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On the total liar's domination of graphs

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Abstract: For a graph G, a set L of vertices is called a *total liar's domination* if $|N_G(u) \cap L| \geq 2$ for any $u \in V(G)$ and $|(N_G(u) \cup N_G(v)) \cap L| \geq 3$ for any distinct vertices $u, v \in V(G)$. The *total liar's domination number* is the cardinality of a minimum total liar's dominating set of G and is denoted by $\gamma_{TLR}(G)$. In this paper we study the total liar's domination numbers of join and products of graphs.

Keywords: Total liar's domination, Join of graphs, Graphs products

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1. Introduction

All graphs under consideration are finite, undirected and without multiple edges and loops. For notation and terminology we follow [2, 3].
Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every vertex $v \in V(G)$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = d_G(v) = |N_G(v)|$. The *minimum degree* and the *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

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The subgraph of a graph G *induced* by $S \subseteq V(G)$, $G[S]$, is a graph with vertex set S and two vertices of S are adjacent in $G[S]$ if and only if they are adjacent in G.
A set $S \subseteq V(G)$ with the property $\bigcup_{v \in S} N_G[v] = V$ is called a *dominating set* and the minimum cardinality of such a set is denoted by $\gamma(G)$. Note that we sometimes refer to a dominating set with minimum cardinality as a γ-set. A set $D \subseteq V(G)$ is called a *total dominating set* of a graph G if for each vertex $v \in V(G), |N_G(v) \cap D| \geq 1$. The *total domination number* of a graph G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Also, a set $S \subseteq V(G)$ is called a *double total dominating set* if for each $v \in V(G)$ we have $|N_G(v) \cap S| \geq 2$. The *double total domination number*, $\gamma_{\times 2,t}(G)$, is the minimum cardinality among all double total dominating sets.
A new and interesting variant of domination in graphs is liar's domination which introduced by Slater in 2019 [7] and since then has been extensively studied by researchers (see for example [1, 5, 6]). A set $D \subseteq V(G)$ of a graph G is called a *liar's dominating* set if for all $v \in V(G), |N_G[v] \cap D| \geq 2$ and for every pair $u, v \in V(G)$ of distinct vertices, $|(N_G[u] \cup N_G[v]) \cap D| \geq 3$. The *liar's domination number* of a graph G, denoted by $\gamma_{LR}(G)$, is the minimum cardinality of a liar's dominating set of G. Liar's dominating set is useful in securing a network which contains at most one intruder and securing devices such that they may misreport or fail to report an intruder location [7].
Based on liar's dominating set, Panda and et. al [4], defined total liar's dominating sets of graphs. A subset $D \subseteq V(G)$ is a *total liar's dominating set* if and only if $|N_G(v) \cap D| \geq 2$ for every $v \in V(G)$ and $|(N_G(u) \cup N_G(v)) \cap D| \geq 3$ for every pair u, v of distinct vertices of G. The *total liar's domination number* is the cardinality of a minimum total liar's dominating set of G and is denoted by $\gamma_{TLR}(G)$. They purpose some algorithmic aspects of this parameter of graph.
Note that a total liar's dominating set of a graph is a total dominating set and a total dominating set exists if and only if $\delta(G) \geq 1$. Following theorem stats that when a graph admits a total liar's dominating set.

Theorem 1. *[4] A connected graph G admits a total liar's dominating set if and only if (i) it has at least three vertices, (ii) $\delta(G) \geq 2$, and (iii) for every non-adjacent pair of vertices $u, v \in V(G), G[S]$ is neither isomorphic to C_4 nor isomorphic to $K_4 \setminus \{e\}$, where $S = N_G(u) \cup N_G(v)$.*

The rest of the paper is organized as follows. Section 2 deals with the total liar's dominating sets of join of two graphs. We show that if G_1 and G_2 have at least three vertices, then the total liar's domination number of join of two graphs is at most six. Also we establish some sharp bounds for total liar's domination number of join of a graph G with graphs K_1 or K_2. We present some sharp bounds for total liar's domination number of direct, cartesian and Lexicographical products of two graphs in Section 3.

2. Join of two graphs

In this section, we study the existence of total liar's dominating set in join of two graphs. Recall that the *join* of two graphs G_1 and G_2, $G = G_1 + G_2$, is a graph with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For example, $K_1 + P_n$ is the *fan* F_n, $K_1 + C_n$ is the *wheel* W_n, and the *friendship graph* Fr_n, is the graph $K_1 + nK_2$. We know that, if a graph G of order n admits a total liar's dominating set, then $3 \leq \gamma_{TLR}(G) \leq n$. Suppose that V_i is the set of vertices of degree i. If G admits a total liar's dominating set, L, then $V_1 = \varnothing$ and $N(V_2) \subseteq L$. This fact implies the following simple result.

Lemma 1. *Let G be a connected graph G of order $n \geq 5$ which admit a total liar's dominating set. If V_2 is a total dominating set of G, then $\gamma_{TLR}(G) = n$.*

Proof. Suppose that V_2 is a total dominating set and let L be a total liar's dominating set. Clearly, $V(G) = N_G(V_2) \subseteq L$ and therefore $L = V(G)$. So $\gamma_{TLR}(G) = n$. □

Following corollary is a simple result of the above lemma.

Corollary 1. $\gamma_{TLR}(Fr_n) = 2n + 1$

Theorem 2. *Let G_1 and G_2 be two graphs with at least 3 vertices. Then $3 \leq \gamma_{TLR}(G_1 + G_2) \leq 6$*

Proof. Clearly any set S of vertices $G_1 + G_2$ which contain 3 elements of $V(G_1)$ and 3 element of $V(G_2)$, is a total liar's dominating set of G. Hence $\gamma_{TLR}(G_1 + G_2) \leq 6$. □

Remark 1. Note that it is possible G_1 (or G_2) has two vertices, and $G_1 + G_2$ has not total liar's dominating set. For example $G = \overline{K_2} + \overline{K_n} = K_{2,n}$ has not total liar's dominating set.

Theorem 3. *Let G be a graph. Then $\gamma_{TLR}(G) = 3$ if and only if $G = K_3 + H$ for some graph H.*

Proof. Let $L = \{x, y, z\}$ be a total liar's dominating set of G. Hence every pair of vertices in $\{x, y, z\}$ are adjacent by the definition of total liar's dominating set. For any $v \in V(G) \backslash \{x, y, z\}$, we have $|N_G(v) \cap L| \geq 2$. If $|N_G(v) \cap L| = 2$ and $N_G(v) \cap L = \{x, y\}$ for a vertex v, then $|(N_G(v) \cup N_G(z)) \cap L| = 2$, which is a contradiction. Therefore $N_G(v) = \{x, y, z\}$ for any $v \in V(G)$. So if $S = V(G) \setminus \{x, y, z\}$ and $H = G[S]$, then $G = K_3 + H$. The converse of theorem is clear. □

Theorem 4. *Let G be a graph of order n. If $\delta(G) \geq 1$, then*

$$1 + \gamma_t(G) \leq \gamma_{TLR}(K_2 + G) \leq 2 + \gamma_t(G).$$

In addition these bounds are sharp.

Proof. Let S be a total dominating set of G and $\{u, v\}$ be the vertex set of K_2. It is not difficult to see that $D = S \cup \{u, v\}$ is a total liar's dominating set of $K_2 + G$. Hence $\gamma_{TLR}(K_2+G) \leq 2+\gamma_t(G)$. Since $\gamma_{TLR}(K_2+C_4) = 4 = 2+\gamma_t(C_4)$, we conclude that the upper bound is sharp.
Let S be a total liar's dominating set of $K_2 + G$. Consider two vertices $x, y \in S$ and set $D = (S \cup \{u, v\}) \setminus \{x, y\}$. Clearly D is a total liar's dominating set of K_2+G with $|D| = |S|$. Set $D' = D \setminus \{u, v\}$. We have $D' \subseteq V(G)$ and $|(N_G(a) \cup N_G(b)) \cap D'| \geq 1$ for any $a, b \in V(G)$. This fact implies that $|N_G(b) \cap D'| \geq 1$ for any vertex $b \in V(G)$ except at most one. If $|N_G(b) \cap D'| \geq 1$ for any $b \in V(G)$, then D' is a total dominating set and we conclude that $2 + \gamma_t(G) \leq \gamma_{TLR}(K_2 + G)$. If there exists $a \in V(G)$ such that $N_G(a) \cap D' = \varnothing$, then choose $b \in N_G(a)$ and set $L = D' \cup \{b\}$. Clearly L is a total dominating set of G and we conclude that $1 + \gamma_t(G) \leq \gamma_{TLR}(K_2 + G)$ and the lower bound holds. The lower bound is sharp since $\gamma_{TLR}(K_2 + C_5) = 4 = 1 + \gamma_t(C_5)$ □

Lemma 2. *Suppose that $n_1 \leq n_2 \leq ... \leq n_k$ and $G = K_{n_1,n_2,...,n_k}$.*
i) If $k = 2$ and $n_1 \leq 2$, then K_{n_1,n_2} dose not admit a total liar's dominating set,
ii) If $k = 2$ and $n_1 \geq 3$, then $\gamma_{TLR}(K_{n_1,n_2}) = 6$,
iii) If $k = 3$ and $n_1 = n_2 = 1$, and $n_3 \geq 2$, then $K_{1,1,n_3}$ dose not admit a total liar's dominating set,
iv) If $k = 3$ and $n_2 \geq 2$, then $\gamma_{TLR}(K_{n_1,n_2,n_3}) = 5$,
v) If $k > 3$, then $\gamma_{TLR}(K_{n_1,...,n_k}) = 4$.

Proof. Let $G = K_{n_1,n_2,...,n_k}$ and $V_1, V_2, \cdots V_k$ be parts of $V(G)$ of sizes $n_1, n_2, \cdots, n_k$, respectively.
Parts (i) and (ii) follows from Lemma 7 of [4].
In case (iii), note that for any $D \subseteq V(G)$ and for any $a, b \in V_3$, we have $|(N_G(a) \cup N_G(b)) \cap D| \leq 2$. Hence G does not admit a total liar's dominating set.
For proving Part (iv), we consider a subset S of $V(G)$, which contain one element from V_1, two elements from V_2 and two elements from V_3. Clearly S is a total liar's dominating set of K_{n_1,n_2,n_3}. Hence $\gamma_{TLR}(K_{n_1,n_2,n_3}) \leq 5$. To prove the inverse inequality, let D be a total liar's dominating set of G and $k_i = |D \cap V_i|$. Since $\gamma_{TLR}(G) \geq 4$, we conclude that $k_1 + k_2 + k_3 \geq 4$ by Theorem 3. Choose $a, b \in V_2$. We have $|(N_G(a) \cup N_G(b)) \cap D| \geq 3$ and this fact implies that $k_1 + k_3 \geq 3$. Similarly $k_1 + k_2 \geq 3$. If $k_1 \leq 1$, then $k_1 + k_2 + k_3 \geq 5$ and hence $\gamma_{TLR}(G) \geq 5$. If $k_1 \geq 2$, then $|V_1| \geq 2$ and hence $k_2 + k_3 \geq 3$. This implies that $k_1 + k_2 + k_3 \geq 5$ and again $\gamma_{TLR}(G) \geq 5$. Therefore $\gamma_{TLR}(G) = 5$.
In case (v), note that $\gamma_{TLR}(G) \geq 4$ by Theorem 3. Now choose $v_i \in V_i$ for $1 \leq i \leq 4$. It is not difficult to see that $D = \{v_1, v_2, v_3, v_4\}$ is a total liar's dominating set of G and hence $\gamma_{TLR}(G) = 4$. □

Theorem 5. *Let G be a graph, which admit a total liar's dominating set. Then*

$$\gamma_t(G) + 1 \leq \gamma_{TLR}(K_1 + G) \leq min\{\gamma_{TLR}(G), \gamma_{\times 2,t}(G) + 1\}.$$

In addition these bounds are sharp.

Proof. First we prove the lower bound. Let S be a total liar's dominating set of K_1+G and $\{x\}$ be the vertex set of K_1. Without loss of generality, we can assume that $x \in S$. Now consider $L = S\backslash\{x\}$. Hence $|L \cap N_G(a)| \geq 1$ for each vertex $a \in V(G)$ and so we conclude that L is a total dominating set. Therefore $\gamma_t(G)+1 \leq \gamma_{TLR}(K_1+G)$. If $G = K_2 + H$ for some graph H, then $\gamma_{TLR}(G + K_1) = 3 = 1 + \gamma_t(G)$ and hence the lower bound is sharp.
Next we prove the upper bound. It is clear that every total liar's dominating set of G is a total liar's dominating set of $G + K_1$ and hence $\gamma_{TLR}(K_1 + G) \leq \gamma_{TLR}(G)$. In addition, if S is a double total dominating set of G, then $S \cup \{x\}$ is a total liar's dominating set of $G+K_1$ and we conclude that $\gamma_{TLR}(K_1+G) \leq \gamma_{\times 2,t}(G)+1$. Hence the upper bound is obtained. For graph C_5, we have $\gamma_{TLR}(C_5) = \gamma_{\times 2,t}(C_5) = 5$ and $\gamma_{TLR}(C_5 + K_1) = \gamma_{TLR}(C_5) = 5$. Also $\gamma_{TLR}(K_{2,2,2}) = 5$ and $\gamma_{\times 2,t}(K_{2,2,2}) = 3$ and

$$\gamma_{TLR}(K_{2,2,2} + K_1) = \gamma_{TLR}(K_{1,2,2,2}) = 4 = \gamma_{TLR}(K_{2,2,2}) + 1.$$

This fact shows that the upper bound is sharp □

3. Graphs Products

In this section we study the total liar's dominating set of some graphs product.
The *direct product* of graphs G and H, $G \times H$, is the graph with the vertex set $V(G) \times V(H)$ and two vertices (a, b) and (a', b') being adjacent in $G \times H$ if and only if $aa' \in E(G)$ and $bb' \in E(H)$.

Theorem 6. *Suppose that G and H are two graphs, which admit double total domination sets. Then*

$$\gamma_{TLR}(G \times H) \leq \gamma_{\times 2,t}(G)\gamma_{\times 2,t}(H).$$

Proof. Let S and T be double total dominating sets of G and H, respectively and set $D = S \times T$. We prove that D is a total liar's dominating set of $G \times H$. Suppose that $(a, b) \in V(G) \times V(H)$. Hence there are two vertices $c, d \in N_G(a) \cap S$, and two vertices $e, f \in N_H(b) \cap T$. Hence

$$(c, e), (c, f), (d, e), (d, f) \in D \cap N_{G\times H}((a, b)),$$

and we conclude that $|D \cap N_{G\times H}((a, b))| \geq 4$. Hence D is a total liar's dominating set of $G \times H$. □

The *cartesian product* of two graphs G and H, denoted by $G\Box H$, is a graph with vertex set $V(G)\times V(H)$, where two vertices (a,b) and (a',b') are adjacent if $aa'\in E(G)$ and $b=b'$ or $a=a'$ and $bb'\in E(H)$.

Theorem 7. *Suppose that G and H are two graphs, which admit total double domination sets. Then*

$$\gamma_{TLR}(G\Box H)\leq min\{|V(G)|\gamma_{\times 2,t}(H),|V(H)|\gamma_{\times 2,t}(G)\}.$$

Proof. Let T be a double total domination of H and consider the set $D=V(G)\times T$. Suppose that $(a,b)\in V(G\Box H)$. There are two distinct vertices $x,y\in N_H(b)\cap T$. Hence

$$(a,x),(a,y)\in N_{G\Box H}((a,b))\cap D,$$

and hence $|N_{G\Box H}((a,b))\cap D|\geq 2$. Now consider two distinct vertices $(a,b),(c,d)\in V(G\Box H)$. If $a=c$, then $b\neq d$, and this fact implies that there are three distinct vertices $x,y,z\in N_H(b)\cup N_H(d)\cap T$. Hence

$$(a,x),(a,y),(a,z)\in (N_{G\Box H}((a,b))\cup N_{G\Box H}((c,d)))\cap D.$$

If $a\neq c$, then consider two vertices $x,y\in N_H(b)$ and two vertices $z,w\in N_H(d)$. Hence

$$(a,x),(a,y),(c,z),(c,w)\in (N_{G\Box H}((a,b))\cup N_{G\Box H}((c,d)))\cap D.$$

In all cases, we conclude that $|(N_{G\Box H}((a,b))\cup N_{G\Box H}((c,d)))\cap D|\geq 3$. Therefore D is a total liar's dominating set. $\Box$

The *Lexicographical product* of graph G and H, denote by $G[H]$, is the graph with $V(G[H])=V(G)\times V(H)$ and two distinct vertices (a,b) and (a',b') are adjacent if and only if $aa'\in E(G)$ or $a=a'$ and $bb'\in E(H)$.

Theorem 8. *Suppose that G and H are two graphs, which admit total liar's domination sets. Then*

$$\gamma_{TLR}(G[H])\leq \gamma_{TLR}(H)\gamma(G).$$

Proof. Let S be a dominating set of G, and T be a total liar's dominating set of H. We show that $D=S\times T$ is a total liar's domination of $G[H]$. Suppose that $(a,b)\in V(G)\times V(H)$. There are distinct vertices $x,y\in N_H(b)\cap T$. If $a\in S$, then $(a,x),(a,y)\in N_{G[H]}((a,b))\cap D$. If $a\notin S$, then there exists $c\in N_G(a)\cap S$. Hence $(c,x),(c,y)\in N_{G[H]}((a,b))\cap D$. In both cases we have

$$|N_{G[H]}((a,b))\cap D|\geq 2.$$

Now consider two distinct vertices $(a,b),(c,d) \in V(G) \times V(H)$. If $a \notin S$, then there exists $a' \in N_G(a) \cap S$. Consider three distinct vertices $x,y,z \in T$. Therefore

$$(a',x),(a',y),(a',z) \in (N_{G[H]}((a,b)) \cup N_{G[H]}((c,d))) \cap D.$$

Suppose that $a,c \in S$. If $a = c$, then $b \neq d$. Hence there are three distinct vertices $x,y,z \in (N_H(b) \cup N_H(d)) \cap D$. Then

$$(a,x),(a,y),(a,z) \in (N_{G[H]}((a,b)) \cup N_{G[H]}((c,d))) \cap D.$$

Now suppose that $a \neq c$. If $b \neq d$, then

$$(a,x),(a,y),(a,z) \in (N_{G[H]}((a,b)) \cup N_{G[H]}((c,d))) \cap D.$$

If $b = d$, then choose $x,y \in N_H b \cap T$ and we have

$$(a,x),(a,y),(c,x),(c,y) \in (N_{G[H]}((a,b)) \cup N_{G[H]}((c,d))) \cap D.$$

In all cases, we have $|(N_{G[H]}((a,b)) \cup N_{G[H]}((c,d))) \cap D| \geq 3$. Hence D is a total liar's dominating set and the proof is complete. □

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