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Eccentric completion of a graph

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Abstract: The eccentric graph G_e of a graph G is a derived graph with the vertex set same as that of G and two vertices in G_e are adjacent if one of them is the eccentric vertex of the other. In this paper, the concepts of iterated eccentric graphs and eccentric completion of a graph are introduced and discussed.

Keywords: Eccentricity, eccentric graphs, iterated eccentric graphs, eccentric completion

AMS Subject classification: 05C12, 05C75

1. Introduction

The problems on distance and eccentricity related concepts in graph theory are significant in many practical situations. In this paper, we discuss one of such problems. All graphs considered in this paper are simple, finite and undirected graphs. For terminologies in graph theory, we refer to [2, 9].
The *distance* between two vertices u and v in the vertex set $V(G)$ of G, denoted by $d(u,v)$, is the length of the shortest path joining them. An *eccentric vertex* of a vertex u is a vertex v such that the distance $d(u,v)$ is maximum. The *eccentricity* of a vertex v, denoted by $ecc(v)$, is the length of the shortest path between v and its eccentric vertex u. The *radius* of a connected graph G, denoted by $rad(G)$, is the minimum of the eccentricities of all the vertices and the *diameter* of G, denoted by $diam(G)$, is the maximum of the eccentricities of all the vertices in G. The eccentricities of all

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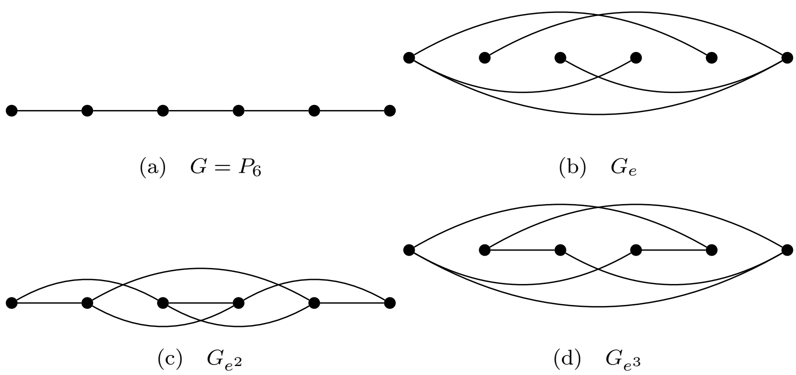
the vertices of a disconnected graph are taken to be $+\infty$ which will also be the radius as well as the diameter of the graph. The articles [6] and [4] elucidate few engrossing studies related to the distance and eccentricity concepts.
The notion of the *eccentric graph* of a graph G, denoted by G_e, is introduced in [1] as a graph on the same set of vertices as that of G and joining two vertices in G_e if and only if one of the vertices has maximum possible distance from the other. That is, two vertices in G_e are adjacent if at least one of them is the eccentric vertex of the other in G. In [1], the eccentric graph is defined as a derived graph, but in [3], it is defined as a graph class. Throughout this discussion, we follow the definition given in [1].

2. Iterated Eccentric Graphs

Motivated by the studies on iterated line graphs, iterated digraphs (see [8]) and iterated eccentric digraphs (see [5]), we introduce the notion of iterated eccentric graphs.

Definition 1. Let G be a graph and G_e be the eccentric graph of G. Then the *iterated eccentric graph* of G, denoted by G_{e^k}, is defined as the derived graph obtained by taking the eccentric graph successively k times, that is, $G_{e^k} = \underbrace{((G_e)_e \ldots)_e}_{k\ times}$.

An illustration of iterated eccentric graphs of a path graph on 6 vertices up to three iterations is given in Figure 1.



(a) $G = P_6$ (b) G_e

(c) G_{e^2} (d) G_{e^3}

Figure 1. Iterated eccentric graphs of P_6 up to three iterations

2.1. Completion of Iterated Eccentric Graphs

Motivated by the studies on the completion number of line graphs (see [7]), in this section, we examine the structural properties of a graph G whose iterated eccentric

graph G_{e^k} is complete for some k.
A graph G is said to be complete when every vertex of G is adjacent to all the other vertices of G. That is, G is complete when the eccentricity of each of the vertices is one. Then, we have:

Lemma 1. *For a graph G, $rad(G) = 1$ if and only if $rad(G_e) = 1$.*

Proof. If $rad(G) = 1$, then there exists at least one vertex which is adjacent to all the other vertices. Thus, in G_e, all such vertices will be adjacent to all the other vertices. Therefore, $rad(G_e) = 1$.
Conversely assume that $rad(G_e) = 1$. Then we have to show that $rad(G) = 1$. If possible, assume the contrary. That is, let $rad(G) \neq 1$. Then, there exists no vertices in G which are adjacent to all the other vertices. Therefore, corresponding to each vertex, in G_e, there exists at least one vertex to which it is not adjacent. Therefore, $rad(G_e) \neq 1$. Hence the result. □

In view of Lemma 1, the following result is immediate.

Corollary 1. *For a graph G, $rad(G) = 1$ if and only if $rad(G_{e^k}) = 1$ for any k.*

Since G_{e^k} is obtained by computing the eccentric graph successively, we can apply Lemma 1 repeatedly to show that $rad(G) = 1$ if and only if $rad(G_{e^k}) = 1$.
Next, we introduce the notion of a k-completing graph as follows:

Definition 2. A graph G is said to be *k-completing* if it becomes a complete graph at the k-th iteration (and not at m-th iteration where m less than k).

In other words, G is a k-completing graph if $G_{e^k} = K_n$ and $G_{e^m} \neq K_n$ for any $m < k$.
The necessary condition for a graph G to be k-completing is discussed below.

Proposition 1. *If G is a k-completing graph, then $rad(G) = 1$.*

The proof follows from the fact that, for a k-completing graph G, $rad(G_{e^k}) = rad(K_n) = 1$ and therefore $rad(G) = 1$, by Corollary 1.
The next theorem gives a necessary and sufficient condition for a vertex of eccentricity 2 in a graph G to attain eccentricity 1 in G_e, where $rad(G) = 1$ and $diam(G) = 2$.

Theorem 1. *For a graph G with $rad(G) = 1$ and $diam(G) = 2$, let $V(G) = X \cup Y$ such that $X = \{x : ecc(x) = 1\}$ and $Y = \{y : ecc(y) = 2\}$. A vertex $u \in Y$ having eccentricity 2 in G attains eccentricity 1 in G_e if and only if u is adjacent only to x such that $x \in X$.*

Proof. Consider the graph G with $rad(G) = 1$ and $diam(G) = 2$. Let us assume that a vertex u of G is adjacent only to each $x \in X$, that is, u is adjacent only to vertices having eccentricity 1. Then, the eccentric vertices of u in G will consist of all the vertices y, where $y \in Y$, and therefore, in G_e, u will be adjacent to all $y \in Y$. In G_e, u will be adjacent to all $x \in X$ as well, since $ecc(x) = 1$ for all $x \in X$. Therefore, in G_e, u is adjacent to all the vertices and therefore $ecc(u) = 1$. This implies that, the vertex u having eccentricity 2 in G attains eccentricity 1 in G_e.
Conversely, suppose that a vertex u having eccentricity 2 in G attains eccentricity 1 in G_e. We need to prove that u is adjacent only to each $x \in X$ in G. Assume the contrary. Then u is also adjacent to some vertex $v \in Y$ in G. Therefore, in G_e, u will not be adjacent to v which means $ecc(u) \neq 1$ in G_e. This is a contradiction to our assumption that u attains eccentricity 1 in G_e. Therefore, if the eccentricity of a vertex changes from 2 in G to 1 in G_e, then it is adjacent only to vertices having eccentricity 1 in G. □

Next, we give a necessary and sufficient condition for a graph G to be k-completing.

Theorem 2. *A graph G is a k-completing graph if and only if at least one additional vertex attains eccentricity* 1 *at each step of finding the iterated eccentric graphs.*

Proof. Let G be a k-completing graph. Then, by Proposition 1, $rad(G) = 1$. We have to prove that, at each step of finding the iterated eccentric graphs, the eccentricity becomes 1 for at least one additional vertex. On the contrary, we assume that $G_{e^{m-1}}$ and G_{e^m} are two iterated eccentric graphs such that the number of vertices with eccentricity 1 is the same in both the graphs. It can be seen as a consequence of Theorem 1 that, if there are no vertices of eccentricity 2 which attain eccentricity 1 on finding the eccentric graph, then there will be no vertex in the original graph which is adjacent only to vertices of eccentricity 1. Therefore, $G_{e^{m-1}}$ has no vertex of eccentricity 2 which is adjacent only to vertices of eccentricity 1. In G_{e^m}, each vertex of eccentricity 2 will be adjacent to all vertices of eccentricity 1, and to those vertices of eccentricity 2 which were non-adjacent in $G_{e^{m-1}}$. Thus, G_{e^m} also doesn't have any vertex which is adjacent only to vertices of eccentricity 1. Thus, on computing $G_{e^{m+1}}$, we obtain the same graph as $G_{e^{m-1}}$, which then implies that G_{e^l} can never be a complete graph for any $l > m$. This is a contradiction to our assumption that G is k-completing. Therefore, if G is k-completing, at each iteration, at least one additional vertex attains eccentricity 1.
Conversely, suppose that at each iteration up to k, at least one more vertex attains eccentricity 1. We have to prove that the graph G is k-completing. The graph under consideration is a finite graph and therefore it has some finite number of vertices, say n. By assumption, at least one more vertex attains eccentricity 1 at each iteration, that is, all the vertices will have attained eccentricity 1 at some level, say k. Therefore, G is k-completing. Hence the result. □

Now, we arrive at the characterisation for a graph G to be k-completing.

Theorem 3. *A graph G is k-completing if and only if the vertex set $V(G)$ of the graph G can be partitioned into $k+1$ subsets V_i such that*

i. $V_0 = \{v \in V(G) : N(v) = V(G)\backslash\{v\}\}$

ii. $V_{2r-1} = \left\{v \in V(G) : N(v) = \bigcup\limits_{i=1}^{r} V_{2i-2}\right\}$ *for* $1 \leq r \leq \lceil \frac{k}{2} \rceil$

iii. $V_{2r} = \left\{v \in V(G) : N(v) = V(G) - \left(\{v\} \cup \left(\bigcup\limits_{i=1}^{r} V_{2i-1}\right)\right)\right\}$ *for* $1 \leq r \leq \lfloor \frac{k}{2} \rfloor$.

Proof. Initially, we need to prove that the given partition consists of $k+1$ sets. V_0 contributes to one partition. When k is even, there are $\lceil \frac{k}{2} \rceil = \frac{k}{2}$ odd suffixed sets and $\lfloor \frac{k}{2} \rfloor = \frac{k}{2}$ even suffixed sets. Therefore, the total number of sets in the partition $= 1 + \frac{k}{2} + \frac{k}{2} = k+1$. When k is odd, there are $\lceil \frac{k}{2} \rceil = \frac{k+1}{2}$ odd suffixed sets and $\lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$ even suffixed sets. Therefore, the total number of sets in the partition $= 1 + \frac{k+1}{2} + \frac{k-1}{2} = k+1$.
Now, let us assume that the vertex set V of the graph G can be partitioned into $k+1$ sets as defined above. We have to prove that G is k-completing. From the statement of the theorem, it is clear that V_0 is the set of vertices of eccentricity 1. As per Theorem 2, it suffices to show that, at each level of iteration, one more vertex attains eccentricity 1 in addition to the vertices in V_0.
Now, consider the eccentric graph G_e of G. In G_e, the vertices in V_0 will be adjacent to all other vertices, as every vertex in V_0 is adjacent to all other vertices in G. The vertices in V_1 are adjacent only to the vertices in V_0 in G and hence by Theorem 1, the vertices in V_1 will be adjacent to all the other vertices in G_e. The vertices in $V_{2r}, 1 \leq r \leq \lfloor \frac{k}{2} \rfloor$ are adjacent only to those vertices in $V_0 \cup \left(\bigcup\limits_{i=1}^{r} V_{2i-1}\right)$ in G_e. Similarly, the vertices in $V_{2r-1}, 2 \leq r \leq \lceil \frac{k}{2} \rceil$ are adjacent to the vertices in $V(G) - \bigcup\limits_{i=2}^{r} V_{2i-2}$. Thus, in the first iteration, the vertices in the set V_1 attain eccentricity 1.
Now, consider the iterated eccentric graph G_{e^2}. The vertices in V_0 and V_1 have eccentricity 1 in G_e and therefore, in G_{e^2}, those vertices will be adjacent to all the other vertices in V. The vertices in V_2 are adjacent only to those vertices in V_0 and V_1 in G_e. Therefore, by Theorem 1, the vertices in V_2 attain eccentricity 1 in G_{e^2}. The vertices in $V_{2r-1}, 2 \leq r \leq \lceil \frac{k}{2} \rceil$ will be adjacent only to the vertices in $V_1 \cup \left(\bigcup\limits_{i=1}^{r} V_{2i-2}\right)$. The vertices in $V_{2r}, 2 \leq r \leq \lfloor \frac{k}{2} \rfloor$ will be adjacent to the vertices in $V(G) - \bigcup\limits_{i=2}^{r} V_{2i-1}$. Thus, at the second iteration, the vertices in V_2 attains eccentricity 1 in addition to the vertices in $V_0 \cup V_1$.
Thus, in general, at the i-th iteration $1 \leq i < k$, the vertices in V_i attain eccentricity 1 in addition to the vertices in $\bigcup\limits_{j=0}^{i-1} V_j$. Finally, at the $(k-1)$-th iteration, all the vertices in $\bigcup\limits_{i=0}^{k-1} V_i$ will attain eccentricity 1. The vertices in V_k will not be adjacent among themselves but will be adjacent to all the remaining vertices which are of

eccentricity 1. Therefore, in G_{e^k}, the vertices in V_k attain eccentricity 1, which means that $G_{e^k} = K_n$ and therefore the graph G is k-completing.
Conversely, suppose that G is a k-completing graph. We have to prove that all k-completing graphs can be characterised as graphs whose vertex set can be partitioned into $k+1$ subsets as given in the statement. Since G is k-completing, by Proposition 1, $rad(G) = 1$, that is, there exists at least one vertex of eccentricity 1. Let the collection of all vertices of eccentricity 1 be called V_0. Now by Theorem 2, G is k-completing if and only if at each iteration, one more vertex attains eccentricity 1. Also, by Theorem 1, such vertices are adjacent only to the vertices of eccentricity 1 in G. Therefore, G has vertices adjacent only to the vertices in V_0. Denote the set of these vertices by V_1. Therefore, in G_e, the vertices in V_1 attain eccentricity 1. Now, applying Theorem 2 and Theorem 1 again, we see that G_e contains vertices that are adjacent only to those vertices in V_0 and V_1 (eccentricity 1 vertices). Denote the set of these vertices by V_2. Since the vertices in V_2 are adjacent only to those vertices in V_0 and V_1 in G_e, the vertices in V_2 will be adjacent to all except those vertices in V_1 in G. Therefore, V_2 is the set of vertices whose eccentricities become 1 in G_{e^2}. Now, the process is repeated to obtain a set, say V_3, in G_{e^2}, whose vertices will be adjacent only to the eccentricity 1 vertices in G_{e^2}, namely V_0, V_1 and V_2. The vertices in V_3 will be adjacent to all except those vertices in V_2 in G_e and therefore, will be adjacent only to the vertices in V_0 and V_2 in G. Proceeding in a similar fashion, by applying Theorem 2 and Theorem 1 repeatedly $k-3$ more times, we obtain the subsets $V_4, V_5, \ldots, V_k$ of $V(G)$ as mentioned above. The vertices in the set V_k attains eccentricity 1 at the $k-$th iteration. Thus, any k-completing graph can be partitioned into $k+1$ subsets of $V(G)$ as mentioned above. This completes the proof. □

Now, we illustrate Theorem 3 with an example.

Example 1. A graph G and its iterated eccentric graphs are illustrated in Figure 2. Here, G is a 5-completing graph on 7 vertices as seen below.
The given graph G is a graph on 7 vertices with the vertex $u_0 \in V_0$, adjacent to all the other vertices. The vertex $u_1 \in V_1$ is adjacent only to u_0. The vertex $u_2 \in V_2$ is adjacent to all the other vertices except u_1. The vertex $u_3 \in V_3$ is adjacent only to u_0 and u_2. The vertex $u_4 \in V_4$ is adjacent to all other vertices except u_1 and u_3. The vertices $u_5, u_6 \in V_5$ are adjacent only to u_0, u_2 and u_4 and they are not adjacent among themselves. Thus, the vertex set $V(G)$ of the graph G is partitioned into 6 subsets, namely $V_0, V_1, \ldots, V_5$ and we see that the graph G is 5-completing. Also, note that the graph G is a 5-completing graph with minimum possible number of vertices which is 7.

Now, the following results give a lower bound for the number of vertices in a k-completing graph.

Corollary 2. *A k-completing graph has at least $k+2$ vertices.*

Proof. By Theorem 3, the vertex set of a k-completing graph can be partitioned into $k+1$ subsets. Each of the $k+1$ sets should have at least one vertex. Now, the last

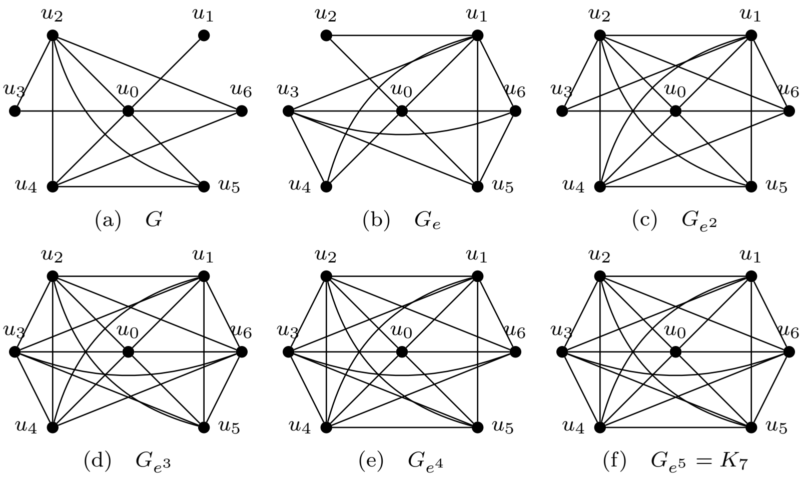


Figure 2. **A 5-completing graph G and its iterated eccentric graphs**

set V_k which attains eccentricity 1 at the k-th level should have at least two elements. This is because, in $G_{e^{k-1}}$, the vertices in V_k will not be adjacent to at least one other vertex. Therefore, a k-completing graph should have at least $k+2$ vertices. □

Remark 1. Since each of the sets can have any non-zero number of vertices, there is no upper limit to the number of vertices that can be there in a k-completing graph.

In Theorem 3, the number of vertices in each of the sets V_i where $1 \le i \le k+1$, is of no significance. Now, assume that, there are p_i elements in each of the sets V_i. Then we can characterise the graphs which are k-completing based on the degree of vertices as follows.

Theorem 4. *A graph G of order n is k-completing if and only if the vertex set $V(G)$ can be partitioned into $k+1$ subsets V_i with $|V_i| = p_i$, where $p_i \neq 0$ and $p_k \geq 2$ such that*

i. $V_0 = \{v \in V(G) : deg(v) = n-1\}$

ii. $V_{2r-1} = \left\{v \in V(G) : deg(v) = \sum\limits_{i=1}^{r} p_{2i-2}\right\}$ *for* $1 \le r \le \lceil \frac{k}{2} \rceil$

iii. $V_{2r} = \left\{v \in V(G) : deg(v) = n-1-\left(\sum\limits_{i=1}^{r} p_{2i-1}\right)\right\}$ *for* $1 \le r \le \lfloor \frac{k}{2} \rfloor$

Proof. Let G be a given graph of order n and V_0,V_{2r-1}, where $1 \le r \le \left\lceil \frac{k}{2} \right\rceil$ and V_{2r}, $1 \le r \le \left\lfloor \frac{k}{2} \right\rfloor$ be the $k+1$ partitions of the vertex set as described in the statement of the theorem.

In order to prove the theorem, it is sufficient to prove that the vertex partitioning mentioned in the statement of the theorem is the same as that of Theorem 3. For this, we consider the partition of the vertex set $V(G)$ of G as in Theorem 3:

i. $U_0 = \{v \in V(G) : N(v) = V(G) \backslash \{v\}\}$

ii. $U_{2r-1} = \left\{ v \in V(G) : N(v) = \bigcup\limits_{i=1}^{r} U_{2i-2} \right\}$ for $1 \leq r \leq \left\lceil \frac{k}{2} \right\rceil$

iii. $U_{2r} = \left\{ v \in V(G) : N(v) = V(G) - \left(\{v\} \cup \bigcup\limits_{i=1}^{r} U_{2i-1} \right) \right\}$ for $1 \leq r \leq \left\lfloor \frac{k}{2} \right\rfloor$

Clearly, $V_0 = U_0$. Now, for all $v \in V_{2r-1}$ where $1 \leq r \leq \left\lceil \frac{k}{2} \right\rceil$, $\deg(v) = \sum\limits_{i=1}^{r} p_{2i-2}$ and so

$$|N(v)| \ = \ \sum_{i=1}^{r} |V_{2i-2}|.$$

By the choice of the set $V_{2i-2}, 1 \leq i \leq r$, the above equation yields $N(v) = \bigcup\limits_{i=1}^{r} V_{2i-2}$. Therefore, we have $V_{2r-1} = U_{2r-1}$.

Consider the sets V_{2r}, where $1 \leq r \leq \left\lfloor \frac{k}{2} \right\rfloor$. Here, we have $\deg(v) = n - 1 - \sum\limits_{i=1}^{r} p_{2i-1}$ and so

$$|N(v)| \ = \ |V(G)| - 1 - |\bigcup_{i=1}^{r} V_{2i-1}|.$$

By the choice of the set $V_{2i-1}, 1 \leq i \leq r$, the above equation yields

$$N(v) \ = \ (V(G) - \{v\}) - \bigcup_{i=1}^{r} V_{2i-1},$$

that is

$$N(v) \ = \ V(G) - \left(\{v\} \cup \left(\bigcup_{i=1}^{r} V_{2i-1} \right) \right)$$

which implies $N(v) \cap \left(\{v\} \cup \left(\bigcup\limits_{i=1}^{r} V_{2i-1} \right) \right) = \emptyset$. That is, $N(v) \cap \{v\} = \emptyset$ and $N(v) \cap \left(\bigcup\limits_{i=1}^{r} V_{2i-1} \right) = \emptyset$. Therefore, $V_{2r} = U_{2r}$. This completes the proof. $\square$

3. Conclusion

In this paper, a study on iterated eccentric graphs has been initiated and the completion of such graphs has been investigated. Characterisations of k-completing graphs based on the adjacency and degrees of vertices have also been discussed. The future studies can be taken up on the structural properties of iterated eccentric graphs that may or may not attain completion.

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References

[1] J. Akiyama, K. Ando, and D. Avis, *Eccentric graphs*, Discrete Math. **56** (1985), no. 1, 1–6.

[2] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, 1990.

[3] G. Chartrand, W. Gu, M. Schultz, and S.J. Winters, *Eccentric graphs*, Networks **34** (1999), no. 2, 115–121.

[4] P. Dankelmann, *New bounds on proximity and remoteness in graphs*, Commun. Comb. Optim. **1** (2016), no. 1, 29–41.

[5] J. Gimbert, M. Miller, F. Ruskey, and J. Ryan, *Iterations of eccentric digraphs*, Bull. Inst. Combin. Appl. **45** (2005), 41–50.

[6] J. T. Hedetniemi, S. T. Hedetniemi, R. C. Laskar, C. Renu, and H. M. Mulder, *Different-distance sets in a graph*, Commun. Comb. Optim. **4** (2019), no. 2, 151–171.

[7] J. V. Kureethara and M. Sebastian, *Line completion number of grid graph* $p_n \times p_m$, Commun. Comb. Optim. **6** (2021), no. 2, 299–313.

[8] E. Prisner, *Graph Dynamics*, vol. 338, CRC Press, 1995.

[9] D. B. West, *Introduction to Graph Theory*, vol. 2, Prentice hall of India, New Delhi, 2001.