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Algorithmic aspects of certified domination in graphs

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Abstract: A dominating set D of a graph $G = (V, E)$ is called a *certified dominating set* of G if $|N(v) \cap (V \setminus D)|$ is either 0 or at least 2 for all $v \in D$. The certified domination number $\gamma_{cer}(G)$ is the minimum cardinality of a certified dominating set of G. In this paper, we prove that the decision problem corresponding to $\gamma_{cer}(G)$ is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs and planar graphs. We also prove that it is linear time solvable for chain graphs, threshold graphs and bounded tree-width graphs.

Keywords: Certified domination, NP-complete, Tree-convex bipartite graphs.

AMS Subject classification: 05C69, 68Q25

1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ are denoted by n and m respectively. For graph theoretic terminology we refer to [15].

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Let $G = (V, E)$ be a graph and let $v \in V$. Then $N(v) = \{u \in V \mid uv \in E\}$ is called the open neighborhood of v. The *degree* $d(v)$ of v is $|N(v)|$. If $d(v) = 1$, then v is called a *pendant vertex* and the unique vertex u which is adjacent to v is called a support vertex. A support vertex u is called a *strong support vertex* if the number of pendant vertices adjacent to u is at least two. Otherwise u is called a *weak support vertex*. For a subset $S \subseteq V$, the subgraph induced by S is defined as $G[S] = (S, E_S)$ where $E_S = \{uv \mid uv \in E \text{ and } u, v \in S\}$. If $G[S]$ is a complete graph, then S is called a clique. If $E_S = \emptyset$, then S is called an independent set.
A graph G is called a split graph if its vertex set can be partitioned into two subsets C and I where C is a clique in G and I is an independent set in G. The graph G is called a bipartite graph if its vertex set can be partitioned into two independent sets A and B. If further every vertex of A is adjacent to every vertex of B, then G is called a complete bipartite graph and is denoted by $K_{r,s}$ where $|A| = r$ and $|B| = s$. A connected acyclic graph is called a tree. The tree $K_{1,n-1}$ is called a star. The tree obtained from a path $P_n = (v_1, v_2, \ldots, v_n)$ by attaching a pendant vertex u_i adjacent to v_i for all i, $1 \leq i \leq n$, is called a comb. Jiang et al. [8] introduced the concept of tree convex bipartite graph.

Definition 1. *([8])* A bipartite graph G with bipartition A, B is called a tree convex bipartite graph if there exists a tree $T = (A, F)$ such that for each $v \in B$, the induced subgraph $T[N(v)]$ is a subtree of T. If T is a star (comb), then G is called a star (comb) convex bipartite graph.

Definition 2. *([10])* A graph $G = (V, E)$ is called a threshold graph if there exists a real number t and a real number $w(v)$ for each $v \in V$ such that a subset S of V is an independent set if and only if $\sum\limits_{v \in S} w(v) \leq t$.

Mahadev and Palad [10] obtained a characterization of threshold graphs.

Theorem 1. ([10]) *A graph G is a threshold graph if and only if G is a split graph with split partition (C, I) and there is an ordering $(x_1, x_2, \ldots, x_r)$ of vertices of C such that $N[x_1] \subseteq N[x_2] \subseteq \ldots \subseteq N[x_r]$ and there is an ordering $(y_1, y_2, \ldots, y_s)$ of the vertices of I such that $N(y_1) \supseteq N(y_2) \supseteq \ldots \supseteq N(y_s)$.*

It follows from Theorem 1 that a graph G is a threshold graph if and only if G is a split graph with split partition (C, I) satisfying the neighborhood chain condition.

Yannakakis [13] introduced the concept of Chain graph.

Definition 3. A bipartite graph G with bipartition (V_1, V_2) is called a chain graph if there exists ordering $(v_1, v_2, \ldots, v_r)$ of the vertices of V_1 such that $N(v_1) \subseteq N(v_2) \subseteq \cdots \subseteq N(v_r)$.

A chain graph also admits an ordering $(w_1, w_2, \ldots, w_s)$ of vertices of V_2 such that $N(w_1) \supseteq N(w_2) \supseteq \cdots \supseteq N(w_s)$.

The study of domination and related subset problems such as irredundance and independence is one of the major research areas within graph theory. A comprehensive treatment of fundamentals of domination in graphs is given in [7]. For a survey of several advanced topics we refer to [6]. Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex not in S is adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. More than 75 models of dominating sets are given in the appendix of [7]. Some recent domination variants have been studied in [1, 2, 11, 12, 14].
In a social network consisting of a set V_1 of officials and a set V_2 of civils where a vertex v in V_1 can serve a civil w in V_2, whenever v is serving w, there exists another civil u that observes v. Thus u serves as a kind of witness so as to avoid any abuse from v. To model this problem Dettlaff et al. [4] introduced the concept of certified domination.

Definition 4. *([4])* A dominating set D of G is called a certified dominating set if every vertex v in D has either no neighbor in $V \setminus D$ or has at least two neighbors in $V \setminus D$. The minimum cardinality of a certified dominating set (CFDS) is called the certified domination number of G and is denoted by $\gamma_{cer}(G)$.

The following basic results on certified domination are given in [4].

Observation 1. *([4])* Every support vertex of a graph G belongs to every certified dominating set of G.

Observation 2. It follows immediately from Observation 1 that $\gamma_{cer}(K_2) = 2$.

Observation 3. *([4])* If G is a graph of order n at least 3, then $\gamma_{cer}(G) = 1$ if and only if $\Delta(G) = n - 1$.

Let G be a connected threshold graph of order n. Let $(x_1, x_2, \dots, x_r)$ and $(y_1, y_2, \dots, y_s)$ be the ordering of the vertices of G as given in Theorem 1. Clearly $\deg(x_r) = n-1$ and hence Observation 2 and Observation 3 give the following result.

Theorem 2. *Let G be a connected threshold graph of order n. Then*

$$\gamma_{cer}(G) = \begin{cases} 2 & \text{if } n = 2 \\ 1 & \text{otherwise.} \end{cases}$$

Let $G = (X, Y, E)$ be a chain graph with chain ordering $X = (x_1, x_2, \dots, x_p)$ and $Y = (y_1, y_2, \dots, y_q)$. If $p = q = 1$, then $G = K_2$ and $\gamma_{cer}(G) = 2$. If $p = 1$ and $q \geq 2$ or if $p \geq 2$ and $q = 1$, then it follows from Observation 3 that $\gamma_{cer}(G) = 1$. If $p, q \geq 2$ then $D = \{x_p, y_1\}$ is a γ_{cer}-set of G. Hence we have the following theorem.

Theorem 3. *Let $G = (X, Y, E)$ be a chain graph with chain ordering $X = (x_1, x_2, \ldots, x_p)$ and $Y = (y_1, y_2, \ldots, y_q)$. Then*

$$\gamma_{cer}(G) = \begin{cases} 2 & \textit{if } p = q = 1 \textit{ or } p, q \geq 2 \\ 1 & \textit{otherwise.} \end{cases}$$

The following is the decision version of the certified domination problem.

Certified Domination Problem (CFDP)

Instance: A graph G and a positive integer k.
Question: Does there exist a CFDS D of G with $|D| \leq k$?

In this paper, we prove that CFDP is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs and planar graphs. We also prove that it is linear time solvable for chain graphs, threshold graphs and bounded tree-width graphs.

2. Complexity Results

In this section, we prove that CFDP is NP-complete for split graphs, star convex bipartite graphs, comb convex bipartite graphs and planar graphs. For the first three cases, the proof is by reduction from the Exact Cover by 3-Sets (X3C) [9], which is given below.

Exact Cover by 3-Sets (X3C)

Instance: A set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
Question: Does C contain an exact cover for X, i.e., a sub-collection $C' \subseteq C$ such that every element in X occurs in exactly one member of C'?

Consider an instance of X3C problem with $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_m\}$. The graph H with vertex set $X \cup C$, where x_i is adjacent to C_j if and only if $x_i \in C_j$, is called the *incidence graph* of the given instance of X3C.

Theorem 4. *CFDP is NP-complete for split graphs.*

Proof. Clearly, CFDP is in NP. Let I be an instance of X3C. Let H be the incidence graph of I. Let G be the split graph obtained from H by making the induced graph $G[C]$ complete. Clearly G can be constructed from I in polynomial time. We now claim that the instance I of X3C has a solution if and only if G has a CFDS of cardinality at most q. Let C' be a solution of the instance I of X3C. Then $D = \{C_j : C_j \in C'\}$ is a dominating set of G and every vertex of D dominates exactly 3 vertices of X. Hence D is a CFDS of G and $|D| = q$.
Conversely, let D' be a CFDS of G with $|D'| \leq q$. If $D' \cap X \neq \emptyset$, then for dominating the vertices of $X - (D' \cap X)$, at least $\frac{3q - |D' \cap X|}{3}$ vertices are needed. Hence $|D'| \geq$

$\frac{3q-|D'\cap X|}{3}+|D'\cap X|>q$, which is a contradiction. Hence $D'\cap X=\emptyset$ and $\{C_j \mid C_j \in D'\}$ is a solution of the instance I of X3C. □

Theorem 5. *CFDP is NP-complete for star convex bipartite graphs.*

Proof. Clearly, CFDP is in NP. Let I be an instance of X3C and let H be the incidence graph of I. Let G be the graph obtained from H by adding an edge ab and joining a to all the vertices of C. Clearly G is a bipartite graph with bipartition $X\cup\{a\}$, $C\cup\{b\}$. Let T denote the star with $V(T)=X\cup\{a\}$ and having a as its central vertex. Let $v\in C\cup\{b\}$. Clearly $a\in N(v)$ and the induced subgraph $T[N(v)]$ is a subtree of T. Hence G is a star convex bipartite graph. Note that the construction of G can be carried out in polynomial time. We claim that the instance I of X3C has a solution if and only if G has a CFDS D with $|D|\le q+1$. If C' is a solution of I, then $D=C'\cup\{a\}$ is a CFDS of G with $|D|=q+1$. Conversely, let D' be a CFDS of G with $|D'|\le q+1$. Since a is a support vertex of G, it follows that $a\in D'$. As in Theorem 4, it can be proved that $D'\cap X=\emptyset$ and $\{C_j \mid C_j\in D'\}$ is a solution to the instance I of X3C. □

Theorem 6. *CFDP is NP-complete for comb convex bipartite graphs.*

Proof. Clearly, CFDP is in NP. Let I be an instance of X3C and let H be the incidence graph of I. Let G be the graph constructed from H as given below.
(i) Add a set $X'=\{x'_1,x'_2,\dots,x'_{3q}\}$ of $3q$ vertices such that $X'\cup C$ is a complete bipartite graph with bipartition X' and C.
(ii) Add three vertices a,a' and b and the edges ab and $aC_i, a'C_i$ for all $C_i\in C$.

Clearly G is a bipartite graph with bipartition $X\cup X'\cup\{a,a'\}$, $C\cup\{b\}$. Let T denote the tree with $V(T)=X\cup X'\cup\{a,a'\}$ such that $(x'_1,x'_2,\dots,x'_{3q},a')$ is a path in T and $x_ix'_i, aa'\in E(T)$ for all i, $1\le i\le 3q$. Clearly T is a comb. Now let $v\in C\cup\{b\}$. Then

$$N(v)=\begin{cases} X'\cup\{a'\} & \text{if } v\in C\\ a & \text{otherwise.}\end{cases}$$

Clearly the induced subgraph $T[N(v)]$ is a subtree of T. Hence G is a comb convex bipartite graph. The proof that the instance I of X3C has a solution if and only if G has a CFDS D with $|D|\le q+1$ is similar to the proof given in Theorem 5. □

X3C problem with the additional restriction that the incidence graph H is planar is called the Planar-X3C problem. It has been proved in [5] that the Planar-X3C is NP-complete. Using Planar-X3C problem and the proof similar to the proof of Theorem 5, we get the following theorem.

Theorem 7. *CFDP is NP-complete for planar graphs.*

We now proceed to prove that the certified domination number $\gamma_{cer}(G)$ can be computed in linear time for several classes of graphs. It follows from Theorem 2 and Theorem 3 that $\gamma_{cer}(G)$ can be computed in linear time for threshold graphs and chain graphs. We now proceed to consider the complexity of CFDP for bounded tree-width graphs.

Let G be a graph, T be a tree and $\mathcal{V}$ be a family of vertex sets $V_t \subseteq V(G)$ indexed by the vertices t of T. The pair $(T, \mathcal{V})$ is called a tree-decomposition of G if it satisfies the following three conditions: (i) $V(G) = \bigcup_{t\in V(T)} V_t$, (ii) for every edge $e \in E(G)$ there exists a $t \in V(T)$ such that both ends of e lie in V_t, (iii) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_1, t_2, $t_3 \in V(T)$ and t_2 is on the path in T from t_1 to t_3. The width of $(T, \mathcal{V})$ is the number $max\{|V_t| - 1 : t \in T\}$, and the tree-width $tw(G)$ of G is the minimum width of any tree-decomposition of G. The following theorem due to Courcelle states that every graph problem that can be described by counting monadic second-order logic (CMSOL) can be solved in linear-time in graphs of bounded tree-width [3].

Theorem 8 (*Courcelle's Theorem*). *([3]) Let P be a graph property expressible in CMSOL and let k be a constant. Then, for any graph G of tree-width at most k, it can be checked in linear-time whether G has property P.*

Theorem 9. *Given a graph G and a positive integer k, CFDP can be expressed in CMSOL.*

Proof. First, we present the CMSOL formula which expresses that the graph G has a dominating set of size at most k.

$$Dominating(S) = (|S| \leq k) \wedge (\forall p)((\exists q)(q \in S \wedge adj(p, q))) \vee (p \in S)$$

where $adj(p, q)$ is the binary adjacency relation which holds if and only if, p, q are two adjacent vertices of G. $Dominating(S)$ ensures that for every vertex $p \in V$, either $p \in S$ or p is adjacent to a vertex in S and the cardinality of S is at most k.

$$\begin{aligned} Certified(S) = (\forall p)((p \notin S) \vee ((\nexists q)(q \in V \setminus S) \wedge adj(p, q))\vee \\ ((\exists q_1)(\exists q_2)(q_1, q_2 \in V \setminus S) \wedge adj(p, q_1) \wedge adj(p, q_2)) \end{aligned}$$

Now, by using the above two CMSOL formulas we can express CFDP in CMSOL formula as follows.

$$CRD(S) = Dominating(S) \wedge Certified(S)$$

Therefore, CFDP can be expressed in CMSOL. □

Now, the following result is immediate from Theorem 8 and Theorem 9.

Theorem 10. *CFDP can be solvable in linear time for bounded tree-width graphs.*

Construction of a Certified Dominating Set

The following algorithm gives a method of constructing a CFDS from any dominating set of a given graph.

Algorithm 1 CONSTRUCTION OF CFDS

Input: A simple and undirected graph G
Output: A CFDS D_1 of G.
$D_1 \leftarrow$ Any dominating set D of G
Let $U \leftarrow \{v : v \in D, |N(v) \cap (V \setminus D)| = 1\}$
while $U \neq \emptyset$ **do**
  Let $v \in U$ and $w = N(v) \cap (V \setminus D)$
  if $d(w) = 1$ **then**
    $D_1 \leftarrow D_1 \cup \{w\}$
  else
    $D_1 \leftarrow (D_1 \setminus \{v\}) \cup \{w\}$
  end if
  $U \leftarrow \{v : v \in D_1, |N(v) \cap (V \setminus D_1)| = 1\}$
end while
return D_1

The above algorithm shows that given any dominating set D of G, we can find a certified dominating set D_1 of G such that $D \subseteq D_1$. The following problem arises naturally.

Remark 1. Which graphs G admit a γ-set D such that the certified dominating set D_1 constructed by the above algorithm is a γ_{cer}-set of G?

3. ILP Formulation of CFDP

In this section, an Integer Linear Programming formulation of certified domination problem (CFDP) is presented. Let $G = (V, E)$ be a graph of order n and let $V = \{v_1, v_2, \ldots, v_n\}$. Let $x = (x_1, x_2, \ldots, x_n)$ be a binary vector. Then x determines a subset of V given by $\{v_i \in V \mid x_i = 1\}$. Conversely, any subset D of V determines a unique binary vector $x = (x_1, x_2, \ldots, x_n)$ where

$$x_i = \begin{cases} 1 & \text{if } v_i \in D \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have a bijection between the set of all binary vectors $(x_1, x_2, \ldots, x_n)$ and the set of all subsets of V. We now define another binary vector $y = (y_1, y_2, \ldots, y_n)$ as follows. Let $N(x_i) = \{x_j : v_j \in N(v_i)\}$.

Let $y_i = \begin{cases} 1 & \text{if } x_i = 1 \text{ and } |\{x_j : x_j \in N(x_i) \text{ and } x_j = 0\}| \text{ is either 0 or at least 2} \\ 0 & \text{otherwise.} \end{cases}$

In other words y_i is a decision variable which takes value 1 if v_i belongs to the subset D of V determined by the vector x and the number of neighbors of v_i in $V \setminus D$ is

either 0 or at least 2. We observe that $\sum_{i=1}^{n} x_i = |D|$.
The following is the ILP model for CFDP.

$$\text{Minimize } z = \sum_{i=1}^{n} x_i$$

subject to:

$$x_i + \sum_{x_j \in N(x_i)} x_j \geq 1, 1 \leq i \leq n \tag{1}$$

$$2(1 - x_i) + \sum_{x_j \in N(x_i)} (1 - x_j) \geq 2y_i, 1 \leq i \leq n \tag{2}$$

$$(1 - x_i) + \sum_{x_j \in N(x_i)} (1 - x_j) \leq ny_i, 1 \leq i \leq n \tag{3}$$

The number of variables in the above ILP is $2n$ and the number of constraints is $3n$. The objective function z is $|D|$, constraint (1) ensures that D is a dominating set and constraints (2) and (3) ensure that any vertex v_i in D is not adjacent to exactly one vertex of $V \setminus D$.

4. Conclusion

In this paper, we have proved that the certified domination problem (CFDP) is NP-complete even when restricted to split graphs, star convex bipartite graphs and comb convex bipartite graphs. On the positive side, we have shown that the problem is linear time solvable for threshold graphs, chain graphs and graphs of bounded tree-width. The investigation of the algorithmic complexity of CFDP for subclasses of chordal graphs and for other bipartite classes is an interesting direction for further research. The ILP formulation of CFDP can be used to compute the value of γ_{cer} for small graphs in a given family $\mathcal{F}$ of graphs and the computed values can be used for determining the value of γ_{cer} for all the graphs of the family $\mathcal{F}$.

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