^{00,2}, j_R^{10,2}, j_R^{20,2}\}$;
end for
return $(i_R^{00}, i_R^{10}, i_R^{20}, i_R^{00,2}, j_R^{00,2}, j_R^{10,2}, j_R^{20,2}, j_R^{00,22})$;

3. Unicyclic graphs

In this section we propose a linear algorithm (Algorithm 3.1) to compute the independent Roman domination number of a given unicyclic graph. For this purpose we need the following notations. Let $G = (V, E)$ be a graph with $u, v \in V$, let w, z be vertices not in V and let $a \in \{0, 1, 2\}$.

- $i'_R(G, v) = \min\{w(f) : f$ is an IRDF on $G + vw$ with $f(v) = 0$ and $f(w) = 2\}$.
- $i_R^{a}(G, v) = \min\{w(f) : f$ is an IRDF on G with $f(v) = a$ (if there exists)$\}$.
- $i_R^{a0}(G, u, v) = \min\{w(f) : f$ is an IRDF on G with $f(u) = 0$ and $f(v) = a\}$.
- $i_R^{a0,2}(G, u, v) = \min\{w(f) : f$ is an IRDF on $G + vw$ with $f(v) = 0$ and $f(u) = f(w) = 2\}$.
- $i_R^{00,22}(G, u, v) = \min\{w(f) : f$ is an IRDF on $G + \{uw, vz\}$ with $f(u) = f(v) = 0$ and $f(w) = f(z) = 2\}$.

Note that two notations $i'_R(G, v)$ and $i_R^{a}(G, v)$ are also defined in the previous section. Through this section assume that $U = (V, E)$ is a unicyclic graph with the unique cycle $v_0, \ldots, v_{k-1}, v_0$ ($k \geq 3$) and let $T = U - v_0v_1$. Clearly, T is a tree with the vertex set V and the edge set $E \setminus \{v_0v_1\}$.

Lemma 2. $i_R(U) = \min\{i_R^{00}(T, v_0, v_1), i_R^{01}(T, v_0, v_1), i_R^{01}(T, v_1, v_0), i_R^{20,2}(T, v_0, v_1) - 2, i_R^{20,2}(T, v_1, v_0) - 2\}$.

Proof. Let $i_R = \min\{i_R^{00}(T, v_0, v_1), i_R^{01}(T, v_0, v_1), i_R^{01}(T, v_1, v_0), i_R^{20,2}(T, v_0, v_1) - 2, i_R^{20,2}(T, v_1, v_0) - 2\}$. We first prove that $i_R \leq i_R(U)$. Let f be an IRDF on U with minimum weight. So, $w(f) = i_R(U)$. Because f is an IRDF on U and $v_0v_1 \in E(U)$, $(f(v_0), f(v_1) \in \{(0,0), (0,1), (1,0), (0,2), (2,0)\}$. In the following we consider these cases. If $(f(v_0), f(v_1)) = (0,0)$, then f is an IRDF on T with $f(v_0) = f(v_1) = 0$ and so $i_R^{00}(T, v_0, v_1) \leq w(f)$. If $(f(v_0), f(v_1)) = (0,1)$, then f is an IRDF on T with $f(v_0) = 0$ and $f(v_1) = 1$ and so $i_R^{01}(T, v_0, v_1) \leq w(f)$. If $(f(v_0), f(v_1)) = (1,0)$, then f is an IRDF on T with $f(v_0) = 1$ and $f(v_1) = 0$ and so $i_R^{01}(T, v_1, v_0) \leq w(f)$. Let w be a vertex not in V. If $(f(v_0), f(v_1)) = (0,2)$, then $h = f \cup \{(w,2)\}$ is an IRDF on $T + v_0w$ with $f(v_0) = 0$ and $f(v_1) = f(w) = 2$ and so $i_R^{20,2}(T, v_1, v_0) \leq w(h) = w(f) + 2$. If $(f(v_0), f(v_1)) = (2,0)$, then $h = f \cup \{(w,2)\}$ is an IRDF on $T + v_1w$ with $f(v_1) = 0$ and $f(v_0) = f(w) = 2$ and so $i_R^{20,2}(T, v_0, v_1) \leq w(h) = w(f) + 2$. So, $i_R \leq i_R(U)$.
Now, we prove that $i_R(U) \leq i_R$. Assume that g is a minimum IRDF on T with $g(v_0) = g(v_1) = 0$. So, $w(g) = i_R^{00}(T, v_0, v_1)$. We deduce that g is an IRDF on U and so $i_R(U) \leq i_R^{00}(T, v_0, v_1)$. Assume that g is a minimum IRDF on T with $g(v_0) = 0$ and $g(v_1) = 1$. So, $w(g) = i_R^{01}(T, v_0, v_1)$. We deduce that g is an IRDF on U and so $i_R(U) \leq i_R^{01}(T, v_0, v_1)$. Assume that g is a minimum IRDF on T with $g(v_0) = 1$ and $g(v_1) = 0$. So, $w(g) = i_R^{01}(T, v_1, v_0)$. We deduce that g is an IRDF on U and so $i_R(U) \leq i_R^{01}(T, v_1, v_0)$. Assume that g is a minimum IRDF on $T + v_0w$ with $g(v_0) = 0$ and $g(v_1) = g(w) = 2$. So, $w(g) = i_R^{20,2}(T, v_1, v_0)$. Let h be the restriction of g to T. We deduce that h is an IRDF on U and so $i_R(U) \leq w(h) = i_R^{20,2}(T, v_1, v_0) - 2$. Assume that g is a minimum IRDF on $T + v_1w$ with $g(v_1) = 0$ and $g(v_0) = g(w) = 2$. So, $w(g) = i_R^{20,2}(T, v_0, v_1)$. Let h be the restriction of g to T. We deduce that h is an IRDF on U and so $i_R(U) \leq w(h) = i_R^{20,2}(T, v_0, v_1) - 2$. So, $i_R(U) \leq i_R$. This completes the proof of the lemma. □

Lemma 3. *Let $T = (V, E)$ be a rooted tree with root u and $v \in V$. Let $(i_R^{00}, i_R^{10}, i_R^{20}, i_R^{00,2}, j_R^{00,2}, j_R^{10,2}, j_R^{20,2}, j_R^{00,22})$ be the output of Algorithm 3.1. Then,*

(i) $i_R^{00} = i_R^{00}(T, u, v)$.

(ii) $i_R^{10} = i_R^{10}(T, u, v)$.

(iii) $i_R^{20} = i_R^{20}(T, u, v)$.

(iv) $i_R^{00,2} = i_R^{00,2}(T, v, u)$.

(v) $j_R^{00,2} = i_R^{00,2}(T, u, v)$.

(vi) $j_R^{10,2} = i_R^{10,2}(T, u, v)$.

(vii) $j_R^{20,2} = i_R^{20,2}(T, u, v)$.

(viii) $j_R^{00,22} = i_R^{00,22}(T, u, v)$.

Proof. Let $P(T, v, u) = w_0(= v), \dots, w_k(= u)$ $(k > 0)$ be the shortest path between v and u in T. The proof is by induction on $k = |P(T, v, u)|$. Let $k = 1$. So, u is the parent of v. Let $T' = T_u - T_v$. Similar to Lemma 2, we can prove that

- $i_R^{00}(T, u, v) = i_R^0(T', u) + i_R^0(T_v, v)$.
- $i_R^{10}(T, u, v) = i_R^1(T', u) + i_R^0(T_v, v)$.
- $i_R^{20}(T, u, v) = i_R^2(T', u) + i'_R(T_v, v) - 2$.
- $i_R^{00,2}(T, v, u) = i'_R(T', u) + i_R^0(T_v, v)$.
- $i_R^{00,2}(T, u, v) = i_R^0(T', u) + i'_R(T_v, v)$.
- $i_R^{10,2}(T, u, v) = i_R^1(T', u) + i'_R(T_v, v)$.
- $i_R^{20,2}(T, u, v) = i_R^2(T', u) + i'_R(T_v, v)$.
- $i_R^{00,22}(T, u, v) = i'_R(T', u) + i'_R(T_v, v)$.

This proves the base case of the induction. Assume that the result is true for all rooted trees T' with the root u, $v \in V(T')$ and $|P(T', v, u)| \leq m$, where $m \geq 1$. Let T be a rooted tree with the root u, $v \in V(T)$, and $P(T, v, u) = w_0(= v), \dots, w_m, w_{m+1}(= u)$. Let $(i_{tR}^{00}, i_{tR}^{10}, i_{tR}^{20}, i_{tR}^{00,2}, j_{tR}^{00,2}, j_{tR}^{10,2}, j_{tR}^{20,2}, j_{tR}^{00,22})$ be values of variables $(i_R^{00}, i_R^{10}, i_R^{20}, i_R^{00,2}, j_R^{00,2}, j_R^{10,2}, j_R^{20,2}, j_R^{00,22})$ of Algorithm 3.1, respectively, after the t-th iteration of the for-loop for each $2 \leq t \leq m + 1$. By the induction hypothesis, we have

- $i_R^{00}(T_{w_m}, w_m, v) = i_{mR}^{00}$.
- $i_R^{10}(T_{w_m}, w_m, v) = i_{mR}^{10}$.
- $i_R^{20}(T_{w_m}, w_m, v) = i_{mR}^{20}$.
- $i_R^{00,2}(T_{w_m}, v, w_m) = i_{mR}^{00,2}$.
- $i_R^{00,2}(T_{w_m}, w_m, v) = j_{mR}^{00,2}$.
- $i_R^{10,2}(T_{w_m}, w_m, v) = j_{mR}^{10,2}$.
- $i_R^{20,2}(T_{w_m}, w_m, v) = j_{mR}^{20,2}$.
- $i_R^{00,22}(T_{w_m}, w_m, v) = j_{mR}^{00,22}$.

Let $T' = T - T_{w_m}$. Since $u(= w_{m+1})$ is the parent of $w_m(\neq v)$ (i.e., u is adjacent to w_m) in T, similar to Lemma 2, we can prove that

- $i_R^{00}(T, u, v) = \{i_R^0(T', u) + i_{mR}^{00}, i_R^0(T', u) + i_{mR}^{10}, i'_R(T', u) + i_{mR}^{20} - 2\}$.

- $i_R^{10}(T,u,v) = i_R^1(T',u) + i_{mR}^{00}$.
- $i_R^{20}(T,u,v) = i_R^2(T',u) + i_{mR}^{00,2} - 2$.
- $i_R^{00,2}(T,v,u) = i'_R(T',u) + \min\{i_{mR}^{00}, i_{mR}^{10}, i_{mR}^{20}\}$.
- $i_R^{00,2}(T,u,v) = \min\{i_R^0(T',u) + j_{mR}^{00,2}, i_R^0(T',u) + j_{mR}^{10,2}, i'_R(T',u) + j_{mR}^{20,2} - 2\}$.
- $i_R^{10,2}(T,u,v) = i_R^1(T',u) + j_{mR}^{00,2}$.
- $i_R^{20,2}(T,u,v) = i_R^2(T',u) + j_{mR}^{00,22} - 2$.
- $i_R^{00,22}(T,u,v) = i'_R(T',u) + \min\{j_{mR}^{00,2}, j_{mR}^{10,2}, j_{mR}^{20,2}\}$.

This completes the proof of the lemma. □

Theorem 2. *There is a linear algorithm for computing the independent Roman domination number of a given unicyclic graph.*

Proof. Let U be a unicyclic graph with the unique cycle $v_0, \ldots, v_{k-1}, v_0$ and let $T = U - v_0v_1$. By Lemma 2, $i_R(U) = \min\{i_R^{00}(T,v_0,v_1), i_R^{01}(T,v_0,v_1), i_R^{01}(T,v_1,v_0), i_R^{20,2}(T,v_0,v_1) - 2, i_R^{20,2}(T,v_1,v_0) - 2\}$. It follows from Lemma 3 that we can compute $i_R(U)$ using the output of Algorithm 3.1 on inputs (T,u,v) and (T,v,u).
Let T be a tree with $u,v \in V(T)$ and let $w_0(=v),\ldots,w_k(=u)$ $(k>0)$ be the shortest path between u and v in T. We can compute the rooted tree T_u with the root u for T and $P(T,v,u)$ in linear time. Let $2 \le m \le k$ and let T_m be the value of variable T' of Algorithm 3.1 after the m-th iteration of the for-loop. Clearly, T_m is a subtree of T_u with the root w_m. By Theorem 1, the running time of lines 2-10 of Algorithm 3.1 is $O(V(T_1))$ and the running time of the m-th iteration of the for-loop of Algorithm 3.1 is $O(V(T_m))$. Clearly, $V(T_i) \cap V(T_j) = \emptyset$ for each $2 \le i < j \le k$. So, the running time of Algorithm 3.1 is $O(V(T_1)) + \sum_{i=2}^{k} O(V(T_m)) = O(V(T))$. This completes the proof of the theorem. □

4. Proper interval graphs

In this section we propose a linear algorithm (Algorithm 4.1) for computing the independent Roman domination number of a given proper interval graph. An ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G is a *consecutive ordering* if $v_iv_k \in E$ for some $1 \le i < k \le n$ implies both $v_iv_j \in E$ and $v_jv_k \in E$ for every $i < j < k$.

Theorem 3 (Looges and Olariu [10]). *A graph G is a proper interval graph if and only if G has a consecutive ordering of its vertices.*

Algorithm 4.1: IRDNPIG$(G, v_1, \ldots, v_n)$

Input: A proper interval graph G of order n and a consecutive ordering $(v_1, \ldots, v_n)$ of vertices of G.

Output: The independent Roman domination number of G.

Compute $\texttt{MIN}(v_1), \ldots, \texttt{MIN}(v_n)$;
$i_R^0(v_1) \leftarrow \infty$; $i_R^1(v_1) \leftarrow 1$; $i_R^2(v_1) \leftarrow 2$; $x \leftarrow v_1$;
while $x < v_n$ **do**
    $x \leftarrow x^+$;
    $u \leftarrow \texttt{MIN}(x)$;
    $i_R^0(x) \leftarrow i_R^2(u)$;
    if $u = x^-$ **then** $i_R^1(x) \leftarrow i_R^0(x^-) + 1$;
    if $\texttt{MIN}(x^-) < u < x^-$ **then** $i_R^1(x) \leftarrow i_R^2(\texttt{MIN}(x^-)) + 1$;
    if $(u < x^-) \wedge (u \leq \texttt{MIN}(x^-))$ **then** $i_R^1(x) \leftarrow \infty$;
    $i_R^2(x) \leftarrow \min\{i_R^0(u^-), i_R^1(u^-), i_R^2(u^-)\} + 2$;
end while
return $\min\{i_R^0(v_n), i_R^1(v_n), i_R^2(v_n)\}$;

Booth and Lueker [3] have proposed a linear-time algorithm for testing whether a graph is a proper interval graph, and give a consecutive ordering if the answer is positive. In the rest of this paper, for a given proper interval G of order n, we assume that a consecutive ordering $(v_1, v_2, \ldots, v_n)$ of vertices of G is given. Let $i \in \{1, 2, \ldots, n\}$, let j and k be positive integers such that $1 \leq j \leq k \leq n$, let $a \in \{0, 1, 2\}$, $b \in \{1, 2\}$ and let v_0 and v_{n+1} be vertices not in V.

- $v_i^+ = v_{i+1}$,
- $v_i^- = v_{i-1}$,
- $v_j \leq v_k$,
- $v_j < v_k$ if $j \neq k$,
- $[v_j, v_k] = \{u \in V : v_j \leq u \leq v_k\}$,
- $G[v_j, v_k] = G[\{u \in V : v_j \leq u \leq v_k\}]$,
- $\texttt{MIN}(v_i) = \min\{u \in N_G[v_i]\}$,
- $\texttt{C}(v_i) = \max\{\texttt{MIN}(u) : u \in [\texttt{MIN}(v_i), v_i]\}$,
- $\gamma_{tR}^a(v_i) = \min\{w(f) : f$ is an TRDF on $G[v_1, v_i]$ with $f(v_i) = a\}$,
- $\gamma_{ptR}^b(v_i) = \min\{w(f) : f$ is an PTRDF on $G[v_1, v_i]$ with $f(v_i) = b\}$,
- $i_R^a(v_i) = \min\{w(f) : f$ is an IRDF on $G[v_1, v_i]$ with $f(v_i) = a$ (if there exists)$\}$.

To prove that Algorithm 4.1 works correctly we need the following results. The following result is clear.

Proposition 2. *Let $G = (V, E)$ be a proper interval graph of order n with a consecutive ordering $(v_1, \ldots, v_n)$ of vertices of G and let $1 \leq i \leq j \leq n$.*

(i) If $v_iv_j \in E$, then $\{v_i, \ldots, v_j\}$ is a clique of G.

(ii) $\mathtt{MIN}(v_i) \leq \mathtt{MIN}(v_j)$.

(iii) $(v_1, \ldots, v_n)$ is also an IG ordering of vertices of G.

Lemma 4. $i^2_R(v_1) \leq i^2_R(v_2) \leq \cdots \leq i^2_R(v_n)$.

Proof. The proof is by induction on n. If $n = 2$, then clearly the claim holds. This proves the base case of the induction. Assume that the lemma is true for all proper interval graphs of order $n \geq 2$. Let H be a proper interval graph of order $n+1$ with a consecutive ordering $(v_1, \ldots, v_{n+1})$ of vertices of H. By Proposition 1, the induced subgraph $H[v_1, v_n]$ is a proper interval graph of order n and so

$$i^2_R(v_1) \leq \cdots \leq i^2_R(v_n). \tag{3}$$

Let g be a minimum IRDF on H with $g(v_{n+1}) = 2$ and so $w(g) = i^2_R(v_{n+1})$. Since g is an IRDF on H, $g(u) = 0$ for each vertex u adjacent to v_{n+1}, that is, $u \in [\mathtt{MIN}(v_{n+1}), v_n]$, see Fig. 1. By Proposition 2 and since H is connected, $\mathtt{MIN}(v_n) \leq \mathtt{MIN}(v_{n+1}) \leq v_n$ and the induced subgraph $H[\mathtt{MIN}(v_n), v_n]$ is a complete graph. So, each vertex adjacent to v_{n+1} is also adjacent to v_n. We distinguish two cases depending on $\mathtt{MIN}(v_n) = \mathtt{MIN}(v_{n+1})$ or $\mathtt{MIN}(v_n) < \mathtt{MIN}(v_{n+1})$. In the following we consider these cases.

Case 1. Assume $\mathtt{MIN}(v_n) = \mathtt{MIN}(v_{n+1})$. Let f be a function from $[v_1, v_n]$ to $\{0, 1, 2\}$ as follows: $f(v) = g(v)$ for all $v \in [v_1, v_{n-1}]$ and $f(v_n) = 2$. So, $w(f) = w(g)$. We obtain that f is an IRDF on $H[v_1, v_n]$ with $f(v_n) = 2$ and so $i^2_R(v_n) \leq w(f) = i^2_R(v_{n+1})$. This, together with Inequality (3), proves that in Case 1 we have $i^2_R(v_1) \leq \cdots \leq i^2_R(v_{n+1})$.

Case 2. Assume $\mathtt{MIN}(v_n) < \mathtt{MIN}(v_{n+1})$. We distinguish two cases depending on $g(x) = 0$ for all vertices $x \in [\mathtt{MIN}(v_n), \mathtt{MIN}(v_{n+1})^-]$ or $g(x) \neq 0$ for some vertex $x \in [\mathtt{MIN}(v_n), \mathtt{MIN}(v_{n+1})^-]$. In the following we consider these cases.

- Assume that $g(x) = 0$ for all vertices $x \in [\mathtt{MIN}(v_n), \mathtt{MIN}(v_{n+1})^-]$, see Fig. 1(a). Let f be a function from $[v_1, v_n]$ to $\{0, 1, 2\}$ as follows: $f(v_n) = 2$ and $f(v) = g(v)$ for all $v \in [v_1, v_{n-1}]$. So, $w(g) = w(f)$. We deduce that f is an IRDF on $H[v_1, v_n]$ with $f(v_n) = 2$ and so

$$i^2_R(v_n) \leq w(f) = w(g) = i^2_R(v_{n+1}) \tag{4}$$

- Assume that $g(x) \neq 0$ for some $x \in [\mathtt{MIN}(v_n), \mathtt{MIN}(v_{n+1})^-]$. Because the set $[\mathtt{MIN}(v_n), v_n]$ is a clique of $H[v_1, v_n]$, there is exactly one vertex $x \in [\mathtt{MIN}(v_n), \mathtt{MIN}(v_{n+1})^-]$ with $g(x) \neq 0$. Clearly, $g(x) \in \{1, 2\}$. If $g(x) = 1$,

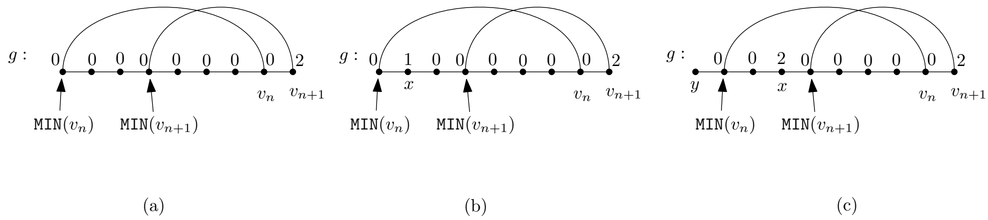

Figure 1. Illustrating a minimum IRDF g on H with $g(v_{n+1}) = 2$ such that (a) $g(x) = 0$ for all $x \in [\texttt{MIN}(v_n), \texttt{MIN}(v_{n+1})^-]$ and (b)–(c) $g(x) \neq 0$ for some $x \in [\texttt{MIN}(v_n), \texttt{MIN}(v_{n+1})^-]$; note that both sets $[\texttt{MIN}(v_n), v_n]$ and $[\texttt{MIN}(v_{n+1}), v_{n+1}]$ are cliques of H.

see Fig. 1(b), then assume that f is a function from $[v_1, v_n]$ to $\{0, 1, 2\}$ as follows: $f(v_n) = 2$, $f(x) = 0$, and $f(v) = g(v)$ for all $v \in [v_1, v_n] \setminus \{x, v_n\}$. So, $w(f) = w(g) - 1$. We deduce that f is an IRDF on $H[v_1, v_n]$ with $f(v_n) = 2$ and so

$$i_R^2(v_n) \leq w(f) < w(g) = i_R^2(v_{n+1}). \tag{5}$$

Assume $g(x) = 2$, see Fig. 1(c). Let g' be the restriction of g to $H[v_1, x]$. We deduce that g' is an IRDF on $H[v_1, x]$ with $g'(v_n) = 2$ and so $i_R^2(x) \leq w(g') = w(g) - 2 = i_R^2(v_{n+1}) - 2$. Let $y = \texttt{MIN}(v_n)^-$ and clearly $y < x$. By Inequality (3), we have $i_R^2(y) \leq i_R^2(x)$. Let f be a minimum IRDF on $H[v_1, y]$ with $f(y) = 2$. So, $w(f) = i_R^2(y)$. Let f' be a function from $[v_1, v_n]$ to $\{0, 1, 2\}$ as follows: $f'(v) = f(v)$ for all vertices $v \in [v_1, y]$, $f'(v) = 0$ for all vertices $v \in [y^+, v_{n-1}]$ and $f'(v_n) = 2$. So, $w(f') = i_R^2(y) + 2$. We deduce that f' is an IRDF on $H[v_1, v_n]$ with $f'(v_n) = 2$ and so

$$i_R^2(v_n) \leq w(f') = i_R^2(y) + 2 \leq i_R^2(x) + 2 \leq i_R^2(v_{n+1}). \tag{6}$$

Inequalities (4)–(6), together with Inequality (3), proves that in Case 2 we have $i_R^2(v_1) \leq \cdots \leq i_R^2(v_{n+1})$. This completes the proof of the lemma. □

Lemma 5. *Let $x \in \{v_2, \ldots, v_n\}$. Then $i_R^0(x) = i_R^2(\texttt{MIN}(x))$.*

Proof. We first prove that $i_R^2(\texttt{MIN}(x)) \leq i_R^0(x)$. Let f be a minimum IRDF on $G[v_1, x]$ with $f(x) = 0$ and so $w(f) = i_R^0(x)$. Clearly, $\texttt{MIN}(x) < x$. Since $N_{G[v_1,x]}[x] = [\texttt{MIN}(x), x]$, we have $f(z) = 2$ for some $z \in [\texttt{MIN}(x), x^-]$. By Lemma 4, $i_R^2(\texttt{MIN}(x)) \leq i_R^2(z)$. Because f is an IRDF on $G[v_1, x]$, we have $f(u) = 0$ for all $u \in N_{G[v_1,x]}(z)$. By Proposition 2, $[\texttt{MIN}(x), x]$ is a clique of G and so $f(u) = 0$ for all $u \in [z^+, x]$. Let f' be the restriction of f to $G[v_1, z]$. We deduce that f' is an IRDF on $G[v_1, z]$ with $f'(z) = 2$ and so $i_R^2(z) \leq w(f') = w(f) = i_R^0(x)$. Therefore, $i_R^2(\texttt{MIN}(x)) \leq i_R^0(x)$.
Now, we prove that $i_R^0(x) \leq i_R^2(\texttt{MIN}(x))$. Assume that g is an IRDF on $G[v_1, \texttt{MIN}(x)]$ with minimum weight and $g(\texttt{MIN}(x)) = 2$ and so $w(g) = i_R^2(\texttt{MIN}(x))$. We construct

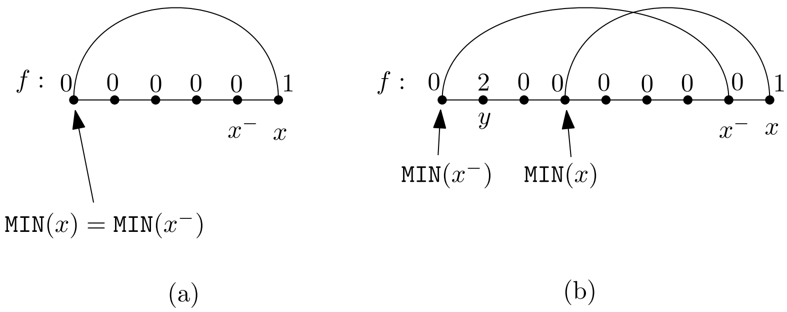

Figure 2. Illustrating a minimum IRDF f on $G[v_1, x]$ with $f(x) = 1$ such that $\mathtt{MIN}(x) < x^-$; note that both sets $[\mathtt{MIN}(x), x]$ and $[\mathtt{MIN}(x^-), x^-]$ are cliques of G.

a function h on $G[v_1, x]$ using g as follows: $h(v) = g(v)$ for all $v \in [v_1, \mathtt{MIN}(x)]$ and $h(v) = 0$ for all $v \in [\mathtt{MIN}(x)^+, x]$. We obtain that h is an IRDF on $G[v_1, x]$ with $h(x) = 0$ and so $i^0_R(x) \le w(h) = w(g) = i^2_R(\mathtt{MIN}(x))$. This completes the proof of the lemma. □

Lemma 6. *Let $x \in \{v_2, \ldots, v_n\}$.*

(i) If $\mathtt{MIN}(x) = x^-$, then $i^1_R(x) = i^0_R(x^-) + 1$.

(ii) If $\mathtt{MIN}(x) < x^-$ and $\mathtt{MIN}(x) \le \mathtt{MIN}(x^-)$, then $i^1_R(x)$ is not defined.

(iii) If $\mathtt{MIN}(x^-) < \mathtt{MIN}(x) < x^-$, then $i^1_R(x) = i^2_R(\mathtt{MIN}(x^-) + 1$.

Proof. Let f be a minimum IRDF on $G[v_1, x]$ with $f(x) = 1$. So, $w(f) = i^1_R(x)$. We first prove (i). Let $\mathtt{MIN}(x) = x^-$. So, x is adjacent only to x^-. Since f is an IRDF on $G[v_1, x]$ with $f(x) = 1$, $f(x^-) = 0$. Let f' be the restriction of f to $G[v_1, x^-]$. Thus, $w(f) = w(f') + 1$. We deduce that f' is an IRDF on $G[v_1, x^-]$ with $f'(x^-) = 0$ and so $i^0_R(x^-) \le w(f') = w(f) - 1 = i^1_R(x) - 1$. Conversely, let g be a minimum IRDF on $G[v_1, x^-]$ with $g(x^-) = 0$. Thus, $w(g) = i^0_R(x^-)$. Assume $h = g \cup \{(x, 1)\}$. We obtain that $w(h) = w(g) + 1$ and h is an IRDF on $G[v_1, x]$ with $h(x) = 1$ and so $i^1_R(x) \le w(h) = i^0_R(x^-) + 1$. This, together with $i^0_R(x^-) \le i^1_R(x) - 1$, completes the proof of (ii). In the rest of the proof we assume that $\mathtt{MIN}(x) < x^-$. So, $|[\mathtt{MIN}(x), x]| > 2$.

Now, we prove (ii). Let $\mathtt{MIN}(x) \le \mathtt{MIN}(x^-)$. By Proposition 2, $[\mathtt{MIN}(x), x]$ is a clique of G. See Fig. 2(a). Since f is an IRDF on $G[v_1, x]$ with $f(x) = 1$, $f(v) = 0$ for all $v \in [\mathtt{MIN}(x), x^-]$. It obtains that $f(x^-) = 0$ and so there is a vertex y with $f(y) = 2$ adjacent to x^- that is not adjacent to x, a contradiction. So, f is not defined. This completes the proof of (iii). In the rest of the proof we assume that $\mathtt{MIN}(x^-) < \mathtt{MIN}(x)$. See Fig. 2(b). The proof of (iii) is similar to the proof of Lemma 5. □

Lemma 7. *Let $x \in \{v_2, \ldots, v_n\}$. Then, $i_R^2(x) = i_R(\mathtt{MIN}(x)^-) + 2$.*

Proof. We first prove that $i_R(\mathtt{MIN}(x)^-) + 2 \leq i_R^2(x)$. Let f be a minimum IRDF on $G[v_1, x]$ with $f(x) = 2$. Hence, $w(f) = i_R^2(x)$. Because $N_{G[v_1,x]}[x] = [\mathtt{MIN}(x), x]$, $f(v) = 0$ for all vertices $v \in [\mathtt{MIN}(x), x^-]$. Let f' be the restriction of f to $G[v_1, \mathtt{MIN}(x)^-]$ and so $w(f') = w(f) - 2 = i_R^2(x) - 2$. We deduce that f' is an IRDF on $G[v_1, \mathtt{MIN}(x)^-]$ and so $i_R(\mathtt{MIN}(x)^-) \leq w(f') = i_R^2(x) - 2$.
Now, we prove that $i_R^2(x) \leq i_R(\mathtt{MIN}(x)^-) + 2$. Let f be a minimum IRDF on $G[v_1, \mathtt{MIN}(x)^-]$ and so $w(f) = i_R(\mathtt{MIN}(x)^-)$. We construct a function g from $[v_1, x]$ to $\{0, 1, 2\}$ as follows: $g(v) = f(v)$ for all $v \in [v_1, \mathtt{MIN}(x)^-]$, $g(v) = 0$ for all $v \in [\mathtt{MIN}(x), x^-]$ and $g(x) = 2$. Hence, $w(g) = w(f) + 2$. We deduce that g is an IRDF on $G[v_1, x]$ with $g(x) = 2$ and so $i_R^2(x) \leq w(g) = i_R(\mathtt{MIN}(x)^-) + 2$. This completes the proof of the lemma. □

Now, we prove that Algorithm 4.1 works correctly.

Theorem 4. *Let $G = (V, E)$ be a proper interval graph of order n with a consecutive ordering $(v_1, \ldots, v_n)$ of vertices of G. Then, Algorithm 4.1 on input $(G, v_1, \ldots, v_n)$ returns $i_R(G)$ in $O(n)$ time.*

Proof. Clearly, $i_R(G) = i_R(v_n) = \min\{i_R^0(v_n), i_R^1(v_n), i_R^2(v_n)\}$. It obtains that $i_R^0(v_1)$ is not defined, $i_R^1(v_1) = 1$ and $i_R^2(v_1) = 2$. Thus, by Lemmas 5, 6 and 7, the output of Algorithm 4.1 on input $(G, v_1, \ldots, v_n)$ is $i_R(G)$. It remains to compute the time complexity of Algorithm 4.1. By (Algorithm 2 of) [1], we can compute all values $\mathtt{MIN}(v_1), \ldots, \mathtt{MIN}(v_n)$ in $O(n)$ time. So, the running time of Algorithm 4.1 is $O(n)$. This completes the proof of the theorem. □

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