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Research Article

Signed total Italian k-domatic number of a graph

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Abstract: Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A *signed total Italian k-dominating function* on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{u\in N(v)} f(u) \geq k$ for every $v \in V(G)$, where $N(v)$ is the neighborhood of v, and each vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Italian k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed total Italian k-dominating family* (of functions) on G. The maximum number of functions in a signed total Italian k-dominating family on G is the *signed total Italian k-domatic number* of G, denoted by $d^k_{stI}(G)$. In this paper we initiate the study of signed total Italian k-domatic numbers in graphs, and we present sharp bounds for $d^k_{stI}(G)$. In addition, we determine the signed total Italian k-domatic number of some graphs.

Keywords: Signed total Italian k-dominating function, Signed total Italian k-domination number, Signed total Italian k-domatic number

AMS Subject classification: 05C69

1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [3]. Specifically, let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A graph G is *regular* or *r-regular* if $d(v) = r$ for each vertex v of G. The complement of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n, $K_{p,q}$ for the *complete bipartite graph* with partite sets X and Y, where $|X| = p$ and $|Y| = q$, C_n for the *cycle* of length n and P_n for the path of order n.

In this paper we continue the study of Roman and Italian dominating functions in graphs and digraphs. If $k \geq 1$ is an integer, then the *signed total Roman k-dominating function* (STRkDF) on a graph G is defined in [9] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for each $v \in V(G)$, and such that every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to at least one vertex w for which $f(w) = 2$. The *weight* of an STRkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *signed total Roman k-domination number* of a graph G, denoted by $\gamma^k_{stR}(G)$, equals the minimum weight of an STRkDF on G. A $\gamma^k_{stR}(G)$-*function* is a signed total Roman k-dominating function of G with weight $\gamma^k_{stR}(G)$. The special case $k = 1$, was introduced and studied in [7] and [6].

A *signed total Italian k-dominating function* (STIkDF) on a graph G is defined in [11] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{u \in N(v)} f(u) \geq k$ for each $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$ the second condition is superfluous. The *weight* of an STIkDF f is the value $\omega(f) = \sum_{v \in V} f(v)$. The *signed total Italian k-domination number* of a graph G, denoted by $\gamma^k_{stI}(G)$, equals the minimum weight of an STIkDF on G. The case $k = 1$ was introduced and studied in [10]. A $\gamma^k_{stI}(G)$-*function* is a signed total Italian k-dominating function of G with weight $\gamma^k_{stI}(G)$.

The signed total Roman and signed total Italian k-domination numbers exist when $\delta(G) \geq \frac{k}{2}$, and the definitions lead to $\gamma^k_{stI}(G) \leq \gamma^k_{stR}(G)$.

A concept dual in a certain sense to the domination number is the domatic number, introduced by Cockayne and Hedetniemi [2]. They have defined the domatic number $d(G)$ of a graph G by means of sets. A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition. The maximum number of classes of a domatic partition of G is the domatic number $d(G)$ of G. But Rall has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. (This was mentioned by Slater and Trees in [5].) Analogous to the fractional domatic number we may define the signed total Roman and signed total Italian k-domatic numbers.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Italian (Roman) k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed total Italian (Roman) k-dominating family* (of functions) on G. The maximum number of functions in a signed total Italian (Roman) k-dominating family (STIkD family (STRkD) family) on G is the *signed total Italian (Roman) k-domatic number* of G, denoted by $d^k_{stI}(G)$ ($d^k_{stR}(G)$). The signed total Italian (Roman) k-domatic numbers are well-defined and $d^k_{stI}(G) \geq d^k_{stR}(G) \geq 1$ for all graphs G with $\delta(G) \geq \frac{k}{2}$. The signed total Roman k-domatic number was introduced and studied in [8]. For more information on the Roman and Italian domatic problem, we refer the reader to the survey article [1]. If $k = 1$, then we also write $\gamma^1_{stR}(G) = \gamma_{stR}(G)$, $\gamma^1_{stI}(G) = \gamma_{stI}(G)$, $d^1_{stR}(G) = d_{stR}(G)$ and $d^1_{stI}(G) = d_{stI}(G)$.

Our purpose in this paper is to initiate the study of the signed total Italian k-domatic number in graphs. We first derive basic properties and sharp bounds for the signed total Italian k-domatic number of a graph. Then we presemt upper bounds on

$\gamma_{stI}^k(G) + d_{stI}^k(G)$ and $d_{stI}^k(G) + d_{stI}^k(\overline{G})$. In addition, we determine the signed total Italian k-domatic number of some classes of graphs.

We make use of the following known results in this paper.

Proposition A. ([11]) *If G is an r-regular graph of order n with $r \geq \frac{k}{2}$, then*

$$\gamma_{stI}^k(G) \geq \frac{kn}{r}.$$

Proposition B. ([11]) *If G is a graph of order n with $\delta(G) \geq k$, then $\gamma_{stI}^k(G) \leq \gamma_{stR}^k(G) \leq n$.*

Proposition C. ([11]) *Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{stI}^k(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2}$, and each vertex of G is adjacent to a vertex of minimum degree.*

Proposition D. ([11]) *If $k \geq 1$ and $n \geq 2$ are integers such that $2n-2 \geq k$, then it holds:*

(i) If $k \geq n$, then $\gamma_{stI}^k(K_n) = k+2$.

(ii) If $k \leq n-1$ and $n-k$ is odd, then $\gamma_{stI}^k(K_n) = k+1$.

(iii) If $k \leq n-1$ and $n-k$ is even, then $\gamma_{stI}^k(K_n) = k+2$.

Proposition E. ([7]) *If $n \geq 3$ is an integer, then $\gamma_{stR}(K_n) = 3$.*

Proposition F. ([6]) *If G is a graph of order n with $\delta(G) \geq 1$, then $\gamma_{stR}(G) \cdot d_{stR}(G) \leq n$.*

Proposition G. ([11],) *If $p, k \geq 1$ are integers such that $p \geq \frac{k}{2}$, then $\gamma_{stI}^k(K_{p,p}) = 2k$.*

Proposition H. ([8]) *If $k \geq 3$ is an integer, then $d_{stR}^k(K_{k,k}) = k$. In addition, $d_{stR}^1(K_{1,1}) = d_{stR}^2(K_{2,2}) = 1$.*

Proposition I. ([8]) *If k, p are integers such that $p \geq k+1 \geq 2$, then $d_{stR}^k(K_{p,p}) = p$, with exception of the case $k = 1$ and $p = 3$, in which case $d_{stR}^1(K_{3,3}) = 1$.*

Proposition J. ([11]) *If C_n is a cycle of length n, then $\gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil + 1$ when $n \equiv 2 \pmod 4$ and $\gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ otherwise.*

Proposition K. ([6]) *Let G be a graph of order n with $\delta(G) \geq 1$. Then $\gamma_{stR}(G) = n$ if and only if the components of G are K_2, K_3, P_3 or C_6.*

Since $\gamma_{stI}(K_2) = 2$, $\gamma_{stI}(K_3) = 3$, $\gamma_{stI}(P_3) = 3$, $\gamma_{stI}(C_6) = 6$ (see also [10]) and $\gamma_{stI}(G) \leq \gamma_{stR}(G)$, Proposition K leads to the next result immediately.

Proposition L. *Let G be a graph of order n with $\delta(G) \geq 1$. Then $\gamma_{stI}(G) = n$ if and only if the components of G are K_2, K_3, P_3 or C_6.*

2. Bounds on the signed total Italian k-domatic number

In this section we present basic properties of $d^k_{stI}(G)$ and sharp bounds on the signed total Italian k-domatic number of a graph.

Theorem 1. If G is a graph with $\delta(G) \geq \frac{k}{2}$, then $d^k_{stI}(G) \leq \delta(G)$. Moreover, if $d^k_{stI}(G) = \delta(G)$, then for each STIkD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d^k_{stI}(G)$ and each vertex v of minimum degree, $\sum_{x \in N(v)} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N(v)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STIkD family on G such that $d = d^k_{stI}(G)$. If v is a vertex of minimum degree $\delta(G)$, then we deduce that

$$kd \leq \sum_{i=1}^{d} \sum_{x \in N(v)} f_i(x) = \sum_{x \in N(v)} \sum_{i=1}^{d} f_i(x) \leq \sum_{x \in N(v)} k = k\delta(G)$$

and thus $d^k_{stI}(G) \leq \delta(G)$.
If $d^k_{stI}(G) = \delta(G)$, then the two inequalities occurring in the proof become equalities. Hence for the STIkD family $\{f_1, f_2, \ldots, f_d\}$ on G and for each vertex v of minimum degree, $\sum_{x \in N(v)} f_i(x) = k$ for each function f_i and $\sum_{i=1}^{d} f_i(x) = k$ for all $x \in N(v)$. □

Theorem 2. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma^k_{stI}(G) \cdot d^k_{stI}(G) \leq kn.$$

Moreover, if $\gamma^k_{stI}(G) \cdot d^k_{stI}(G) = kn$, then for each STIkD family $\{f_1, f_2, \ldots, f_d\}$ on G with $d = d^k_{stI}(G)$, each function f_i is a $\gamma^k_{stI}(G)$-function and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(G)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be an STIkD family on G such that $d = d^k_{stI}(G)$ and let $v \in V(G)$. Then

$$\begin{aligned} d \cdot \gamma^k_{stI}(G) &= \sum_{i=1}^{d} \gamma^k_{stI}(G) \leq \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) \\ &= \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \leq \sum_{v \in V(G)} k = kn. \end{aligned}$$

If $\gamma^k_{stI}(G) \cdot d^k_{stI}(G) = kn$, then the two inequalities occurring in the proof become equalities. Hence for the STIkD family $\{f_1, f_2, \dots, f_d\}$ on G and for each i, $\sum_{v \in V(G)} f_i(v) = \gamma^k_{stI}(G)$. Thus each function f_i is a $\gamma^k_{stI}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V(G)$. □

Example 1. *If $n = 4p+2$ with an integer $p \geq 1$, then $d_{stI}(K_n) = \frac{n}{2}$.*

Proof. Proposition D and Theorem 2 imply $d_{stI}(K_n) \leq n/\gamma_{stI}(K_n) = n/2$. Now let $x_1, x_2, \dots x_n$ be the vertices of K_n. Define the function f_1 by $f_1(x_i) = -1$ for $1 \leq i \leq 2p$ and $f_1(x_i) = 1$ for $2p+1 \leq i \leq 4p+2$. For $2 \leq j \leq 2p+1$ define $f_j(x_i) = -1$ for $2j-1 \leq i \leq 2p+2j-2$ and $f_j(x_i) = 1$ for $2p+2j-1 \leq i \leq 4p+2j$, where the indices are taken modulo $n = 4p+2$. It is straightforwad to verify that f_i is a signed total Italian dominating function of K_n for $1 \leq i \leq 2p+1$ and $\{f_1, f_2, \dots, f_{2p+1}\}$ is a signed total Italian dominating family on K_n. Hence $d_{stI}(K_n) \geq n/2$ and thus $d_{stI}(K_n) = n/2$. □

It follows from Propositions E and F that $d_{stR}(K_n) \leq n/\gamma_{stR}(K_n) = n/3$. Therefore we deduce from Example 1 for $n = 4p+2$ that

$$d_{stI}(K_n) - d_{stR}(K_n) \geq \frac{n}{2} - \frac{n}{3} = \frac{n}{6}.$$

Thus the difference $d_{stI}(G) - d_{stR}(G)$ can be arbitrarily large.

Example 2. *If $n = 6p+3$ with an integer $p \geq 1$, then $d_{stI}(K_n) = \frac{n}{3}$.*

Proof. Proposition D and Theorem 2 imply $d_{stI}(K_n) \leq n/\gamma_{stI}(K_n) = n/3$. Now let $x_1, x_2, \dots x_n$ be the vertices of K_n. Define the function f_1 by $f_1(x_i) = -1$ for $1 \leq i \leq 3p$ and $f_1(x_i) = 1$ for $3p+1 \leq i \leq 6p+3$. For $2 \leq j \leq 2p+1$ define $f_j(x_i) = -1$ for $3j-2 \leq i \leq 3p+3j-3$ and $f_j(x_i) = 1$ for $3p+3j-2 \leq i \leq 6p+3j$, where the indices are taken modulo $n = 6p+3$. It is straightforwad to verify that f_i is a signed total Italian dominating function of K_n for $1 \leq i \leq 2p+1$ and $\{f_1, f_2, \dots, f_{2p+1}\}$ is a signed total Italian dominating family on K_n. Hence $d_{stI}(K_n) \geq n/3$ and thus $d_{stI}(K_n) = n/3$. □

Example 3. *Let $k \geq 1$ and $p \geq \frac{k}{2}$ be integers. Then $d^k_{stI}(K_{p,p}) = p$, with exception of the cases $p = \frac{k}{2}$, in which case $d^k_{stI}(K_{\frac{k}{2},\frac{k}{2}}) = 1$, or $p = k = 2$, in which case $d^2_{stI}(K_{2,2}) = 1$.*

Proof. Theorem 1 implies $d^k_{stI}(K_{p,p}) \leq p$. If $p = \frac{k}{2}$, then the function f with $f(x) = 2$ for each vertex x is the unique STIkDF and thus $d^k_{stI}(K_{\frac{k}{2},\frac{k}{2}}) = 1$. If $p = k = 2$, then le f be an STI2DF on $K_{2,2}$. Then $f(x) \geq 1$ for each vertex x. If we suppose that $\{f_1, f_2\}$ is a STI2D family on $K_{2,2}$, then the condition $f_1(x) + f_2(x) \leq 2$ leads to $f_1(x) = f_2(x) = 1$ for each vertex x, a contradiction. Therefore $d^2_{stI}(K_{2,2}) = 1$.

In the remaining cases we now show that $d^k_{stI}(K_{p,p}) \ge p$. Let $A = \{a_1, a_2, \dots, a_p\}$ and $B = \{b_1, b_2, \dots, b_p\}$ be a bipartition of $K_{p,p}$.
If $p = 1$, then $d^k_{stI}(K_{1,1}) = 1$ is immediate for $1 \le k \le 2$. If $p = 2$ and $k = 1$, then define f_1 and f_2 by $f_1(a_1) = f_1(b_1) = -1$, $f_1(a_2) = f_1(b_2) = 2$, $f_2(a_1) = f_2(b_1) = 2$ and $f_2(a_2) = f_2(b_2) = -1$. Then $\{f_1, f_2\}$ is a signed total Italian 1-dominating family on $K_{2,2}$ and thus $d_{stI}(K_{2,2}) \ge 2$. If $p = 2$ and $k = 3$, then define f_1 and f_2 by $f_1(a_1) = f_1(b_1) = 1$, $f_1(a_2) = f_1(b_2) = 2$, $f_2(a_1) = f_2(b_1) = 2$ and $f_2(a_2) = f_2(b_2) = 1$. Then $\{f_1, f_2\}$ is a signed total Italian 3-dominating family on $K_{2,2}$ and thus $d^3_{stI}(K_{2,2}) \ge 2$.
Let now $p \ge 3$. If $p = 3$ and $k = 1$, then define the functions f_1, f_2 and f_3 by $f_1(a_1) = f_1(b_1) = -1$, $f_1(a_2) = f_1(b_2) = f_1(a_3) = f_1(b_3) = 1$, $f_2(a_2) = f(b_2) = -1$, $f_2(a_1) = f_2(b_1) = f_2(a_3) = f_2(b_3) = 1$, $f_3(a_3) = f(b_3) = -1$ and $f_3(a_1) = f_3(b_1) = f_3(a_2) = f_3(b_2) = 1$. Then $\{f_1, f_2, f_3\}$ is a signed total Italian 1-dominating family on $K_{3,3}$ and thus $d_{stI}(K_{3,3}) \ge 3$. If $p \ne 3$, and $p \ge k \ge 1$, then it follows from Propositions H amd I that $d^k_{stR}(K_{p,p}) = p$ and therefore $d^k_{stI}(K_{p,p}) \ge d^k_{stR}(K_{p,p}) = p$.
Finally, let $\frac{k}{2} < p \le k - 1$. Define the function f_1 by $f_1(a_i) = f_1(b_i) = 1$ for $1 \le i \le 2p - k$ and $f_1(a_i) = f_1(b_i) = 2$ for $2p - k + 1 \le i \le p$. For $2 \le j \le p$ and $1 \le i \le p$ define $f_j(a_i) = f_j(b_i) = f_1(a_{i+j-1})$, where the indices are taken modulo p. It is easy to see that f_i is a signed total Italian k-dominating function of $K_{p,p}$ and $\{f_1, f_2, \dots, f_p\}$ is a signed total Italian k-dominating family on $K_{p,p}$. Hence $d^k_{stI}(K_{p,p}) \ge p$ also in this case, and the proof is complete. □

Example 3 and Proposition G demonstrate that Theorems 1 and 2 are both sharp.
For some regular graphs we will improve the upper bound $d^k_{stI}(G) \le \delta(G)$, given in Theorem 1.

Theorem 3. Let G be a δ-regular graph of order n with $\delta \ge \frac{k}{2}$ such that $n = p\delta + r$ with integers $p \ge 0$ and $1 \le r \le \delta - 1$ and $kr = t\delta + s$ with integers $t \ge 0$ and $1 \le s \le \delta - 1$. Then $d^k_{stI}(G) \le \delta - 1$.

Proof. Let $\{f_1, f_2, \dots, f_d\}$ be a STIkD family on G such that $d = d^k_{stI}(G)$. It follows that

$$\sum_{i=1}^{d} \omega(f_i) = \sum_{i=1}^{d} \sum_{v \in V(G)} f_i(v) = \sum_{v \in V(G)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(G)} k = kn.$$

Proposition A implies

$$\begin{aligned} \omega(f_i) &\ge \gamma^k_{stI}(G) \ge \left\lceil \frac{kn}{\delta} \right\rceil = \left\lceil \frac{kp\delta + kr}{\delta} \right\rceil \\ &= kp + \left\lceil \frac{kr}{\delta} \right\rceil = kp + \left\lceil \frac{t\delta + s}{\delta} \right\rceil = kp + t + 1 \end{aligned}$$

for each $i \in \{1, 2, \dots, d\}$. If we suppose to the contrary that $d = \delta$, then the above inequality chains lead to the contradiction

$$\begin{aligned} kn &\geq \sum_{i=1}^{d} \omega(f_i) \geq d(kp+t+1) = \delta(kp+t+1) \\ &= kp\delta + \delta t + \delta = kp\delta + kr - s + \delta > kp\delta + kr = k(p\delta + r) = kn. \end{aligned}$$

Thus $d \leq \delta - 1$, and the proof is complete. □

Example 3 demonstrates that Theorem 3 is not valid in general.

Corollary 1. If C_n is a cycle of odd length n, then $d^3_{stI}(C_n) = 1$.

Proof. Let $n = 2p+1$ for an integer $p \geq 1$. With $\delta = 2$, $r = 1$, $k = 3$, $t = 1$ and $s = 1$, we deduce from Theorem 3 that $d^3_{stI}(C_n) \leq \delta - 1 = 1$ and therefore $d^3_{stI}(C_n) = 1$. □

Example 4. *If C_n is a cycle of even length n, then $d^3_{stI}(C_n) = 1$ if $n \equiv 2 \pmod 4$ and $d^3_{stI}(C_n) = 2$ if $n \equiv 0 \pmod 4$.*

Proof. First assume that $n = 4p+2$ for an integer $p \geq 1$. Proposition J implies $\gamma^3_{stI}(C_n) = \lceil \frac{3(4p+2)}{2} \rceil + 1 = 6p+4$. So it follows from Theorem 2 that $d^3_{stI}(C_n) \leq \frac{3(4p+2)}{6p+4} < 2$ and therefore $d^3_{stI}(C_n) = 1$ in this case.
Second assume that $n = 4p$ for an integer $p \geq 1$. Let $C_n = (x_1, x_2, \ldots, x_n, x_1)$. Define the function f_1 by $f_1(x_{4i+1}) = f_1(x_{4i+2}) = 1$ and $f_1(x_{4i+3}) = f_1(x_{4i+4}) = 2$ and the function f_2 by $f_2(x_{4i+1}) = f_2(x_{4i+2}) = 2$ and $f_2(x_{4i+3}) = f_2(x_{4i+4}) = 1$ for $0 \leq i \leq p-1$. Then f_1 and f_2 are signed total Italian 3-dominating functions of C_n and $\{f_1, f_2\}$ is a signed total Italian 3-dominating family on C_n. Hence $d^3_{stI}(C_n) \geq 2$ and thus $d^3_{stI}(C_n) = 2$ according to Theorem 1. □

Theorem 4. Let $k \geq 4$ be an integer, and let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. If $\gamma^k_{stI}(G) \leq 2n-1$, then $d^k_{stI}(G) \geq 2$. If $\gamma^k_{stI}(G) \leq 2n-1$ and $\delta(G) = 2$, then $d^4_{stI}(G) = \delta(G) = 2$.

Proof. Since $\gamma^k_{stI}(G) \leq 2n-1$, there exists an STIkDF f_1 with $f_1(v) \leq 1$ for at least one vertex $v \in V(G)$. Note that $f_2 : V(G) \longrightarrow \{-1, 1, 2\}$ with $f_2(x) = 2$ for each vertex $x \in V(G)$ is another STIkDF on G. As $f_1(x) + f_2(x) \leq 4 \leq k$ for each vertex $x \in V(G)$, $\{f_1, f_2\}$ is a signed total Italian k-dominating family on G and thus $d^k_{stI}(G) \geq 2$.
If $\delta(G) = 2$, then Theorem 1 implies $d^4_{stI}(G) \leq \delta(G) = 2$ and thus $d^4_{stI}(G) = \delta(G) = 2$. □

Corollary 1 demonstrates that Theorem 4 is not valid for $k = 3$ in general. In addition, if H is a graph with $\delta(H) = 1$, then it follows from Theorem 1 that $\gamma^k_{stI}(H) = 1$ for $k = 1$ or $k = 2$. Thus Theorem 4 is also not valid for $k = 1$ or $k = 2$ in general.

Corollary 2. Let G be a graph of order n with $\delta(G) \geq 2$. If $2n - 1 \geq \gamma^4_{stI}(G) > \frac{4n}{3}$, then $d^4_{stI}(G) = 2$.

Proof. Theorem 4 implies $d^4_{stI}(G) \geq 2$.
Conversely, it follows from Theorem 2 that

$$d^4_{stI}(G) \leq \frac{4n}{\gamma^4_{stI}(G)} < 3.$$

Thus $d^4_{stI}(G) \leq 2$, and the proof is complete. □

Theorem 5. If G is a graph of order $n \geq 3$ with $\delta(G) \geq \frac{k}{2}$, then $d^k_{stI}(G) \leq n - 2$. In the case $k = 1$, we have $d_{stI}(G) = n - 2$ if and only if $G \in \{K_3, P_3, C_4\}$.

Proof. Let $\delta = \delta(G)$. If $\delta \leq n - 2$, then Theorem 1 implies $d^k_{stI}(G) \leq \delta \leq n - 2$. If $\delta = n - 1$, then $G = K_n$, and we deduce from Proposition D that $\gamma^k_{stI}(K_n) = k + 2$ if $k \geq n$ or $k \leq n - 1$ and $n - k$ is even and $\gamma^k_{stI}(K_n) = k + 1$ if $k \leq n - 1$ and $n - k$ is odd. If $k = 2n - 2$, then $d^k_{stI}(K_n) = 1 \leq n - 2$. Assume next that $k \leq 2n - 3$. In the case $\gamma^k_{stI}(K_n) = k + 2$, it follows from Theorem 2 and $k \leq 2n - 3$ that $d^k_{stI}(K_n) \leq kn/\gamma^k_{stI}(K_n) = kn/(k + 2) < n - 1$. Thus $d^k_{stI}(K_n) \leq n - 2$. Assume next that $\gamma^k_{stI}(K_n) = k + 1$. If $k < n - 1$, then Theorem 2 implies $d^k_{stI}(K_n) \leq kn/\gamma^k_{stI}(K_n) = kn/(k + 1) < n - 1$ and so $d^k_{stI}(K_n) \leq n - 2$. Finally, assume that $k = n - 1$. Then $\gamma^k_{stI}(K_n) = k + 1 = n$. Suppose that $d^k_{stI}(K_n) = n - 1 = k$, and let $\{f_1, f_2, \ldots, f_k\}$ be an STIkD family on K_n. As $\gamma^k_{stI}(K_n) \cdot d^k_{stI}(K_n) = kn$, it follows from Theorem 2 that each function f_i is a $\gamma^k_{stI}(K_n)$-function. If there exists a function f_i and a vertex w such that $f_i(w) = 2$, then $\omega(f_i) = f_i(w)+f_i(N(w)) \geq 2+k$. Consequently, f_i is not a $\gamma^k_{stI}(K_n)$-function. Thus $f_i(x) \in \{-1, 1\}$ for $1 \leq i \leq d$ and $x \in V(K_n)$. However, only the function f with $f(x) = 1$ for each vertex $x \in V(K_n)$ has weight n if $f(x) \in \{-1, 1\}$. Therefore $d^k_{stI}(K_n) = n - 1$ is not possible.
If $G \in \{K_3, P_3, C_4\}$, then it is easy to verify that $d_{stI}(G) = n - 2$.
Conversely, assume that $d_{stI}(G) = n - 2$. If $\delta(G) \leq n - 3$, then Theorem 1 leads to the contradiction $n - 2 = d_{stI}(G) \leq n - 3$. Thus there remain the cases $\delta = n - 1$ and $\delta = n - 2$. If $\delta = n - 1$, then we observe as above that $n - 2 = d_{stI}(G) = n/2$ when n is even and $n - 2 = d_{stI}(G) = n/3$ when n is odd. This yields to $n = 4$ or $n = 3$. Since $d_{stI}(K_4) = 1 < 2 = n - 2$, we obtain $G = K_3$. In the case $\delta = n - 2$, we distinguish two cases.

Case 1. Assume that G is δ-regular. If $\delta = 1$, then $n = 3$, a contradiction. If $\delta = 2$, then $n = 4$ and so $G = C_4$. Let now $\delta \geq 3$. Then $n = (n - 2) + 2 = \delta + 2$, and thus Theorem 3 leads to the contradiction $n - 2 = d_{stI}(G) \leq \delta - 1 = n - 3$.

Case 2. Assume that $\delta = n-2$ and $\Delta(G) = n-1$. If $n = 3$, then $G = P_3$. If $n = 4$, then $G = K_4 - e$, where e is an arbitrary edge of K_4. However, since $d_{stI}(K_4 - e) = 1$, this is not possible. Let now $n \geq 5$, and let f be a $\gamma_{stI}(G)$-function. We will show that $\gamma_{stI}(G) \geq 2$.
If $f(x) \geq 1$ for all $x \in V(G)$, then $\gamma_{stI}(G) \geq n > 2$. Assume next that $f(v) = -1$ for at least one vertex $v \in V(G)$.
Assume first that there exists a vertex w with $f(w) = 2$. If $d(w) = n-1$, then it follows that $\gamma_{stI}(G) = f(w) + f(N(w)) \geq 2+1 = 3 > 2$. If $d(w) = n-2$, then let u be a vertex not adjacent to w. This leads to $\gamma_{stI}(G) = f(w) + f(N(w)) + f(u) \geq 2+1-1 = 2$.
Finally, assume that $f(x) \in \{-1, 1\}$ for all $x \in V(G)$. Assume next that n is even. Let now w be a vertex with $f(w) = 1$. If $d(w) = n-1$, then $\gamma_{stI}(G) = f(w) + f(N(w)) \geq 1+1 = 2$. If $d(w) = n-2$, then $f(N(w)) \geq 1$ and the condition that $n-2$ is even shows that $f(N(w)) \geq 2$. If z is the vertex not adjacent to w, then we obtain $\gamma_{stI}(G) = f(w) + f(N(w)) + f(u) \geq 2+1-1 = 2$. Assume now that $n = 2p+1$ is odd. Clearly, there exist at least $p+1$ vertices x with $f(x) = 1$. If there are at least $p+2$ vertices x with $f(x) = 1$, then $\gamma_{stI}(G) \geq p+2-(n-(p+2)) \geq 3$. Now suppose that there exist exactly $p+1$ vertices x with $f(x) = 1$ and p vertices y with $f(y) = -1$. Let $X = \{x_1, x_2, \ldots, x_{p+1}\}$ such that $f(x_i) = 1$ for $1 \leq i \leq p+1$ and $Y = \{y_1, y_2, \ldots, y_p\}$ such that $f(y_i) = -1$ for $1 \leq i \leq p$. First we observe that $d(x_i) = \delta = 2p-1$ for $1 \leq i \leq p+1$. Therefore there exists a vertex y_i, say y_1 such $d(y_1) = \Delta(G) = 2p$. The condition $f(N(x_i)) \geq 1$ shows that there are at most $(p+1)(p-1)$ edges from X to Y. In addition, the condition $\delta = 2p-1$ shows that there are at least $p \cdot p$ edges from Y to X. This leads to the contradiction $p^2 \leq (p+1)(p-1) = p^2 - 1$, and therefore this case is not possible.
Consequently, we have $\gamma_{stI}(G) \geq 2$. Using again Theorem 2, we obtain $n-2 = d_{stI}(G) \leq n/\gamma_{stI}(G) \leq n/2$, a contradiction to $n \geq 5$. This completes the proof. □

The inequality $d_{stR}(G) \leq d_{stI}(G)$ leads to the following known result.

Corollary 3. *([6])* Let G be a graph of order $n \geq 3$ with $\delta(G) \geq 1$. Then $d_{stR}(G) \leq n-2$, with equality if and only if $G \in \{K_3, P_3, C_4\}$.

3. Upper bounds on the sum $\gamma^k_{stI}(G) + d^k_{stI}(G)$

Theorem 6. If G is a graph of order n with $\delta(G) \geq k$, then

$$\gamma^k_{stI}(G) + d^k_{stI}(G) \leq n + k,$$

with equality if and only if $d^k_{stI}(G) = k$ and $\gamma^k_{stI}(G) = n$.

Proof. If $d^k_{stI}(G) \leq k-1$, then Proposition B implies $\gamma^k_{stI}(G) + d^k_{stI}(G) \leq n+k-1$ immediately. If $d^k_{stI}(G) = k$, then Proposition B implies $\gamma^k_{stI}(G) + d^k_{stI}(G) \leq n+k$, with equality if and only if $\gamma^k_{stI}(G) = n$. Let now $d^k_{stI}(G) \geq k+1$. Then Theorem 1 leads to $n-1 \geq \delta(G) \geq d_{stI}^k(G) \geq k+1$ and thus $n \geq k+2$. It follows from Theorem 2 that

$$\gamma_{stI}^k(G) + d_{stI}^k(G) \leq \frac{kn}{d_{stI}^k(G)} + d_{stI}^k(G).$$

According to Theorem 1, we have $k+1 \leq d_{stI}^k(G) \leq n-1$. Using these bounds, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $k+1 \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq n-1$, we obtain

$$\gamma_{stI}^k(G) + d_{stI}^k(G) \leq \frac{kn}{d_{stI}^k(G)} + d_{stI}^k(G) \leq \max\left\{\frac{kn}{k+1} + k + 1, \frac{kn}{n-1} + n - 1\right\}.$$

Since $n \geq k+2$, we observe that

$$\left\{\frac{kn}{k+1} + k + 1, \frac{kn}{n-1} + n - 1\right\} < n + k,$$

and therefore $\gamma_{stI}^k(G) + d_{stI}^k(G) \leq n + k - 1$ in this case. □

If $p = k \geq 3$ in Example 3, then $d_{stI}^k(K_{p,p}) = p$ and $\gamma_{stI}^k(K_{p,p}) = 2k$ by Proposition G and $n = 2p = 2k$. It follows that $d_{stI}^k(K_{p,p}) + \gamma_{stI}^k(K_{p,p}) = p + 2k = 2p + k = n + k$ and thus equality in Theorem 6 holds.
For $k = 1$ we have the following more precise version of Theorem 6.

Corollary 4. If G is a graph of order n with $\delta(G) \geq 1$, then $\gamma_{stI}(G) + d_{stI}(G) \leq n + 1$, with equality if and only if the components of G are K_2, K_3, P_3 or C_6.

Proof. Theorem 6 implies $\gamma_{stI}(G) + d_{stI}(G) \leq n + 1$, with equality if and only if $d_{stI}(G) = 1$ and $\gamma_{stI}(G) = n$. Since $\gamma_{stI}(G) = n$, Theorem 2 leads to $d_{stI}(G) = 1$, and it follows from Proposition L that the components of G are K_2, K_3, P_3 or C_6. □

Theorem 7. Let G be a graph of order n and $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then

$$\gamma_{stI}^k(G) + d_{stI}^k(G) \leq 2n + k - 1,$$

with equality if and only if $k = 2$, n is even and $G = \frac{n}{2}K_2$.

Proof. If $\delta = \delta(G) \geq k$, then Theorem 6 implies

$$\gamma_{stI}^k(G) + d_{stI}^k(G) \leq n + k < 2n + k - 1.$$

Assume next that $\lceil \frac{k}{2} \rceil \leq \delta \leq k - 1$. Then $k \geq 2$ and according to Proposition C and Theorem 1, we obtain

$$\gamma_{stI}^k(G) + d_{stI}^k(G) \leq 2n + \delta \leq 2n + k - 1. \tag{1}$$

If we have equality in (1), then $\gamma_{stI}^k(G)=2n$ and $d_{stI}^k(G)=k-1$. Therefore Theorem 2 leads to $2n(k-1)=\gamma_{stI}^k(G)\cdot d_{stI}^k(G)\le kn$ and so $k=2$. Thus Proposition C yields to $\delta=1$ and each vertex is adjacent to a vertex of degree 1. Hence $G=\frac{n}{2}K_2$.
Clearly, if n is even and $G=\frac{n}{2}K_2$, then $\gamma_{stI}^2(G)=2n$ and $d_{stI}^2(G)=1$ and thus $\gamma_{stI}^2(G)+d_{stI}^2(G)=2n+1=2n+2-1$. □

Theorem 8. Let $k\ge 3$ be an integer, and let G be a graph of order n with $\delta(G)\ge\lceil\frac{k}{2}\rceil$. If $\gamma_{stI}^k(G)=2n$, then $\gamma_{stI}^k(G)+d_{stI}^k(G)=2n+1$. If $\gamma_{stI}^k(G)\le 2n-1$, then

$$\gamma_{stI}^k(G)+d_{stI}^k(G)\le 2n+\left\lceil\frac{k}{2}\right\rceil-1,$$

with equality if and only if k is even, $\gamma_{stI}^k(G)=2n-1$ and $d_{stI}^k(G)=\frac{k}{2}$.

Proof. If $\gamma_{stI}^k(G)=2n$, then $d_{stI}^k(G)=1$ and thus $\gamma_{stI}^k(G)+d_{stI}^k(G)=2n+1$. Let now $\gamma_{stI}^k(G)\le 2n-1$. Since $n\ge\delta(G)+1\ge\lceil\frac{k}{2}\rceil+1\ge\frac{k}{2}+1$, we observe that $k\le 2n-2$. If $k=2n-2$, then $\gamma_{stI}^k(G)=2n$, a contradiction. Therefore $k\le 2n-3$. If $\delta=\delta(G)\ge k$, then Theorem 6 and $k\le 2n-3$ lead to

$$\gamma_{stI}^k(G)+d_{stI}^k(G)\le n+k<2n+\left\lceil\frac{k}{2}\right\rceil-1$$

and hence $\gamma_{stI}^k(G)+d_{stI}^k(G)\le 2n+\lceil\frac{k}{2}\rceil-2$.
Assume next that $\lceil\frac{k}{2}\rceil\le\delta\le k-1$. If $d_{stI}^k(G)\le\lceil\frac{k}{2}\rceil-1$, then we deduce that $\gamma_{stI}^k(G)+d_{stI}^k(G)\le 2n-1+\lceil\frac{k}{2}\rceil-1=2n+\lceil\frac{k}{2}\rceil-2$.
If $d_{stI}^k(G)=\lceil\frac{k}{2}\rceil$, then we deduce that

$$\gamma_{stI}^k(G)+d_{stI}^k(G)\le 2n+\left\lceil\frac{k}{2}\right\rceil-1,$$

with equality if and only $\gamma_{stI}^k(G)=2n-1$. However, if k is odd, then $d_{stI}^k(G)=\lceil\frac{k}{2}\rceil=\frac{k+1}{2}$, and Theorem 2 leads to the contradiction

$$2n-1=\gamma_{stI}^k(G)\le\frac{kn}{d_{stI}^k(G)}=\frac{2kn}{k+1}.$$

Finally, let $\lceil\frac{k}{2}\rceil+1\le d_{stI}^k(G)\le k-1$. Then $k\ge 4$ and so $n\ge 4$. We deduce from Theorem 2 that

$$\gamma_{stI}^k(G)+d_{stI}^k(G)\le\frac{kn}{d_{stI}^k(G)}+d_{stI}^k(G).$$

Using these bounds, we obtain analogously to the proof of Theorem 6 that

$$\gamma_{stI}^k(G)+d_{stI}^k(G)\le\max\left\{\frac{kn}{\lceil k/2\rceil+1}+\left\lceil\frac{k}{2}\right\rceil+1,\frac{kn}{k-1}+k-1\right\}.$$

Now we show that

$$\max\left\{\frac{kn}{\lceil k/2\rceil+1}+\left\lceil\frac{k}{2}\right\rceil+1,\frac{kn}{k-1}+k-1\right\}<2n+\left\lceil\frac{k}{2}\right\rceil-1.$$

The inequality

$$\frac{kn}{\lceil k/2\rceil+1}+\left\lceil\frac{k}{2}\right\rceil+1<2n+\left\lceil\frac{k}{2}\right\rceil-1$$

is equivalent with $kn<(2n-2)(\lceil\frac{k}{2}\rceil+1)$. If k is even, then the last inequality is equivalent with $k<2n-2$, and if k is odd then this inequality is equivalent with $k<3n-3$. Since $k<2n-2<3n-3$, the desired inequality is valid.
If k is even, then the inequality

$$\frac{kn}{k-1}+k-1<2n+\left\lceil\frac{k}{2}\right\rceil-1$$

is equivalent with $k^2-(2n+1)k+4n<0$ for $4\le k\le 2n-3$. Using the fact that the function $g(x)=x^2-(2n+1)x+4n$ is decreasing for $4\le x\le n+\frac{1}{2}$ and increasing for $n+\frac{1}{2}\le x\le 2n-3$, we obtain

$$k^2-(2n+1)k+4n\le\max\{g(4),g(2n-3)\}=\max\{12-4n,12-4n\}<0,$$

and the desired bound is proved. If k is odd, then the proof is similar and is therefore omitted. □

Example 5. *Let $C_n=x_1x_2\ldots x_nx_1$ be a cycle of order $n\ge 4$. If we add the edge x_1x_3, then we denote the resulting graph by H. The function f with $f(x_2)=1$, $f(x_1)=2$ and $f(x_i)=2$ for $3\le i\le n$ is a $\gamma^4_{stI}(H)$-function of weight $2n-1$. According to Corollary 2, we have $d^4_{stI}(H)=2$ and thus $\gamma^4_{stI}(H)+d^4_{stI}(H)=2n+1=2n+\lceil\frac{k}{2}\rceil-1$ for $k=4$. Thus equality in the bound of Theorem 8 is possible, at least for $k=4$.*

4. Nordhaus-Gaddum type results

Results of Nordhaus-Gaddum type study the extreme values of the sum or the product of a parameter on a graph and its complement. In their current classical paper [4], Nordhaus and Gaddum discussed this problem for the chromatic number. We present such inequalities for the signed total Italian k-domatic number.

Theorem 9. If G is a graph of order n with $\delta(G),\delta(\overline{G})\ge\lceil\frac{k}{2}\rceil$, then $d^k_{stI}(G)+d^k_{stI}(\overline{G})\le n-1$. Furthermore, if $d^k_{stI}(G)+d^k_{stI}(\overline{G})=n-1$, then G is regular.

Proof. It follows from Theorem 1 that

$$d^k_{stI}(G) + d^k_{stI}(\overline{G}) \le \delta(G) + \delta(\overline{G}) = \delta(G) + (n - \Delta(G) - 1) \le n - 1.$$

If G is not regular, then $\Delta(G) - \delta(G) \ge 1$, and hence the above inequality chain implies the better bound $d^k_{stI}(G) + d^k_{stI}(\overline{G}) \le n - 2$. □

For $k = 1$ we will improve Theorem 9.

Theorem 10. If G is a graph of order n with $\delta(G), \delta(\overline{G}) \ge 1$, then $d_{stI}(G) + d_{stI}(\overline{G}) \le n - 1$, with equality if and only if $G = C_4$ or $\overline{G} = C_4$.

Proof. If G is not regular, then Theorem 9 implies $d_{stI}(G) + d_{stI}(\overline{G}) \le n - 2$.
If $G = C_4$ or $\overline{G} = C_4$, say $G = C_4$, then it follows from Example 3 that $d_{stI}(G) = 2$ and $d_{stI}(\overline{G}) = 1$ and therefore $d_{stI}(G) + d_{stI}(\overline{G}) = 3 = n - 1$.
Conversely, assume that $d_{stI}(G) + d_{stI}(\overline{G}) = n - 1$. Then G is δ-regular and $\overline{G}$ is $(n-\delta-1)$-regular with $1 \le \delta \le n-2$ and $1 \le n-\delta-1 \le n-2$. We assume, without loss of generality, that $\delta \le (n-1)/2$.
If $n \not\equiv 0 \pmod{\delta}$, then we deduce from Theorems 1 and 3 that

$$d_{stI}(G) + d_{stI}(\overline{G}) \le (\delta - 1) + (n - \delta - 1) = n - 2,$$

a contradiction. Next assume that $n \equiv 0 \pmod{\delta}$. Since $\delta \le (n-1)/2$, we have $n = p\delta$ with an integer $p \ge 3$. If $n \not\equiv 0 \pmod{(n - \delta - 1)}$, then Theorems 1 and 3 lead to

$$d_{stI}(G) + d_{stI}(\overline{G}) \le \delta + (n - \delta - 2) = n - 2,$$

a contradiction. Therefore assume that $n \equiv 0 \pmod{(n-\delta-1)}$. Then $n = q(n-\delta-1)$ with an integer $q \ge 2$. However, since $n - \delta - 1 \ge (n-1)/2$, we note that $q = 2$. Altogether, we have $n = p\delta = 2(n - \delta - 1)$ and thus $n = p\delta = 2\delta + 2$. The conditions $p \ge 3$ and $\delta \ge 1$ yield to $p = 3$ and $\delta = 2$ or $p = 4$ and $\delta = 1$. If $p = 4$ and $\delta = 1$, then $G = \overline{C_4}$ and $\overline{G} = C_4$ as desired. If $\delta = 2$ and $p = 3$, then $G = C_6$ or $G = 2C_3$. Now it is straightforward to verify that $d_{stI}(C_6) = d_{stI}(2C_3) = 1$ and consequently $d_{stI}(C_6) + d_{stI}(\overline{C_6}) \le 4 = n - 2$ and $d_{stI}(2C_3) + d_{stI}(\overline{2C_3}) \le 4 = n - 2$. □

Since $d_{stR}(G) \le d_{stI}(G)$, Theorem 10 yields to the next known result.

Corollary 5. *([6])* If G is a graph of order n with $\delta(G), \delta(\overline{G}) \ge 1$, then $d_{stR}(G) + d_{stR}(\overline{G}) \le n - 1$, with equality if and only if $G = C_4$ or $\overline{G} = C_4$.

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