Communications in Combinatorics and Optimization
Vol. 8, No. 1 (2023), pp. 77-101
DOI: 10.22049/CCO.2021.26989.1175

CCO
Commun. Comb. Optim.

Research Article

Some families of α-labeled subgraphs of the integral grid

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Received: 30 October 2020; Accepted: 9 October 2021
Published Online: 11 October 2021

Abstract: In this work we study the most restrictive variety of graceful labeling, that is, we study the existence of an α-labeling for some families of graphs that can be embedded in the integral grid. Among the categories of graphs considered here we have a subfamily of 2-link fences, a subfamily of column-convex polyominoes, and a subfamily of irregular cyclic-snakes. We prove that under some conditions, the α-labeling of these graphs can be transformed into a harmonious labeling. We also present a closed formula for the number of 2-link fences examined here.

Keywords: graceful, α-labeling, integral grid, harmonious, polyomino, fence

AMS Subject classification: 05C78, 05C30

1. Introduction

Since its introduction, almost six decades ago, the concept of difference vertex labeling has received a significant amount of attention. A *difference vertex labeling* of a graph G of order n and size m is an injective function $f : V(G) \to S$, where S is a set on non-negative integers, with the property that every edge of G has assigned a *weight* defined as the absolute difference of the labels of its end-vertices. In his groundbreaking paper, Rosa [14] defined four of these functions, being graceful labeling the most studied type. A difference vertex labeling f is said to be *graceful* when $S = \{0, 1, \ldots, m\}$ and the set of weights induced by f is $W_f = \{1, 2, \ldots, m\}$; when such a labeling exists, the graph G is called *graceful*. A difference vertex labeling f is called *d-graceful* if

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$W_f = \{d, d+1, \ldots, d+m-1\}$ for some positive integer d. Thus, a graceful labeling is 1-graceful.

In general terms, if f is a labeling, the *complementary* labeling of f, denoted by $\overline{f}$, is the function defined for every $v \in V(G)$ by $\overline{f}(v) = \max(S) - f(v)$. Assuming that f is a graceful labeling of a graph G of size m and uv is an edge of G such that $|f(u) - f(v)| = x$, then $|\overline{f}(u) - \overline{f}(v)| = |m - f(u) - m + f(v)| = |f(v) - f(u)| = x$; in other terms, the weight of the edge uv is the same under both labelings and $\overline{f}$ is also a graceful labeling of G. A *shifting* of f is a function g such that, for every $v \in V(G)$, $g(v) = f(v) + c$ for some positive constant c. Note that $W_f = W_g$.

When G is a bipartite graph, some difference vertex labelings satisfy an interesting property: suppose that G is a bipartite graph with stable sets A and B, a difference vertex labeling f of G is said to be *bipartite* if there exists an integer λ, called *boundary value* of f, such that $\max\{f(v) : v \in A\} \leq \lambda < \min\{f(v) : v \in B\}$; this definition was introduced by Rosa and Širáň [16]. A graceful bipartite labeling of G is called an *α-labeling* and the graph G is said to be an *α-graph*. Let G be a bipartite graph of size m with stable sets A and B. If G is an α-graph, there exists an α-labeling f of G, such that $f(v) = 0$ for some $v \in V(G)$. Note that either f or $\overline{f}$ assigns the label 0 to a vertex of A. Assuming that $f(v) = 0$ for $v \in A$, then the labels assigned by f to the vertices of each stable set are in the sets $L_A \subseteq \{0, 1, \ldots, \lambda\}$ and $L_B \subseteq \{\lambda+1, \lambda+2, \ldots, m\}$. Moreover, if G has size m and order $m+1$, then $L_A = \{0, 1, \ldots, \lambda\}$ and $L_B = \{\lambda+1, \lambda+2, \ldots, m\}$. The α-labeling is the most restrictive among the labelings introduced by Rosa in [14]; in the area of graph labelings, they are located in the most prominent place. A detailed study of the interconnections between α-labelings and other types of graph labeling can be found in the book of López and Muntaner-Batle [12].

A graph G is said to be *arbitrarily graceful* if admits a d-graceful labeling for every positive integer d. This definition was introduced in 1982 independently by Maheo and Thuillier [13] and Slater [17]. Every α-graph is arbitrarily graceful. The process used to transform an α-labeling into a d-graceful labeling is the following: let f be an α-labeling of a graph G such that the vertex labeled 0 by f is in the stable set A, the function g defined by $g(v) = f(v)$ when $v \in A$ and $g(v) = f(v) + d - 1$ when $v \in B$ is in fact a d-graceful labeling; to prove this statement it is enough to observe that for every $w \in \{1, 2, \ldots, m\}$ such that $f(v) - f(u) = w$, $g(v) - g(u) = f(v) + d - 1 - f(u) = w + d - 1$, since $1 \leq w \leq m$, we get $d \leq w + d - 1 \leq d + m - 1$. Observe that for the labeling g, $L_A \subseteq \{0, 1, \ldots, \lambda\}$ and $L_B \subseteq \{d+\lambda, d+\lambda+1, \ldots, d+m-1\}$. This transformation of the α-labelings is used extensively in this work.

An additional property of the α-labelings is the existence of the reverse labeling. Let f be an α-labeling of G; there exists another α-labeling of G associated to f, we call this function the *reverse* labeling, it is defined by

$$f_r(v) = \begin{cases} \lambda - f(v) & \text{if } f(v) \leq \lambda, \\ m + \lambda + 1 - f(v) & \text{if } f(v) > \lambda, \end{cases}$$

where λ is the boundary value of f. This labeling was introduced by Rosa [15] under the name of *inverse* labeling. Thus, from a general perspective, for any given graph G, the number of graceful labelings of G is divisible by two and the number of α-labelings of G is divisible by four, because if f is graceful, so it is $\overline{f}$, and if f is an α-labeling, then $\overline{f}, f_r$, and $\overline{f}_r$ are also α-labelings of the same graph. In order to clarify these definitions, let us consider an example of an α-labeling f of the cycle C_8. If $f = (0, 8, 3, 4, 2, 5, 1, 7)$, then $\overline{f} = (8, 0, 5, 4, 6, 3, 7, 1)$, $f_r = (3, 4, 0, 8, 1, 7, 2, 5)$, $\overline{f}_r = (5, 4, 8, 0, 7, 1, 6, 3)$, and $g = (0, 13, 3, 9, 2, 10, 1, 12)$, where g is the 6-graceful labeling obtained from f.

In this work we study α-labelings of some graphs that can be embedded in the integral grid, that is, they are subgraphs of the Cartesian product $P_m \times P_n$. In [7], Gallian mentioned that problems related to the labeling of grids such $P_m \times P_n, P_m \times C_n$, and $C_m \times C_n$, are the most interesting to him. In [1], Acharya considered a specific kind of subgraphs of $P_m \times P_n$, called polyominoes; he proved that all convex polyominoes admit an α-labeling and asked whether or not the same is true for any polyomino. Jungreis and Reid [11] proved that the grid $P_m \times P_n$ is an α-graph; some subgraphs of this grid have been studied, see for instance the graphs considered in [1], [3], and [4]. In Section 2 we show the existence of an α-labeling for any graph formed by r copies of the path P_n, where two consecutive copies are connected by exactly two edges that determine a cycle of size divisible by four. The last result in that section is the enumeration of the non-isomorphic graphs that can be built in this way. In Section 3 we introduce the concept of zig-zag polyomino. These graphs form a subfamily of the vertical (or horizontal) convex polyominoes. We present an α-labeling for a robust set of these polyominoes. In Section 4 we give a new technique to concatenate α-labeled graphs obtaining a chain graph with an α-labeling. As a consequence of this result, we have a subgraph of $P_m \times P_n$ formed by concatenating α-cycles, expanding in this way the number of cyclic-snakes admitting either a graceful or an α-labeling. We conclude this work in Section 5, where we study how some of the labelings of these graphs can be transformed into a harmonious labeling.

All graphs considered in this paper are simple, i.e., finite with no loops nor multiple edges. The notation and terminology not given here is taken from [5] and [8].

2. α-Labeling of the Family $\mathscr{H}_{n,r}$

A *2-link fence* is a bipartite graph obtained with r copies of the path P_n by connecting two vertices of the ith copy to the corresponding two vertices in the $(i+1)$th copy. The main result of this section is the introduction of an α-labeling for any member of a subfamily of 2-link fences.

Suppose that $v_1, v_2, \ldots, v_n$ are the consecutive vertices of P_n, where $A = \{v_1, v_3 \ldots\}$ and $B = \{v_2, v_4, \ldots\}$. Let $f : V(P_n) \to \{0, 1, \ldots, n-1\}$ be the α-labeling of P_n, given by Rosa [14]; thus,

$$f(v_j) = \begin{cases} \dfrac{j-1}{2} & \text{if } j \text{ is odd,} \\ n - \dfrac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

Hence, when n is odd, $L_A = [0, \frac{n-1}{2}]$ and $L_B = [\frac{n+1}{2}, n-1]$; when n is even, $L_A = [0, \frac{n-2}{2}]$ and $L_B = [\frac{n}{2}, n-1]$.
Consequently, the reverse of the complementary labeling is defined by

$$\overline{f}_r(v_j) = \begin{cases} \dfrac{n-1-\delta+j}{2} & \text{if } j \text{ is odd,} \\ \dfrac{n-\delta-j}{2} & \text{if } j \text{ is even,} \end{cases}$$

where $\delta = 1$ if n is odd, and $\delta = 0$ otherwise. Therefore, when n is odd, $L_A = [0, \frac{n-3}{2}]$ and $L_B = [\frac{n-1}{2}, n-1]$; when n is even, $L_A = [0, \frac{n-2}{2}]$ and $L_B = [\frac{n}{2}, n-1]$.

Let $C^1, C^2, \dots, C^r$ be disjoint copies of P_n. For each $1 \leq i \leq r$, $V(C^i) = \{v_1^i, v_2^i, \dots, v_n^i\}$. We denote by $\mathscr{H}_{n,r}$ the family of all graphs of order rn and size $r(n+1)-2$, obtained by connecting, for all $1 \leq i \leq r-1$, the vertices $v_{j_i}^i$ and $v_{k_i}^i$ of C^i to the vertices $v_{j_i}^{i+1}$ and $v_{k_i}^{i+1}$ of C^{i+1} respectively, where $j_i, k_i \in \{1, 2, \dots, n\}$ and $|j_i - k_i|$ is odd. We refer to $v_{j_i}^i$ and $v_{k_i}^i$ as the *links* connecting C^i and C^{i+1}. Note that $\mathscr{H}_{n,r}$ is a robust family, for instance, when $r = 3$ and $n = 7$, we have identified 20 non-isomorphic elements of $\mathscr{H}_{3,7}$. At the end of this section we investigate the number of non-isomorphic elements in $\mathscr{H}_{n,r}$. We claim that all the elements of $\mathscr{H}_{n,r}$ are α-graphs.

Theorem 1. *Let $n \geq 2$, if $G \in \mathscr{H}_{n,r}$, then G is an α-graph for every $r \geq 2$.*

Proof. Let $G \in \mathscr{H}_{n,r}$ and $C^1, C^2, \dots, C^r$ be the disjoint copies of P_n used to construct G, being $v_1^i, v_2^i, \dots, v_n^i$ the consecutive vertices of C^i. The labelings f and $\overline{f}_r$ given at the begining of the section are used to label the copies of P_n; thus, the copy C^i is labeled using f when i is odd and is labeled using $\overline{f}_r$ when i is even. The α-labeling of C^i is transformed, following the steps given in the Introduction, into a $((n+1)(r-i)+1)$-graceful labeling. Thus, $W_i = \{(n+1)(r-i)+1, (n+1)(r-i)+2, \dots, (n+1)(r-i)+n-1\}$ is the set formed by the weights of the edges of C^i. Since $\min(W_i) = (n+1)(r-i)+1$ and $\max(W_{i+1}) = (n+1)(r-(i+1))+n-1 = (n+1)(r-i)-2$, we get that $\min(W_i) - \max(W_{i+1}) = 2$; i.e., these sets are pairwise disjoint. In addition, $\cup_{i=1}^r W_i \subset \{1, 2, \dots, (n+1)r-2\}$.
Let A_i and B_i denote the stable sets of C^i, where A_i is the set which vertices have

the smaller labels. Therefore,

$$L_{A_i} = \begin{cases} [0, \frac{n-1}{2}] & \text{if } n \text{ is odd and } i \text{ is odd,} \\ [0, \frac{n-3}{2}] & \text{if } n \text{ is odd and } i \text{ is even,} \\ [0, \frac{n-2}{2}] & \text{if } n \text{ is even,} \end{cases}$$

and

$$L_{B_i} = \begin{cases} [\frac{n+1}{2} + (n+1)(r-i), n-1+(n+1)(r-i)] & \text{if } n \text{ is odd and } i \text{ is odd,} \\ [\frac{n-1}{2} + (n+1)(r-i), n-1+(n+1)(r-i)] & \text{if } n \text{ is odd and } i \text{ is even,} \\ [\frac{n}{2} + (n+1)(r-i), n-1+(n+1)(r-i)] & \text{if } n \text{ is even.} \end{cases}$$

For each $i \geq 1$, the $((n+1)(r-i)+1)$-graceful labeling of C^i is shifted c_i units, where

$$c_i = \begin{cases} \frac{1}{2}n(i-1) + \left\lfloor \frac{i-1}{2} \right\rfloor & \text{if } n \text{ is even,} \\ \frac{1}{2}(n+1)(i-1)) & \text{if } n \text{ is odd.} \end{cases}$$

Since the constant c_i is added to each label on C^i, the set of weights induced by the new labeling is still W_i; but the value of c_i depends on i, therefore the labels on the stable sets of C^i change.
Thus, when n is even,

$$L_{A_i} = \left[\frac{1}{2}n(i-1) + \left\lfloor \frac{i-1}{2} \right\rfloor, \frac{1}{2}ni + \left\lfloor \frac{i-1}{2} \right\rfloor - 1\right]$$

and

$$L_{B_i} = \left[\frac{1}{2}ni + \left\lfloor \frac{i-1}{2} \right\rfloor + (n+1)(r-i), \frac{1}{2}n(i+1) + \left\lfloor \frac{i-1}{2} \right\rfloor + (n+1)(r-i) - 1\right].$$

In view of the fact that $i \leq r$, we get that $(n+1)(r-i) \geq 0$, which implies that $\max(L_{A_i}) < \min(L_{B_i})$; in particular, when $i = r$ we have that $\min(L_{B_r}) < \max(L_{A_r}) = 1$, which is consistent with the fact that the labeling of C^r is just a shifting of the original α-labeling. Because $\min(L_{A_{i+1}}) = \frac{1}{2}ni + \left\lfloor \frac{i}{2} \right\rfloor > \max(L_{A_i})$ and $\max(L_{B_{i+1}}) = \frac{1}{2}ni + \left\lfloor \frac{i}{2} \right\rfloor + (n+1)(r-i) - 2 < \min(L_{B_i})$, we have that $L_{A_i} \cap L_{A_{i+1}} = \varnothing$ and $L_{B_i} \cap L_{B_{i+1}} = \varnothing$ for each $i \geq 1$. That is, there is no repetition of labels, i.e., we have an injective assignment of labels.
When n is odd, we need to analyze two cases:

- When i is odd,
$$L_{A_i} = \left[\frac{1}{2}(n+1)(i-1), \frac{1}{2}(n+1)i - 1\right]$$
and
$$L_{B_i} = \left[\frac{1}{2}(n+1)(2r-i), \frac{1}{2}(n+1)(2r-i) + \frac{1}{2}(n-3)\right].$$
Since $i \leq r$ and $n \geq 2$, we get that
$$\begin{aligned}\min(L_{B_i}) - \max(L_{A_i}) &= \frac{1}{2}(n+1)(2r-i) - \frac{1}{2}(n+1)i + 1 \\ &= (n+1)r - \frac{1}{2}(n+1)i - \frac{1}{2}(n+1)i + 1 \\ &= (n+1)(r-i) + 1 > 0.\end{aligned}$$
Hence, $\max(L_{A_i}) < \min(L_{B_i})$. When $i = r$, these two values are consecutive numbers.

- When i is even,
$$L_{A_i} = \left[\frac{1}{2}(n+1)(i-1), \frac{1}{2}(n+1)i - 2\right]$$
and
$$L_{B_i} = \left[\frac{1}{2}(n+1)(2r-i) - 1, \frac{1}{2}(n+1)(2r-i) + \frac{1}{2}(n-3)\right].$$
As in the previous case,
$$\begin{aligned}\min(L_{B_i}) - \max(L_{A_i}) &= \frac{1}{2}(n+1)(2r-i) - 1 - \frac{1}{2}(n+1)i + 2 \\ &= (n+1)(r-i) + 1 > 0.\end{aligned}$$
In other terms, $\max(L_{A_i}) < \min(L_{B_i})$. When $i = r$, these values differ by one.

Regardless the parity of i,
$$\min(L_{A_{i+1}}) = \frac{1}{2}(n+1)i > \max(L_{A_i})$$
and
$$\max(L_{B_{i+1}}) = \frac{1}{2}(n+1)(2r-i) - 2 < \min(L_{B_i}).$$
Consequently, $L_{A_i} \cap L_{A_{i+1}} = \varnothing$ and $L_{B_i} \cap L_{B_{i+1}} = \varnothing$. This implies that the assignment of labels is injective.

Now we turn our attention to the weight of the edges connecting C^i with C^{i+1}. Since the labels of the links on C^i depend on the parity of the parameters n and i, we analyze the four existing cases. Recall that the links of C^i are v^i_j and v^i_k and that j and k have different parity; without loss of generality we assume that j is odd.

- When n is even and i is even:

  The label of v_j^i is: $\frac{n-1+j}{2} + (n+1)(r-i) + \frac{1}{2}n(i-1) + \frac{i-2}{2}$.

  The label of v_k^i is: $\frac{n-k}{2} + \frac{1}{2}n(i-1) + \frac{i-2}{2}$.

  The label of v_j^{i+1} is: $\frac{j-1}{2} + \frac{1}{2}ni + \frac{i}{2}$.

  The label of v_k^{i+1} is: $n - \frac{k}{2} + (n+1)(r-i-1) + \frac{1}{2}ni + \frac{i}{2}$.

  The weight of $v_j^i v_j^{i+1}$ is:

  $\left(\frac{n-1+j}{2} + (n+1)(r-i) + \frac{1}{2}n(i-1) + \frac{i-2}{2}\right) - \left(\frac{j-1}{2} + \frac{1}{2}ni + \frac{i}{2}\right) = (n+1)(r-i) - 1$.

  The weight of $v_k^i v_k^{i+1}$ is:

  $\left(n - \frac{k}{2} + (n+1)(r-i-1) + \frac{1}{2}ni + \frac{i}{2}\right) - \left(\frac{n-k}{2} + \frac{1}{2}n(i-1) + \frac{i-2}{2}\right) = (n+1)(r-i)$.

- When n is even and i is odd:

  The label of v_j^i is: $\frac{j-1}{2} + \frac{1}{2}(n-i) + \frac{i-1}{2}$.

  The label of v_k^i is: $n - \frac{k}{2} + (n+1)(r-i) + \frac{1}{2}n(i-1) + \frac{i-1}{2}$.

  The label of v_j^{i+1} is: $\frac{n-1+j}{2} + (n+1)(r-i-1) + \frac{1}{2}ni + \frac{i-1}{2}$.

  The label of v_k^{i+1} is: $\frac{n-k}{2} + \frac{1}{2}ni + \frac{i-1}{2}$.

  The weight of $v_j^i v_j^{i+1}$ is:

  $\left(\frac{n-1+j}{2} + (n+1)(r-i-1) + \frac{1}{2}ni + \frac{i-1}{2}\right) - \left(\frac{j-1}{2} + \frac{1}{2}(n-i) + \frac{i-1}{2}\right) = (n+1)(r-i) - 1$.

  The weight of $v_k^i v_k^{i+1}$ is:

  $\left(n - \frac{k}{2} + (n+1)(r-i) + \frac{1}{2}n(i-1) + \frac{i-1}{2}\right) - \left(\frac{n-k}{2} + \frac{1}{2}ni + \frac{i-1}{2}\right) = (n+1)(r-i)$.

- When n is odd and i is even:

  The label of v_j^i is: $\frac{n-2+j}{2} + (n+1)(r-i) + \frac{1}{2}(n+1)(i-1)$.

  The label of v_k^i is: $\frac{n-1-k}{2} + \frac{1}{2}(n+1)(i-1)$.

  The label of v_j^{i+1} is: $\frac{j-1}{2} + \frac{1}{2}(n+1)(i+1-1)$.

  The label of v_k^{i+1} is: $n - \frac{k}{2} + (n+1)(r-i-1) + \frac{1}{2}(n+1)(i+1-1)$.

  The weight of $v_j^i v_j^{i+1}$ is:

  $\left(\frac{n-2+j}{2} + (n+1)(r-i) + \frac{1}{2}(n+1)(i-1)\right) - \left(\frac{j-1}{2} + \frac{1}{2}(n+1)(i+1-1)\right) = (n+1)(r-i) - 1$.

  The weight of $v_k^i v_k^{i+1}$ is:

  $\left(n - \frac{k}{2} + (n+1)(r-i-1) + \frac{1}{2}(n+1)(i+1-1)\right) - \left(\frac{n-1-k}{2} + \frac{1}{2}(n+1)(i-1)\right) = (n+1)(r-i)$.

- When n is odd and i is odd:

  The label of v_j^i is: $\frac{j-1}{2} + \frac{1}{2}(n+1)(i-1)$.

  The label of v_k^i is: $n - \frac{k}{2} + (n+1)(r-i) + \frac{1}{2}(n+1)(i-1)$.

  The label of v_j^{i+1} is: $\frac{n-2+j}{2} + (n+1)(r-i-1) + \frac{1}{2}(n+1)(i+1-1)$.

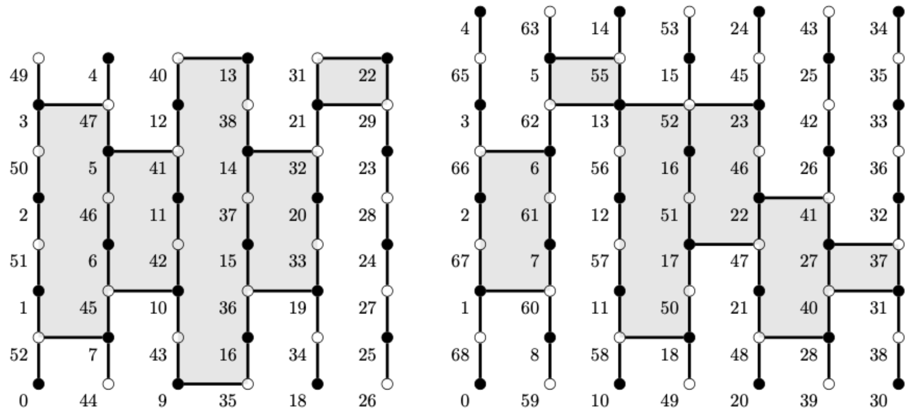

Figure 1. **α-labeled members of $\mathscr{H}_{8,6}$ and $\mathscr{H}_{9,7}$.**

The label of v_k^{i+1} is: $\frac{n-1-k}{2} + \frac{1}{2}(n+1)(i+1-1)$.

The weight of $v_j^i v_j^{i+1}$ is:

$\left(\frac{n-2+j}{2} + (n+1)(r-i-1) + \frac{1}{2}(n+1)(i+1-1)\right) - \left(\frac{j-1}{2} + \frac{1}{2}(n+1)(i-1)\right) = (n+1)(r-i) - 1.$

The weight of $v_k^i v_k^{i+1}$ is:

$\left(n-\frac{k}{2}+(n+1)(r-i)+\frac{1}{2}(n+1)(i-1)\right)-\left(\frac{n-1-k}{2}+\frac{1}{2}(n+1)(i+1-1)\right) = (n+1)(r-i).$

Since $\min(W_i) = (n+1)(r-i)+1$ and $\max(W_{i+1}) = (n+1)(r-i)-2$, the weights induced on the edges connecting C^i and C^{i+1}, are the numbers required to conclude that the set of all weights on the edges of C^i and C^{i+1}, together with the edges connecting them, is a set of consecutive numbers. This implies that the weights induced on the edges of G are $1, 2, \ldots, (n+1)r-2$.
Considering the fact that the final labeling of C^i is bipartite, because it is a shifting of a d-graceful labeling obtained from an α-labeling, and the edges linking the copies of P_n always connect vertices on different stable sets, we conclude that the labeling of G is definitely an α-labeling. □

In Figure 1 we show two examples of the α-labeling obtained with this procedure, for a member of $\mathscr{H}_{8,6}$ and a member of $\mathscr{H}_{9,7}$.

As we mentioned earlier, the way used to construct the elements of $\mathscr{H}_{n,r}$ produces graphs that are isomorphic. We are interested in the number of non-isomorphic graphs in this family; we denote this number by $a(n,r)$. In order to determine this quantity, we use Pólya's enumeration theorem; we start assuming that a generic element G of $\mathscr{H}_{n,r}$, can be enclosed in a rectangle of height $n-1$ and width $r-1$, and that the geometric center of this rectangle corresponds to the center of a rectangular coordinate

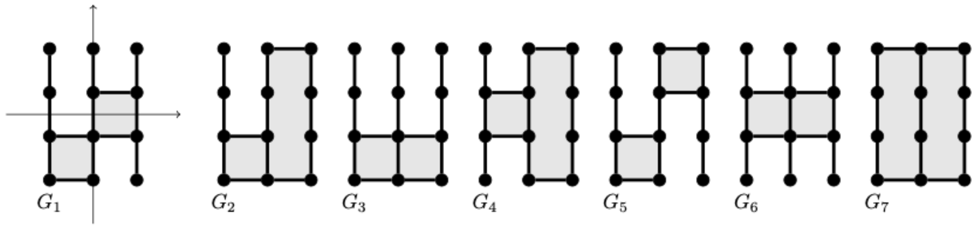

Figure 2. **All non-isomorphic members of $\mathscr{H}_{4,3}$.**

system. This representation is used to explain the "symmetries" of this graph that allow us to determine when two members of $\mathscr{H}_{n,r}$ are isomorphic.
Let $F, G \in \mathscr{H}_{n,r}$, we say that $F \sim G$ if and only if $F \cong G$. Clearly, this is an equivalence relation in $\mathscr{H}_{n,r}$. Thus, $a(n,r)$ is the number of equivalence classes in the partition of $\mathscr{H}_{n,r}$ induced by the relation $\sim$. In Figure 2 we show the seven non-isomorphic elements of $\mathscr{H}_{4,3}$.
Among the graphs in Figure 2, only G_1 and G_2 do not present any symmetry. In contrast, G_3, G_6, and G_7 have a symmetry with respect to the vertical axis; G_4, G_6, and G_7 have a symmetry with respect to the horizontal axis; G_5, G_6, and G_7 have a symmetry with respect to the origin. In general if $[G] = \{F \in \mathscr{H}_{n,r} : F \sim G\}$, then

i. $\big|[G]\big| = 4$ when G has no symmetries, i.e., G together with its two reflections and $180°$ rotation are in $[G]$.

ii. $\big|[G]\big| = 2$ when G has exactly one symmetry, i.e., G together with a reflection or $180°$ rotation are in $[G]$.

iii. $\big|[G]\big| = 1$ when G in indistinguishable from any of its reflections or $180°$ rotation.

Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be subfamilies of $\mathscr{H}_{n,r}$, where $\mathscr{A}$ contains all the elements symmetric with respect to the vertical axis, $\mathscr{B}$ contains all the elements symmetric with respect to the horizontal axis, and $\mathscr{C}$ contains all the elements symmetric with respect to the origin. Thus, every element of $\mathscr{H}_{n,r}$ has multiplicity four in the multiset $\mathscr{H}_{n,r} \cup \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$. Indeed, let $G \in \mathscr{H}_{n,r}$ such that

- $G \notin \mathscr{A} \cup \mathscr{B} \cup \mathscr{C}$, then G is counted four times in $\mathscr{H}_{n,r}$.
- $G \in \mathscr{A}$, then G is counted twice in $\mathscr{A}$ and twice in $\mathscr{H}_{n,r}$.
- $G \in \mathscr{B}$, then G is counted twice in $\mathscr{B}$ and twice in $\mathscr{H}_{n,r}$.
- $G \in \mathscr{C}$, then G is counted twice in $\mathscr{C}$ and twice in $\mathscr{H}_{n,r}$.
- $G \in \mathscr{A} \cap \mathscr{B} \cap \mathscr{C}$, then G is counted once on each of $\mathscr{A}, \mathscr{B}, \mathscr{C}$, and $\mathscr{H}_{n,r}$.

Consequently,

$$a(n,r) = \frac{1}{4}\left(\left|\mathscr{H}_{n,r}\right| + |\mathscr{A}| + |\mathscr{B}| + |\mathscr{C}|\right).$$

This formula is the support of the coming theorems. We use the notation $C(p,q)$ to represent the binomial coefficient, that is, $C(p,q) = \frac{p!}{q!(p-q)!}$.

Theorem 2. *For every $n \geq 1$ and $r \geq 2$, the number of non-isomorphic graphs in $\mathscr{H}_{2n,r}$ is*

$$a(2n,r) = \begin{cases} \dfrac{n^{r-1}\left(n^{r-1}+n+2\right)}{4} & \text{when } r \text{ is even,} \\ \dfrac{n^{r-1}\left(n^{r-1}+3\right)}{4} & \text{when } r \text{ is odd.} \end{cases}$$

Proof. Let $C^1, C^2, \ldots, C^r$ be the copies of P_{2n} used to form any graph in $\mathscr{H}_{2n,r}$. Let $v_1^i, v_2^i, \ldots, v_{2n}^i$ be the consecutive vertices of C^i. Suppose that $G \in \mathscr{H}_{2n,r}$ and $i \in \{1,2,\ldots,r-1\}$; if v_j^i and v_k^i are the links connecting C^i and C^{i+1}, then $\text{dist}(v_j^i, v_k^i)$ is odd, this implies that the indices j and k have different parity. Therefore, $C(n,1)C(n,1) = n^2$ is the number of different ways to select these links. Consequently, $\left|\mathscr{H}_{2n,r}\right| = (n^2)^{r-1}$.
In order to determine the cardinality of the subfamilies, we analyze two cases that depend on the parity of r.
Case I: When r is even. Suppose that $G \in \mathscr{A}$, then for every $i \in \{1,2,\ldots,\frac{r}{2}\}$, the subgraph of G induced by $C^i \cup C^{i+1}$ is isomorphic to the subgraph induced by $C^{r-i} \cup C^{r-i+1}$. Note that when $i = \frac{r}{2}$, $C^i = C^{r-i}$ and $C^{i+1} = C^{r-i+1}$; it follows that $\left(n^2\right)^{\frac{r}{2}} = n^r$ is the number of graphs in $\mathscr{A}$. Suppose now that $G \in \mathscr{B}$; then for every $i \in \{1,2,\ldots,r-1\}$, the indices of the links connecting C^i and C^{i+1} must satisfy $j+k = 2n+1$; since there are n pairs j,k in $\{1,2,\ldots,2n\}$ satisfying this condition, we conclude that n^{r-1} is the number of graphs in $\mathscr{B}$. Finally, suppose that $G \in \mathscr{C}$; since G is symmetric with respect to the origin, the links connecting $C^{\frac{r}{2}}$ and $C^{\frac{r}{2}+1}$ must satisfy the same equation that the graphs in $\mathscr{B}$, i.e., there are n different options to select them. For each $i \in \{1,2,\ldots,\frac{r}{2}-1\}$, if v_j^i and v_k^i are the links connecting C^i and C^{i+1}, then the links connecting C^{r-i} and C^{r-i+1} are v_{2n+1-j}^{r-i} and v_{2n+1-k}^{r-i}; hence, there are $\left(n^2\right)^{\frac{r}{2}-1} \cdot n = n^{r-1}$ graphs in $\mathscr{C}$. Therefore, when r is even,

$$a(2n,r) = \frac{n^{2(r-1)} + n^r + n^{r-1} + n^{r-1}}{4} = \frac{n^{r-1}\left(n^{r-1}+n+2\right)}{4}.$$

Case II: When r is odd. If $G \in \mathscr{A}$, then for every $i \in \{1,2,\ldots,\frac{r-1}{2}\}$, the selection of v_j^i and v_k^i forces a unique selection for v_j^{r-i} and v_k^{r-i}. In other terms, we have $\left(n^2\right)^{\frac{r-1}{2}} = n^{r-1}$ graphs in $\mathscr{A}$. If $G \in \mathscr{B}$, we have the same situation that in the case r even, except that $i \in \{1,2,\ldots,r-1\}$, which implies that there are n^{r-1} elements in $\mathscr{B}$. If $G \in \mathscr{C}$, for each $i \in \{1,2,\ldots,\frac{r-1}{2}\}$, there are n^2 ways to select the links connecting

$n\backslash r$	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1
2	3	7	24	76	288	1,072	4,224	16,576
3	6	27	216	1,701	15,066	133,407	1,198,476	10,766,601
4	10	76	1,120	16,576	263,680	4,197,376	67,133,440	1,073,790,976
5	15	175	4,125	98,125	2,446,875	61,046,875	1,526,015,625	38,147,265,625
6	21	351	12,096	420,876	15,132,096	544,230,576	19,591,600,896	705,278,736,576
7	28	637	30,184	1,443,001	70,656,628	3,460,410,037	169,557,621,184	8,308,236,966,001
8	36	1,072	66,816	4,197,376	268,517,376	17,180,065,792	1,099,516,870,656	70,368,756,760,576

Table 1. **Number of non-isomorphic graphs in $\mathscr{H}_{2n,r}$.**

C^i to C^{i+1}; each of these selections imposes a unique selection of the links between C^{r-i} and C^{r-i+1}; thus, there are $\left(n^2\right)^{\frac{r-1}{2}} = n^{r-1}$ graphs in $\mathscr{C}$. Consequently, when r is odd,

$$a(2n,r) = \frac{n^{2(r-1)} + n^{r-1} + n^{r-1} + n^{r-1}}{4} = \frac{n^{r-1}\left(n^{r-1}+3\right)}{4}.$$

$\square$

In Table 1 we show the initial values of $a(2n,r)$. Note that the first column consists of the triangular numbers, sequence A000217 in OEIS (The On-Line Encyclopedia of Integer Sequences). We can also find in OEIS, the second column as well as the first two rows, A039623, A225826, A225828, respectively.

Theorem 3. *For every $n \geq 1$ and $r \geq 2$, the number of non-isomorphic graphs in $\mathscr{H}_{2n+1,r}$ is*

$$a(2n+1,r) = \begin{cases} \dfrac{\left(n(n+1)\right)^{r-1} + \left(n(n+1)\right)^{\frac{r}{2}}}{4} & \text{when } r \text{ is even,} \\ \dfrac{\left(n(n+1)\right)^{r-1} + 2\left(n(n+1)\right)^{\frac{r-1}{2}}}{4} & \text{when } r \text{ is odd.} \end{cases}$$

Proof. Within the set $\{1,2,\ldots,2n+1\}$, n and $n+1$ are the corresponding amounts of even and odd numbers. Then, $C(n,1)C(n+1,1) = n(n+1)$ is the number of different ways to select the two links between C^i and C^{i+1} for each $i \in \{1,2,\ldots,r-1\}$. Thus, $\left|\mathscr{H}_{2n+1,r}\right| = \left(n(n+1)\right)^{r-1}$. Note that $\mathscr{B} = \varnothing$. To prove this statement we argue by contradiction. Suppose that $\mathscr{B} \neq \varnothing$, then there exists $G \in \mathscr{B}$ such that for every $i \in \{1,2,\ldots,r-1\}$, the links v^i_j and v^i_k must satisfy $j+k = 2n+2$, which implies that j and k have the same parity contradicting the fact that $|j-k|$ is odd. Since $\mathscr{B} = \varnothing$, then $\mathscr{C} = \varnothing$ when r is even because, for every $G \in \mathscr{C}$, the subgraph of G induced by the vertices of $C^{\frac{r}{2}} \cup C^{\frac{r}{2}+1}$ must be in $\mathscr{B}$ which is empty.

$n\backslash r$	2	3	4	5	6	7	8	9
1	1	2	3	6	10	20	36	72
2	3	12	63	342	1,998	11,772	70,308	420,552
3	6	42	468	5,256	62,640	747,630	8,963,136	107,505,792
4	10	110	2,100	40,200	802,000	16,004,000	320,040,0000	6,400,080,000
5	15	240	6,975	202,950	6,081,750	182,263,500	5,467,702,500	164,025,405,000
6	21	462	18,963	778,806	32,691,330	1,372,294,980	57,635,611,236	2,420,664,554,952
7	28	812	44,688	2,460,192	137,726,848	7,719,332,672	431,776,171,008	24,179,332,810,752
8	36	1,332	94,608	6,721,056	483,822,720	34,828,704,000	2,507,659,969,536	180,551,047,514,112

Table 2. **Number of non-isomorphic graphs in $\mathscr{H}_{2n+1,r}$.**

Case I: When r is even. Suppose that $G \in \mathscr{A}$; for each $i \in \{1, 2, \dots, \frac{r}{2}\}$, the selection of the links between C^i and C^{i+1} forces a unique option for the links between C^{r-i} and C^{r-i+1}. Thus, there are $\big(n(n+1)\big)^{\frac{r}{2}}$ graphs in $\mathscr{A}$. Since $\mathscr{B}$ and $\mathscr{C}$ are empty, we conclude that

$$\begin{aligned} a(2n+1, r) &= \frac{\big(n(n+1)\big)^{r-1} + \big(n(n+1)\big)^{\frac{r}{2}} + 0 + 0}{4} \\ &= \frac{\big(n(n+1)\big)^{r-1} + \big(n(n+1)\big)^{\frac{r}{2}}}{4}. \end{aligned}$$

Case II: When r is odd. The only difference with the previous case is that now $\mathscr{C} \neq \varnothing$. If $G \in \mathscr{C}$, then for every $i \in \{1, 2, \dots, \frac{r-1}{2}\}$, the selection of the links between C^i and C^{i+1} determines, uniquely, the links between C^{r-i} and C^{r-i+1}. So, $\big(n(n+1)\big)^{\frac{r-1}{2}}$ is the number of graphs in $\mathscr{C}$. Consequently, when r is odd,

$$\begin{aligned} a(2n+1, r) &= \frac{\big(n(n+1)\big)^{r-1} + \big(n(n+1)\big)^{\frac{r-1}{2}} + 0 + \big(n(n+1)\big)^{\frac{r-1}{2}}}{4} \\ &= \frac{\big(n(n+1)\big)^{r-1} + 2\left(n(n+1)\right)^{\frac{r-1}{2}}}{4}. \end{aligned}$$

$\square$

In Table 2 we show the initial values of $a(2n+1, r)$. As in Table 1, the first column consists of the triangular numbers. The second column and first row are the sequences A001621 and A005418 in OEIS.

3. α-Labeling of Zig-Zag Polyominoes

The basic definition of a polyomino [9] states that *polyominoes* are 2-dimensional shapes made by connecting a certain number of congruent squares, each joined together with at least one other square along a side. From the perspective of graph theory, a polyomino is the skeleton of the 2-dimensional shape, i.e., the graph induced by the corners of the squares used to build the shape. Therefore, polyominoes are subgraphs of the integral grid, that is, subgraphs of $P_n \times P_m$. For any given polyomino G, the minimal $P_n \times P_m$ that contains G as a subgraph is called the *bounding box* of G. A polyomino is said to be *convex* if its perimeter equals the perimeter of its bounding box. The study of α-labelings for this kind of graph was initiated by Acharya [1], where it was asked whether or not all polyominoes admit a d-graceful labeling for every $d \geq 1$. Acharya partially answered this question by proving that all convex polyominoes have a d-graceful labeling. Another family of polyominoes with the same property is the family formed by the snake polyominoes, that is, by those polyominoes where each square shares at most two sides with other squares. In [3], Barrientos and Minion presented a method to construct an α-labeling for any snake polyomino based on the use of conveniently labeled building blocks. All polyominoes with up to seven squares are α-graphs, a big portion of them can be labeled using the labelings presented in [1] and [3]; the α-labeling for those polyominoes that are not snakes or convex was found case by case. So, we repeat Acharya's question: Are all polyominoes arbitrarily graceful? The evidence indicates that the answer to this question is affirmative.
A *column-convex polyomino* is a self-avoiding polyomino such that the intersection of any vertical line with the polyomino results in at most two components. The main result of this section is the proof of the existence of an α-labeling for every member of a subfamily of column-convex polyominoes.

It is well-known that the ladder L_n admits an α-labeling [11]. In the following proposition we show an α-labeling of L_n, different of the one given in [11], that will be used in the construction of new harmonious and α-graphs. Before presenting the labeling we introduce some conventions. The *ladder* L_n is the result of the Cartesian product of the paths P_n and P_2. Thus, $L_n = P_n \times P_2$ is a bipartite graph of order $2n$ and size $3n-2$. In order to facilitate the description of the labeling, we define the vertex and edge sets of L_n in the following way: assuming that $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are the consecutive vertices of two copies of P_n, then $V(L_n) = \{u_j, v_j : 1 \leq j \leq n\}$ and $E(L_n) = \{u_j u_{j+1}, v_j v_{j+1} : 1 \leq j \leq n-1\} \cup \{u_j v_j : 1 \leq j \leq n\}$. Let A and B be the stable sets of L_n, without loss of generality, we assume that $u_1 \in A$. We refer to the elements of A and B as the *black* and *white* vertices, respectively.

Proposition 1. *For $n \geq 1$, the ladder L_n is an α-graph.*

Proof. Consider the following labeling of the vertices of L_n:

$$f(u_j) = \begin{cases} 2j-2 & \text{if } j \text{ is odd,} \\ 3n-1-j & \text{if } j \text{ is even,} \end{cases}$$

and

$$f(v_j) = \begin{cases} 3n-1-j & \text{if } j \text{ is odd,} \\ 2j-2 & \text{if } j \text{ is even.} \end{cases}$$

Note that each part of this function is injective; moreover, when n is odd,

$$\{f(u_j) : j \text{ is odd}\} \cup \{f(v_j) : j \text{ is even}\} = \\ \{0, 4, \ldots, 2n-2\} \cup \{2, 6, \ldots, 2n-2\} = \{0, 2, \ldots, 2n-2\},$$

and

$$\{f(u_j) : j \text{ is even}\} \cup \{f(v_j) : j \text{ is odd}\} = \\ \{2n, 2n+2, \ldots, 3n-3\} \cup \{2n-1, 2n+1, \ldots, 3n-2\} = \{2n-1, 2n, \ldots, 3n-2\}.$$

When n is even,

$$\{f(u_j) : j \text{ is odd}\} \cup \{f(v_j) : j \text{ is even}\} = \\ \{0, 4, \ldots, 2n-4\} \cup \{2, 6, \ldots, 2n-2\} = \{0, 2, \ldots, 2n-2\},$$

and

$$\{f(u_j) : j \text{ is even}\} \cup \{f(v_j) : i \text{ is odd}\} = \\ \{2n-1, 2n+1, \ldots, 3n-3\} \cup \{2n, 2n+2, \ldots, 3n-2\} = \{2n-1, 2n, \ldots, 3n-2\}.$$

So, independently of the parity of n, f is an injective function which range is a subset of $\{0, 1, \ldots, 3n-2\}$.
Suppose that j is any odd number in $\{1, 2, \ldots, n\}$. Assuming that the coming edges exist, the following hold:

Edge	Weight
$u_j v_j$	$f(v_j) - f(u_j) = 3(n-j) + 1$
$u_j u_{j+1}$	$f(u_{j+1}) - f(u_j) = 3(n-j)$
$v_j v_{j+1}$	$f(v_j) - f(v_{j+1}) = 3(n-j) - 1$
$u_{j+1} v_{j+1}$	$f(u_{j+1}) - f(v_{j+1}) = 3(n-j) - 2$
$v_{j+1} v_{j+2}$	$f(v_{j+2}) - f(v_{j+1}) = 3(n-j) - 3$
$u_{j+1} u_{j+2}$	$f(u_{j+1}) - f(u_{j+2}) = 3(n-j) - 4$

In other terms, the weights of these edges are consecutive integers. In addition, note that the edge $u_{j+2}v_{j+2}$ has weight $f(v_{j+2}) - f(u_{j+2}) = 3(n-j) - 5$. This fact implies that the set of induced weights is a set of consecutive integers, being $3(n-1)+1 = 3n-2$ the largest weight and 1 the smallest weight, which is always on the edge $u_n v_n$. Since the smaller labels are assigned to the stable set containing u_1, we have that f is in fact an α-labeling which boundary value is $\lambda = n-1$. □

Remark 1. Note that the integers 1 and $\lambda - 1 = n-2$ are not assigned by f as labels to L_n. This is an important fact that is used in Section 5 to transform an α-graph into a harmonious graph.

Let $n_1, n_2, \ldots, n_t$ and $r_1, r_2, \ldots, r_{t-1}$ be positive integers such that $n_i \geq 2$ and $2 \leq r_i \leq \min\{n_i, n_{i+1}\}$. We use these numbers to build a column-convex polyomino which α-labeling has not been studied before. Consider the collection of ladders $L_{n_1}, L_{n_2}, \ldots, L_{n_t}$, where for each $1 \leq i \leq t$, $V(L_{n_i}) = \{u^i_j, v^i_j : 1 \leq j \leq n_i\}$ and $E(L_{n_i}) = \{u^i_j u^i_{j+1}, v^i_j v^i_{j+1} : 1 \leq j \leq n_i - 1\} \cup \{u^i_j v^i_j : 1 \leq j \leq n_i\}$. The ladders L_{n_i} and $L_{n_{i+1}}$ are connected by r_i edges; for each $j \in \{1, 2, \ldots, r_i\}$, these edges are determined by the parity of n_i:

1. If n_i is even, either
   - **(a)** $v^i_{n_i+1-j}$ is connected to v^{i+1}_j, or
   - **(b)** $u^i_{n_i+1-j}$ is connected to u^{i+1}_j.
2. If n_i is odd, either
   - **(a)** $v^i_{n_i+1-j}$ is connected to u^{i+1}_j, or
   - **(b)** $u^i_{n_i+1-j}$ is connected to v^{i+1}_j.

Once all the connections have been made, we obtain a polyomino G of order $N = \sum_{i=1}^{t} 2n_i$ and size $M = \sum_{i=1}^{t}(3n_i - 2) + \sum_{i=1}^{t-1} r_k$. Any graph obtained in this way is a column-convex polyomino. Clearly, not every column-convex polyomino can be constructed in this way; in order to avoid any possible confusion, we say that any graph produced by our construction is a *zig-zag polyomino*. In Figure 3 we show one of these polyominoes obtained with the parameters $n_1 = 7, n_2 = 5, n_3 = 7, n_4 = 8, n_5 = 5, r_1 = 3, r_2 = 5, r_3 = 5,$ and $r_4 = 3$. Thus, $N = 64$ and $M = 102$. Observe that the grid $P_{2n+1} \times P_{2m+1}$ is in fact a zig-zag polyomino. In the next theorem we prove that any zig-zag polyomino G admits an α-labeling if for each pair of consecutive ladders within G, there is an odd number of edges connecting them. The labeling of the zig-zag in Figure 3 is the one described in the proof of this theorem.

Theorem 4. *All zig-zag polyominoes are α-graphs when the amount of edges between two consecutive ladders is odd.*

Proof. Let $n_1, n_2, \ldots, n_t$ and $r_1, r_2, \ldots, r_{t-1}$ be positive integers such that $n_i \geq 3$ and $3 \leq r_i \leq \min\{n_i, n_{i+1}\}$. Let G be the zig-zag polyomino built using these parameters. Assume that for each $i \in \{1, 2, \ldots, t\}$, the ladder L_{n_i} is initially labeled with the α-labeling f given in Proposition 1. Each of these labelings is transformed into a d_i-graceful labeling shifted c_i units, where

$$d_i = 1 + \sum_{k=i+1}^{t} (3n_k - 2) + \sum_{k=i}^{t-1} r_k,$$

with $c_1 = 0$ and, for each $i \geq 2$,

$$c_i = 2\sum_{k=1}^{i-1} n_k - (i-1).$$

In this way, the labeling of L_{n_i} is given by

$$g(u_j^i) = \begin{cases} c_i + 2j - 2 & \text{if } j \text{ is odd,} \\ c_i + d_i + 3n_i - 2 - j & \text{if } j \text{ is even,} \end{cases}$$

and

$$g(v_j^i) = \begin{cases} c_i + d_i + 3n_i - 2 - j & \text{if } j \text{ is odd,} \\ c_i + 2j - 2 & \text{if } j \text{ is even.} \end{cases}$$

Thus, on L_{n_i}, the set of labels assigned on the stable set A_i is $L_{A_i} = \{c_i, c_i+2, \ldots, c_i+2(n_i-1)\}$ and on B_i is $L_{b_i} = \{c_i+d_i+2n_i-2, c_i+d_i+2n_i-1, \ldots, c_i+d_i+3n_i-3\}$. If we consider two consecutive ladders, say L_{n_i} and $L_{n_{i+1}}$, we get

$$\max(L_{A_i}) = c_i + 2(n_i - 1) = 2\sum_{k=1}^{i-1} n_k - (i-1) + 2(n_i - 1) = 2\sum_{k=1}^{i} n_k - (i+1),$$

and

$$\min(L_{B_{i+1}}) = c_{i+1} = 2\sum_{k=1}^{i-1} n_k - (i-1) + (i+1-1) = 2\sum_{k=1}^{i} n_k - i.$$

This means that on the black vertices there is no repetition of labels and these labels increase along with the subindex. On the other side,

$$\max(L_{B_{i+1}}) = c_{i+1} + d_{i+1} + 3n_{i+1} - 3$$

and

$$\begin{aligned}
\min(L_{B_i}) &= c_i + d_i + 2n_i - 2 \\
&= 2\sum_{k=1}^{i-1} n_k - (i-1) + 1 + \sum_{k=i+1}^{t} (3n_k - 2) + \sum_{k=i}^{t-1} r_k + 2n_i - 2 \\
&= 2\sum_{k=1}^{i} n_k - 1 + \sum_{k=i+1}^{t} (3n_k - 2) + \sum_{k=i}^{t-1} r_k \\
&= 2\sum_{k=1}^{i} n_k - i + 1 + \sum_{k=i+2}^{t} (3n_k - 2) + \sum_{k=i+1}^{t-1} r_k + 3n_{i+1} - 3 + r_i \\
&= c_{i+1} + d_{i+1} + 3n_{i+1} - 3 + r_i.
\end{aligned}$$

Consequently, on the white vertices, there is no repetition of labels neither, and these labels decrease inversely along with the subindex, with a gap of r_i units between these two ladders.
Note that L_{n_1} has the vertices with the minimum and maximum labels, i.e., 0 and

$$\begin{aligned}
c_1 + d_1 + 3n_1 - 3 &= 0 + 1 + \sum_{k=2}^{t}(3n_k - 2) + \sum_{k=1}^{t-1} r_k + 3n_1 - 3 \\
&= \sum_{k=1}^{t}(3n_k - 2) + \sum_{k=1}^{t-1} r_k = M.
\end{aligned}$$

Thus, the labels assigned to the vertices of G are in the range required by any α-labeling of this graph.

Now we study the weights induced by this labeling. Since the labeling of L_{n_i} is a d_i-graceful labeling shifted c_i units, the weights on the edges of this ladder form the set $W_i = \{d_i, d_i + 1, \ldots, d_i + 3n_i - 3\}$. Then, $\min(W_i) = d_i$ and $\max(W_{i+1}) = d_i - r_i - 1$ because

$$\begin{aligned}
d_{i+1} + 3n_{i+1} - 3 &= 1 + \sum_{k=i+2}^{t} (3n_k - 2) + \sum_{k=i+1}^{t-1} r_k + 3n_{i+1} - 3 \\
&= 1 + \sum_{k=i+1}^{t} (3n_k - 2) + \sum_{k=i}^{t-1} r_k - (r_i + 1) \\
&= d_i - (r_i + 1).
\end{aligned}$$

Therefore, we just need to prove that the r_i edges connecting L_{n_i} to $L_{n_{i+1}}$ have weights $d_i - r_i, d_i - r_i + 1, \ldots, d_i - 1$. We analyze each of the four possible ways to connect these two ladders.

Case 1a: When n_i is even and $v^i_{n_i+1-j}$ is connected to v^{i+1}_j. Note that for every $j \in \{1, 2, \dots, r_i\}$, the indices j and $n_i + 1 - j$ have different parity.

- When j is odd, the set formed by the weights on the edges of the form $v^i_{n_i+1-j} v^{i+1}_j$ is $\{d_i - r_i, d_i - r_i + 2, \dots, d_i - 1\}$, because

$$
\begin{aligned}
g(v^{i+1}_j) - g(v^i_{n_i+1-j}) &= (c_{i+1} + d_{i+1} + 3n_{i+1} - 2 - j) - (c_i + 2(n_i + 1 - j) - 2) \\
&= c_{i+1} - c_i + d_{i+1} + 3n_{i+1} - 2 - j - 2n_i - 2 + 2j + 2 \\
&= 2\sum_{k=1}^{i} n_k - 1 - 2\sum_{k=1}^{i-1} n_k + (i-1) + 1 + \sum_{k=i+2}^{t} (3n_k - 2) \\
&+ \sum_{k=i+1}^{t-1} r_k + (3n_{i+1} - 2) - 2n_i + j \\
&= 2n_i - 1 + 1 + \sum_{k=i+1}^{t} (3n_k - 2) + \sum_{k=i}^{t-1} r_k - r_i - 2n_i + j \\
&= d_i - r_i + j - 1.
\end{aligned}
$$

- When j is even, the set formed by the weights on the edges of the form $v^i_{n_i+1-j} v^{i+1}_j$ is $\{d_i - 2, d_i - 4, \dots, d_i - r_i + 1\}$, because

$$
\begin{aligned}
g(v^i_{n_i+1-j}) - g(v^{i+1}_j) &= (c_i + d_i + 3n_i - 2 - n_i - 1 + j) - (c_{i+1} + 2j - 2) \\
&= c_i - c_{i+1} + d_i + 2n_i - 1 - j \\
&= 2\sum_{k=1}^{i-1} n_k - (i-1) - 2\sum_{k=1}^{i} n_k + i + 1 + \sum_{k=i+1}^{t} (3n_k - 2) \\
&+ \sum_{k=i}^{t-1} r_k + 2n_i - 1 - j \\
&= 1 - 2n_i + d_i + 2n_i - 1 - j \\
&= d_i - j.
\end{aligned}
$$

Case 1b: When n_i is even and $u^i_{n_i+1-j}$ is connected to u^{i+1}_j. As in the previous case, for every $j \in \{1, 2, \dots, r_i\}$, the indices j and $n_i + 1 - j$ have different parity.

- When j is odd, the set formed by the weights on the edges of the form $u^i_{n_i+1-j} u^{i+1}_j$ is $\{d_i - 1, d_i - 3, \dots, d_i - r_i\}$, because

$$
\begin{aligned}
g(u^i_{n_i+1-j}) - g(u^{i+1}_j) &= (c_i + d_i + 3n_i - 2 - n_i - 1 - j) - (c_{i+1} + 2j - 2) \\
&= d_i - j.
\end{aligned}
$$

- When j is even, the set formed by the weights on the edges of the form $u^i_{n_i+1-j}u^{i+1}_j$ is $\{d_i - r_i + 1, d_i - r_i + 3, \ldots, d_i - 2\}$, because

$$\begin{aligned} g(u^{i+1}_j) - g(u^i_{n_i+1-j}) &= (c_{i+1} + d_{i+1} + 3n_{i+1} - 2 - j) - (c_i + 2(n_i + 1 - j) - 2) \\ &= d_i - r_i + j - 1. \end{aligned}$$

Case 2a: When n_i is odd and $v^i_{n_i+1-j}$ is connected to u^{i+1}_j. Note that for every $j \in \{1, 2, \ldots, r_i\}$, the indices j and $n_i + 1 - j$ have the same parity.

- When j is odd, the set formed by the weights on the edges of the form $v^i_{n_i+1-j}u^{i+1}_j$ is $\{d_i - 1, d_i - 3, \ldots, d_i - r_i\}$, because

$$\begin{aligned} g(v^i_{n_i+1-j}) - g(u^{i+1}_j) &= (c_i + d_i + 3n_i - 2 - n_i - 1 - j) - (c_{i+1} + 2j - 2) \\ &= d_i - j. \end{aligned}$$

- When j is even, the set formed by the weights on the edges of the form $u^i_{n_i+1-j}u^{i+1}_j$ is $\{d_i - r_i + 1, d_i - r_i + 3, \ldots, d_i - 2\}$, because

$$\begin{aligned} g(u^{i+1}_j) - g(u^i_{n_i+1-j}) &= (c_{i+1} + d_{i+1} + 3n_{i+1} - 2 - j) - (c_i + 2(n_i + 1 - j) - 2) \\ &= d_i - r_i + j - 1. \end{aligned}$$

Case 2b: When n_i is odd and $u^i_{n_i+1-j}$ is connected to v^{i+1}_j. As in the previous case, for every $j \in \{1, 2, \ldots, r_i\}$, the indices j and $n_i + 1 - j$ have the same parity.

- When j is odd, the set formed by the weights on the edges of the form $u^i_{n_i+1-j}v^{i+1}_j$ is $\{d_i - r_i, d_i - r_i + 2, \ldots, d_i - 1\}$, because

$$\begin{aligned} g(v^{i+1}_j) - g(u^i_{n_i+1-j}) &= (c_{i+1} + d_{i+1} + 3n_{i+1} - 2 - j) - (c_i + 2(n_i + 1 - j) - 2) \\ &= d_i - r_i + j - 1. \end{aligned}$$

- When j is even, the set formed by the weights on the edges of the form $u^i_{n_i+1-j}u^{i+1}_j$ is $\{d_i - 2, d_i - 4, \ldots, d_i - r_i + 1\}$, because

$$\begin{aligned} g(u^i_{n_i+1-j}) - g(v^{i+1}_j) &= (c_i + d_i + 3n_i - 2 - n_i - 1 + j) - (c_{i+1} + 2j - 2) \\ &= d_i - j. \end{aligned}$$

Therefore, independently of the case, the weights of the edges connecting these two ladders is $\{d_i - r_i, d_i - r_i + 1, \ldots, d_i - 1\}$. As a result of this, we have that the labeling of G satisfies all the conditions to be, in fact, an α-labeling.

□

In Figure 3 we show an example of the α-labeling of a zig-zag polyomino obtained using the construction explained in the proof of this last theorem.

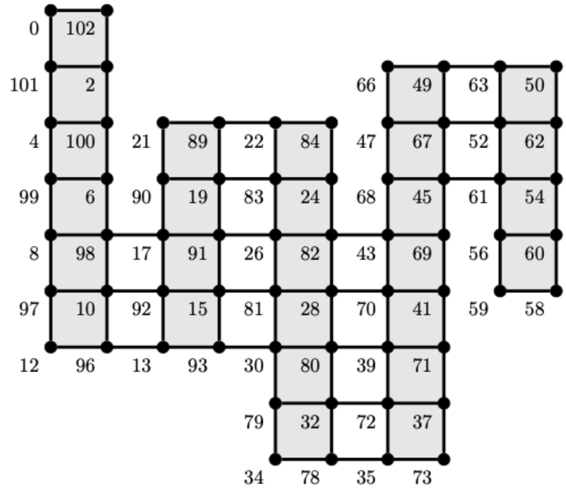

Figure 3. **An α-labeling of a zig-zag polyomino.**

4. A New α-Labeled Chain Graph

The following definition is a slight modification of the concept of chain graph given in [2], the goal is to expand the family of the graphs under consideration. For each $i \in \{1, 2, \ldots, t\}$, let G_i be a connected graph of order n_i and size m_i, let u_i, v_i be two distinct vertices of G_i. A graph G of order $\sum_{i=1}^{t} n_i - (t-1)$ and size $\sum_{i=1}^{t} m_i$ is called a *chain graph* if it is formed by identifying (via vertex amalgamation), for every $2 \leq j \leq t-1$, the vertex u_{j-1} with the vertex u_j, and the vertex v_j with the vertex v_{j+1}; the difference with the definition given in [2] is that before we imposed the condition that each G_i was a block. We proved in [2], that the chain graph G is an α-graph when each G_i is an α-labeled graph and the distinguished vertices u_i and v_i are chosen to be the vertices labeled 0 and λ_i, respectively, where λ_i is the boundary value of the α-labeling of G_i. In the next theorem we prove that another α-labeled chain graph can be obtained using the graphs G_i.

Theorem 5. *Suppose that for each $i \in \{1, 2, \ldots, t\}$, f_i is an α-labeling of a connected graph G_i of size m_i. The chain graph G is an α-graph if for each even value of i the labels of the amalgamated vertices satisfy the following conditions: $f_{i-1}(u_{i-1}) = \lambda_{i-1}$, $f_i(u_i) = 0$, $f_i(v_i) = \lambda_i + 1$, and $f_{i+1}(v_{i+1}) = m_{i+1}$.*

Proof. Suppose that for each $i \in \{1, 2, \ldots, t\}$, G_i is a connected α-graph of order n_i and size m_i. Let f_i be an α-labeling with boundary value λ_i of G_i. We transform f_i into a d_i-graceful labeling g_i, where

$$d_i = \begin{cases} 1 + \sum_{j=i+1}^{t} m_j & \text{if } i < t, \\ 1 & \text{if } i = t. \end{cases}$$

In this way, the set of weights induced by g_i is $W_{g_i} = \{d_i, d_i + 1, \ldots, d_i + m_i - 1\}$; observe that $\min\left(W_{g_i}\right) - \max\left(W_{g_{i+1}}\right) = 1$; in fact,

$$\begin{aligned} \min\left(W_{g_i}\right) - \max\left(W_{g_{i+1}}\right) &= d_i - (d_{i+1} + m_{i+1} - 1) \\ &= 1 + \sum_{j=i+1}^{t} m_j - 1 - \sum_{j=i+2}^{t} m_j - m_{i+1} + 1 \\ &= 1 + \sum_{j=i+1}^{t} m_j - \sum_{j=i+1}^{t} m_j = 1. \end{aligned}$$

This fact implies that the set of induced weights on the graph $\cup_{i=1}^{t} G_i$ consists of $\sum_{i=1}^{t} m_i$ consecutive integers. Since $d_t = 1$, the smallest of these consecutive integers is 1, which implies that $\sum_{i=1}^{t} m_i$ is the largest induced weight.

With the only exception of g_1, all these labelings are shifted to avoid the use of a label more than once. Recall that $g_1(u_1) = f_1(u_1) = \lambda_1$ and $g_2(u_2) = f_2(u_2) = 0$; since these two vertices are identified in G, the labeling g_2 of G_2 is shifted λ_1 units. Once this shifting is done, the new label of v_2 is fixed and this number determines the shifting on G_3; as in a chain reaction, the shifting on G_i determines the shifting on G_{i+1}. So, to form G we just identify the vertices with the same label.

Now that the final labeling of each G_i is completed, i.e., that the labeling of G is in place, we need to check that its labels satisfy the conditions required to be an α-labeling. Considering the shiftings together with the identification of the vertices with the same label, we know that the labeling restricted to each stable set of G is in fact injective; however, it is necessary to check that it is injective over the entire vertex set. Suppose that for each $i \in \{1, 2, \ldots, t\}$, A_i and B_i are the stable sets of G_i. The original α-labeling of G_i satisfies the inequality $f(u) < f(v)$ for each pair $(u, v) \in A_i \times B_i$. Its intermediate labeling, g_i, is obtained by adding a non-negative constant to each label on a vertex of B_i; thus, $g_i(u) < g_i(v)$ too. The final labeling of G_i is obtained by adding a non-negative constant to each label on a vertex of G_i; consequently, the last inequality is still valid. In particular, the labels on the vertices of A_t are smaller than the labels on the vertices of B_t; implying that the labeling of the chain graph G is injective. The way in which G is constructed guarantees that this labeling is in fact an α-labeling. Therefore, the chain graph G is an α-graph. $\square$

A well-known family of α-graphs is the family of cycles of size m, where $m \equiv 0 (\mathrm{mod}\ 4)$. In [14], Rosa provided an α-labeling of these cycles, where the neighbors of the vertex labeled 0 are the vertices labeled m and $\lambda + 1 = \frac{m}{2}$. If the graphs $G_1, G_2, \ldots, G_t$ used to construct G are α-cycles, then the block-cutpoint graph is the path P_t. The graph G, obtained with these cycles, is what we call an *irregular cyclic-snake*, because its

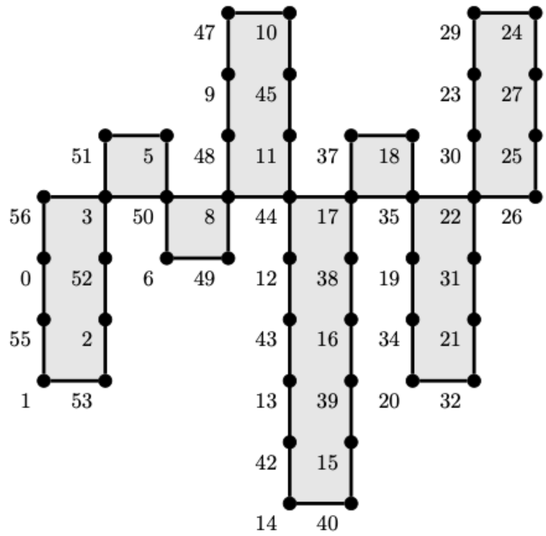

Figure 4. **An α-labeling of an irregular cyclic-snake.**

blocks, that are cycles, do not need to be isomorphic. Note that G is another example of an α-graph that can be embedded in the integral grid.

Corollary 1. *Any cyclic-snake formed with α-cycles, not necessarily isomorphic, is an α-graph when the distance between consecutive cut vertices is 1.*

In Figure 4 we show an example of this corollary, where G is formed with the sequence of cycles $C_8, C_4, C_4, C_8, C_{12}, C_4, C_8, C_8$.

5. From Alpha to Harmonious

The study of harmonious graphs started in 1980 with the work of Graham and Sloane [10]. A graph G of size m is said to be *harmonious* if there exists an injective function $f : V(G) \to \{0, 1, \ldots, m-1\}$ such that when each edge uv of G is assigned the *weight* $\big(f(u) + f(v)\big)(\text{mod } m)$, the set of all weights is $\{0, 1, \ldots, m-1\}$. Among other results, they proved that a cycle is harmonious if and only if its size is odd. In this section we show a procedure to transform the α-labeling of some of the graphs considered in this work into a harmonious labeling. Before that, we need a transitional labeling, this labeling was introduced by Figueroa-Centeno et al. [6], they said that a bipartite graph G of size m is *strongly felicitous* if there is an injective function $f : V(G) \to \{0, 1, \ldots, m\}$, where the weight of an edge uv of G is defined as $\big(f(u) + f(v)\big)(\text{mod } m + 1)$, all weights are distinct, and there exists an integer κ such that $\min\{f(u), f(v)\} \le \kappa < \max\{f(u), f(v)\}$ for every edge uv of G. They proved that G

is strongly felicitous if and only if is an α-graph. If f is an α-labeling with boundary value λ of a graph G of size m, then a strongly felicitous labeling is given by

$$g(v) = \begin{cases} f(v) & \text{if } f(v) \leq \lambda, \\ m + \lambda + 1 - f(v) & \text{if } f(v) > \lambda. \end{cases}$$

Although these two definitions are quite similar, not every α-graph admits a harmonious labeling, an easy example is given by the cycles of size divisible by four that are α-graphs but are not harmonious. In the following theorems we prove that some of the zig-zag polyominoes in Section 3 and the some of the cyclic-snakes in Section 4 are harmonious graphs.

Theorem 6. *Let $r_1, r_2, \ldots, r_{t-1}$ be odd integers greater than 3. All zig-zag polyominoes built with the ladders $L_{n_1}, L_{n_2}, \ldots, L_{n_t}$, where L_{n_i} is connected to $L_{n_{i+1}}$ with r_i edges, are harmonious when $r_{t-1} < n_t$.*

Proof. Suppose that G is a zig-zag polyomino of size m that satisfies the theorem's hypotheses. Let f be the α-labeling of G described in Theorem 4 and g be the strongly felicitous labeling of G obtained from f. Then, the labels assigned by g are in the set $\{0, 1, \ldots, m\}$ and the set of weights induced by g, before the reduction modulo $m+1$, is $W_g = \{\lambda+1, \lambda+2, \ldots, \lambda+m\}$. Observe that neither f nor g assigns the numbers 1 and $\lambda - 1$ as a label of G. Since $r_{t-1} < n_t$, the ladder L_{n_t} has two vertices of degree 2. On the last C_4 in L_{n_t}, i.e., the C_4 that contains the vertices of degree 2, the consecutive labels are $\lambda - 2, m-1, \lambda, m$, where λ and m are labels of the vertices of degree 2. The weights in this cycle are $m+\lambda-3, m+\lambda-1, m+\lambda,$ and $m+\lambda-2$. Replace the labels λ and m with $\lambda - 1$ and 1, respectively. Thus, the weights on this cycle are $m+\lambda-3, m+\lambda-2, \lambda,$ and $\lambda-1$. Consequently, the new labeling of G is an injective function that assigns labels from $\{0, 1, \ldots, m-1\}$ to induce the weights $\lambda - 1, \lambda, \ldots, m+\lambda-2$, that is, a set of m consecutive integers. Therefore, G is a harmonious graph. □

In Figure 5 we show an example of this harmonious labeling for a zig-zag polyomino of size 42 built with the ladders $L_5, L_4,$ and L_5, where $r_1 = r_2 = 3$. From the left to the right the labelings exhibited are the α-labeling, the strongly felicitous, and the harmonious labeling.
Using the same technique, we can transform the α-labeling of any of the chain graphs in Theorem 5 if the extremes of the chain are graphs that admit an α-labeling that do not use the integers 1 and $\lambda - 1$ as labels. Recall that these are the numbers used to replace the labels m and $\Lambda - 1$ as labels, where Λ is the boundary value of the α-labeling of the chain graph. One may ask for α-labeled graphs satisfying this condition, one simple example is given by the α-labeling of the complete bipartite graph $K_{m,n}$ for $m, n \geq 2$, where the labels of one stable set are the first non-negative multiples of either m or n.

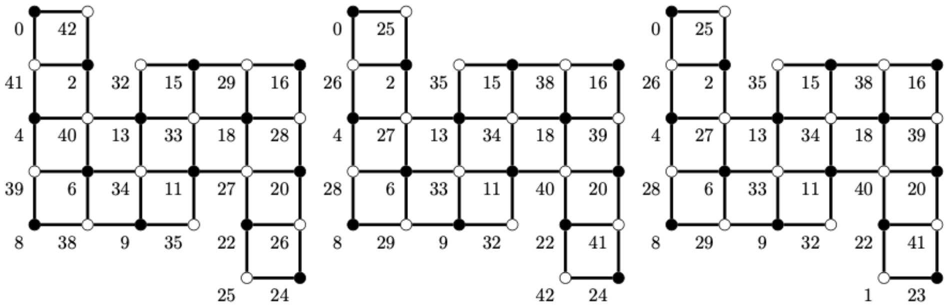

Figure 5. The α-, strongly felicitous, and harmonious labelings of a zig-zag polyomino of size 42.

Theorem 7. *Suppose that for each $i \in \{1, 2, \ldots, t\}$, f_i is an α-labeling of a connected graph G_i of size m_i. Let G be a chain graph formed with these graphs. The graph G is a harmonious graph if f_i does not assign the label 1, f_t does not assign the label $\lambda_t - 1$, and for each value of $i \in \{2, 3, \ldots, t-1\}$, $f_{i-1}(u_{i-1}) = \lambda_{i-1}$, $f_i(u_i) = 0$, $f_i(v_i) = \lambda_i + 1$, and $f_{i+1}(v_{i+1}) = m_{i+1}$.*

Corollary 2. *Any cyclic-snake formed with α-cycles, not necessarily isomorphic, is an α-graph when the extreme cycles are copies of C_4 and the distance between consecutive cut vertices is 1.*

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