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# $\mathscr{P}$-energy of generalized Petersen graphs

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**Abstract:** For a given graph $G$, its $\mathscr{P}$-energy is the sum of the absolute values of the eigenvalues of the $\mathscr{P}$-matrix of $G$. In this article, we explore the $\mathscr{P}$-energy of generalized Petersen graphs $G(p,k)$ for various vertex partitions such as independent, domatic, total domatic and $k$-ply domatic partitions and partition containing a perfect matching in $G(p,k)$. Further, we present a python program to obtain the $\mathscr{P}$-energy of $G(p,k)$ for the vertex partitions under consideration and examine the relation between them.

**Keywords:** Graph energy, partition matrix, $\mathscr{P}$-matrix, independent partition, domatic partition, total domatic partition, $k$-ply domatic partition

**AMS Subject classification:** 05C15, 05C50, 05C69

## 1. Terminology and introduction

In the present article we consider simple, finite and undirected graphs of order $n$ and size $m$. The graph and spectral theoretic terminologies are taken from Cvetković et al. and West [5, 18]. For the notations and concepts dealing with domination and related concepts, we refer to Hedetniemi et al. [9], Cockayne et al. [2], Cockayne and Hedetniemi [3] and Zelinka [19].
Earlier to 1940's, the concept of energy of a molecular graph was restricted to chemistry. Later, in 1978 Gutman brought this concept to graph theory for an arbitrary graph [7]. He defined the *energy* of a graph $G$ as the sum of the absolute values of

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the eigenvalues of the adjacency matrix of $G$. This concept gained popularity among researchers due to its applications and scope for further expansions and variations. For more details about spectral graph theory and related studies one may refer to Cvetković et al. [5] and Li et al. [8].
One of the variations of graph energy that considers the vertex partitions of a graph is the *k-partition energy* $E_{P_k}(G)$ defined by Sampathkumar et al. [16]. Recently, Prajakta and Mayamma [10], introduced *$\mathscr{P}$-energy* $E_{\mathscr{P}}(G)$ of a graph $G$ as an extension of $E_{P_k}(G)$, wherein the number of elements in the sets in the partition $\mathscr{P}$ is taken into consideration. They defined it as the sum of the absolute values of the eigenvalues of the *$\mathscr{P}$-matrix* $A_{\mathscr{P}}(G)$ [10]. If $G$ is a graph having a vertex partition $\mathscr{P}$ consisting of $k$ elements, then the matrix $A_{\mathscr{P}}(G)$ is of order $n \times n$ with the elements

$$a_{ij} = \begin{cases} |V_r| & \text{if } v_i = v_j \in V_r, \text{ for } r = 1, 2, \dots k \\ 2 & \text{if } v_iv_j \in E(G) \text{ with } v_i, v_j \in V_r, \\ 1 & \text{if } v_iv_j \in E(G) \text{ with } v_i \in V_r \text{ and } v_j \in V_s \text{ for } r \neq s, \\ -1 & \text{if } v_iv_j \notin E(G) \text{ with } v_i, v_j \in V_r, \\ 0 & \text{otherwise.} \end{cases}$$

The authors have obtained a few bounds for $\mathscr{P}$-energy in [10] along with its exact values for a variety of graph classes such as complete graphs, star graphs, complete bipartite graphs and double star graph as well as $\mathscr{P}$-energy of join of graphs [11, 12].

Vertex partition problems have several applications in networks such as wireless sensor network system. In such networks, nodes are represented by vertices, and an edge between two vertices exists if and only if they are in each other's communication range. One challenge that such networks face is the limited battery power and life time of the nodes. Hence the energy conservation is a core issue to be addressed in sensor network systems. Moscibroda and Wattenhofer [13] provided a solution to this by maximizing the number of distinct dominating sets and rotating the active state for each domatic partition. For instance, if $\{V_1, V_2, \dots, V_k\}$ is a partition of the vertices of a network into dominating sets, then vertices in $V_1$ are active or awake and take the responsibility of all the activities in the network for a specific period of time while other nodes are asleep. After a certain time interval the active state will be transferred to $V_2$ and the process continues. This process of successive activation of nodes in each $V_i$'s results in less energy consumption. It has been observed in [13, 14] that this process works better if we use maximum number of disjoint domatic partitions. This observation motivates us to study the concept of energy with respect to various types of vertex partitions.
In the present study, we examine the $\mathscr{P}$-energy of $G(p, k)$ with respect to vertex partitions of various types: independent, domatic, total domatic, $k$-ply domatic partitions, and partition containing a perfect matching in $G(p, k)$.
Petersen graph was generalized by Coxeter [4] and later Watkin [17] named its generalization as *generalized Petersen graph* $G(p, k)$. It is a graph of order $n = 2p$

where $p \geq 3$ with vertex and edge set as follows: For $1 \leq k \leq p-1$ and $2k \neq n$, $V[G(p,k)] = \{u_i, v_i : 0 \leq i \leq p-1\}$ and $E[G(p,k)] = \{u_iv_i, u_iu_{i+1}, v_iv_{i+k} : i \in Z_p,$ group of integers modulo $p\}$. It is a connected cubic graph consisting of an inner star polygon with vertices $u_0, u_1, \ldots, u_{p-1}$ and an outer regular polygon on vertices $v_0, v_1, \ldots, v_{p-1}$. The edges $\bigcup_{i=0}^{p-1} u_iv_i$ are called spokes.
We limit the present study to the family of generalized Petersen graphs because of the significance of these graphs in wireless sensor networks. Zitnik et al. [21] in 2009 have proved that all generalized Petersen graphs are unit distance graphs which are important in the study of networks in the context of maximizing the lifespan of a network.

## 2. $\mathscr{P}$-energy of $G(p,k)$

As evident from the definition, the value $\mathscr{P}$-energy of a graph depends on the nature of the partition $\mathscr{P}$ of the vertex set. We examine the $\mathscr{P}$-energy of generalized Petersen graphs $G(p,k)$ by considering the trivial partitions $\mathscr{P}_r$ and $\mathscr{P}_s$, partition $\mathscr{P}(pm)$ containing a perfect matching, independent partition $\mathscr{P}(i)$, domatic partition $\mathscr{P}(d)$, total domatic partition $\mathscr{P}(td)$, and $k$-ply domatic partitions $\mathscr{P}(d^k)$ for $k = 2, 3$.
We begin by examining $E_{\mathscr{P}}(G)$ for the trivial partitions. It is to be noted that, the $\mathscr{P}$-energy of a graph $G$ with vertex partition $\mathscr{P}_r = V(G)$ is the robust $\mathscr{P}$-energy $E_{\mathscr{P}_r}(G)$ and the $\mathscr{P}$-energy of $G$ with vertex partition $\mathscr{P}_s$ containing all the singleton set of vertices is the shear $\mathscr{P}$-energy $E_{\mathscr{P}_s}(G)$ [10].
We would be using the following result for further discussions.

**Lemma 1.** [5] *If* $C = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ *is a symmetric block matrix of order* $2 \times 2$*, then the spectrum of* $C$ *is the union of the spectra of* $A + B$ *and* $A - B$*.*

**Theorem 1.** *For the generalized Petersen graph* $G(p,k)$*,* $E_{\mathscr{P}_r}(G(p,k)) = n^2$*.*

*Proof.* For $\mathscr{P}_r = V(G)$, the corresponding $\mathscr{P}$-matrix is positive semi-definite and hence the result is direct.

$$E_{\mathscr{P}_r}(G(p,k)) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} \lambda_i = trace(A_{\mathscr{P}_r}(G(p,k))) = n^2.$$

□

In the next theorem we consider the vertex partition

$$\mathscr{P}_s = \{\{u_0\}, \{u_1\}, \{u_2\}, \ldots, \{u_{p-1}\}, \{v_0\}, \{v_1\}, \{v_2\}, \ldots, \{v_{p-1}\}\}$$

of $V(G(p,1))$ and investigate the corresponding $\mathscr{P}$-energy.

**Theorem 2.** *Let $G(p,1)$ be the generalized Petersen graph. Then*

$$E_{\mathscr{P}_s}(G(p,1)) = \sum_{j=1}^{p} \left| 2 + 2cos\left(\frac{2\pi j}{p}\right) \right| + \sum_{l=1}^{p} \left| 2cos\left(\frac{2\pi l}{p}\right) \right|. \tag{1}$$

*Proof.* The $\mathscr{P}$-matrix of $G(p,1)$ is of the form $\begin{pmatrix} A & I \\ I & A \end{pmatrix}$, where $A$ is a block matrix of order $p \times p$ such that

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

and $I$ be an identity matrix of order $p \times p$. Therefore by Lemma 1, it is sufficient to find eigenvalues of $(A+I)$ and $(A-I)$. It has been shown in [1] that the eigenvalues of a circulant matrix of order $p$ are

$$\lambda_j = a_1 + a_2 w^j + a_3 w^{2j} + \dots + a_p w^{(n-1)j} \text{ for } 1 \le j \le p \tag{2}$$

where $w$ is a primitive $p^{th}$ root of unity and $(a_1, a_2, a_3, \dots, a_p)$ is the first row of the given circulant matrix. Hence, by Equality (2) we know that the eigenvalues of $(A+I)$ are

$$\begin{aligned} \lambda_j =& 2 + w^j + w^{(p-1)j} \\ =& 2 + cos\left(\frac{2\pi j}{p}\right) + cos\left(\frac{2\pi (p-1)j}{p}\right) + i\left[sin\left(\frac{2\pi j}{p}\right) + sin\left(\frac{2\pi (p-1)j}{p}\right)\right]. \end{aligned}$$

On simplifying, we get

$$\lambda_j = 2 + 2cos\left(\frac{2\pi j}{p}\right); j = 1, 2, \dots, p. \tag{3}$$

Similarly, the eigenvalues of $(A-I)$ are

$$\lambda_l = 2cos\left(\frac{2\pi l}{p}\right); l = 1, 2, \dots, p. \tag{4}$$

Hence by Equations (3), (4) and Lemma 1, the result holds. □

### 2.1. Partition with perfect matching

A perfect matching is a matching in a graph $G$ such that it saturates all its vertices [18]. Now, we consider the perfect matching containing all spokes in $G(p,k)$ as a vertex partition $\mathscr{P}(pm)$ and compute the corresponding $\mathscr{P}$-energy.

**Theorem 3.** *For the generalized Petersen graph $G(p,1)$ with vertex partition $\mathscr{P}(pm) = \bigcup_{i=0}^{p-1}\{\{u_i,v_i\}\}$,*

$$E_{\mathscr{P}(pm)}(G(p,1)) = \sum_{i=1}^{p}\left|4+2cos\left(\frac{2\pi i}{p}\right)\right| + \sum_{j=1}^{p}\left|2cos\left(\frac{2\pi j}{p}\right)\right|. \tag{5}$$

*Proof.* The $\mathscr{P}$-matrix of $G(p,1)$ is $\begin{pmatrix} A & 2I \\ 2I & A \end{pmatrix}$, where $I$ is an identity matrix of order $p\times p$ and $A$ is a block matrix which is also of order $p\times p$

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{pmatrix}.$$

Note that,

$$A+2I = \begin{pmatrix} 4 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 4 \end{pmatrix}$$

and

$$A-2I = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Both these matrices are circulant. Thus by Equation (2), the eigenvalues of $A+2I$ are given by

$$\lambda_i = 4 + w^i + w^{(n-1)i}, \text{ for } 1\le i\le p \tag{6}$$

so that

$$\lambda_i = 4 + 2cos\left(\frac{2\pi i}{p}\right), \text{ for } 1\le i\le p. \tag{7}$$

On the other hand, the matrix $A - 2I$ can be interpreted as the adjacency matrix of a cycle of order $p$. Therefore, its eigenvalues are

$$\lambda_j = 2cos\left(\frac{2\pi j}{p}\right), \text{ for } 1 \leq j \leq p. \tag{8}$$

Thus, by Equations (7), (8) and Lemma 1, we have

$$E_{\mathscr{P}(pm)}(G(p,1)) = \sum_{i=1}^{p}\left|4 + 2cos\left(\frac{2\pi i}{p}\right)\right| + \sum_{j=1}^{p}\left|2cos\left(\frac{2\pi j}{p}\right)\right|.$$

□

It has been verified that for $p \geq 3$ and $k = 1$, there is a beautiful relation between $E_{\mathscr{P}(pm)}(G(p,k))$ and the shear $\mathscr{P}$-energy $E_{\mathscr{P}s}(G(p,k))$. However it has not yet been proved theoretically. Hence we present it as a conjecture.

**Conjecture 4.** For $p \geq 3$ and $k = 1$, $E_{\mathscr{P}(pm)}(G(p,k)) - E_{\mathscr{P}s}(G(p,k)) = 2p$.

Now, we consider another perfect matching in $G(p,k)$ and let $\mathscr{P}(pm) = \left\{\{\bigcup_i\{u_i, u_{i+1}\}, \bigcup_i\{v_i, v_{i+1}\}\} : i \equiv 0(\mod 2), 0 \leq i \leq p-2\right\}$. In the next theorem, we obtain its corresponding $\mathscr{P}$-energy. We omit its proof as it is similar to the proof of Theorem 3.

**Theorem 5.** *For the generalized Petersen graph $G(p,1)$ such that $p$ is even,*

$$E_{\mathscr{P}(pm)}(G(p,1)) = \sum_{t=1}^{p}\left|3 + 3cos\left(\frac{2\pi t}{p}\right) + isin\left(\frac{2\pi t}{p}\right)\right| + \sum_{l=1}^{p}\left|1 + 3cos\left(\frac{2\pi l}{p}\right) + isin\left(\frac{2\pi l}{p}\right)\right|. \tag{9}$$

*where* $\mathscr{P}(pm) = \left\{\{\bigcup_i\{u_i, u_{i+1}\}, \bigcup_i\{v_i, v_{i+1}\}\} : i \equiv 0 \pmod 2, 0 \leq i \leq p-2\right\}$.

## 2.2. Independent Partition

Now we proceed to examine $\mathscr{P}$-energy when the vertex partitions are independent sets. We consider the special case when $G(p,k)$ is bipartite with the bipartition $\mathscr{P}(i) = \{V_1, V_2\}$ where $V_1 = \{u_0, u_2, u_4, \ldots, u_{p-2}, v_1, v_3, v_5, \ldots, v_{p-1}\}$ and $V_2 = \{u_1, u_3, u_5, \ldots, u_{p-1}, v_0, v_2, v_4, \ldots, v_{p-2}\}$ which is in fact an independent partition. It has been proved in [15] that $G(p,k)$ is bipartite if and only if $p$ is even and $k$ is odd. Thus in Theorem 6, we determine the $\mathscr{P}$-energy of $G(p,k)$ with respect to the vertex partition $\mathscr{P}(i)$ for even $p$ and odd $k$.

**Theorem 6.** *If $G(p,k)$ is a generalized Petersen graph such that $p \equiv 0(\mod 2)$ and $k = 1$ and $\mathscr{P}(i)$ is an independent vertex partition with minimum cardinality $\chi(G(p,k))$, then*

$$E_{\mathscr{P}(i)}(G(p,k)) = \sum_{l=1}^{p} \left| p + 6cos\left(\frac{2\pi l}{p}\right) + 2cos\left(\frac{6\pi l}{p}\right) \right| + \sum_{j=1}^{p} \left| p + 2 + 2cos\left(\frac{2\pi j}{p}\right) \right|. \quad (10)$$

*Proof.* The $\mathscr{P}$-matrix of $G(p,1)$ is in the form $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$, where $A$ and $B$ are block matrices of order $p \times p$ such that

$$A = \begin{pmatrix} p & 1 & -1 & 0 & -1 & \dots & 1 \\ 1 & p & 1 & -1 & 0 & \dots & -1 \\ -1 & 1 & p & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & -1 & 0 & -1 & 0 & \dots & p \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & -1 & 0 & -1 & \dots & -1 \\ -1 & 1 & -1 & 0 & \dots & 0 \\ 0 & -1 & 1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & -1 & 0 & \dots & 1 \end{pmatrix}.$$

Now we consider the the matrices $A + B$ and $A - B$ to determine the eigenvalues of the $\mathscr{P}$-matrix $\left(\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right)$.

$$A + B = \begin{pmatrix} p+1 & 0 & -1 & -1 & -1 & \dots & -1 & 0 \\ 0 & p+1 & 0 & -1 & -1 & \dots & -1 & -1 \\ -1 & 0 & p+1 & 0 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & -1 & -1 & -1 & \dots & 0 & p+1 \end{pmatrix}$$

and

$$A - B = \begin{pmatrix} p-1 & 2 & -1 & 1 & -1 & \dots & -1 & 2 \\ 2 & p-1 & 2 & -1 & 1 & \dots & -1 & -1 \\ -1 & 2 & p-1 & 2 & -1 & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & -1 & -1 & -1 & -1 & \dots & 2 & p-1 \end{pmatrix}.$$

As in the preceding cases, both $A + B$ and $A - B$ are circulant, so that by Equation (2) the eigenvalues of $A + B$ are

$$\lambda_j = p + 1 - [w^{2j} + w^{3j} + \cdots + w^{(p-2)j}] \text{ for } 1 \le j \le p.$$

Simplifying this expression we have

$$\lambda_j = p + 2 + 2cos\left(\frac{2\pi j}{p}\right) \text{ for } 1 \leq j \leq p. \tag{11}$$

Similarly, the eigenvalues of $A - B$ are

$$\lambda_l = p - 1 + 2w^l + w^{3l} - [w^{2l} + w^{3l} + \cdots + w^{(p-2)l}] + 2w^{(p-1)l} \text{ for } 1 \leq l \leq p.$$

Therefore,

$$\lambda_l = p + 6cos\left(\frac{2\pi l}{p}\right) + 2cos\left(\frac{6\pi l}{p}\right) \text{ for } 1 \leq l \leq p. \tag{12}$$

Hence by Equations (11), (12) and Lemma 1, we get

$$E_{\mathscr{P}(i)}(G(p,k)) = \sum_{l=1}^{p} \left| p + 6cos\left(\frac{2\pi l}{p}\right) + 2cos\left(\frac{6\pi l}{p}\right) \right| + \sum_{j=1}^{p} \left| p + 2 + 2cos\left(\frac{2\pi j}{p}\right) \right|.$$

□

**Observation 7.** For even values of $p$, it has been verified that $E_{\mathscr{P}(i)}(G(p,1)) = 2p^2 + 4$.

### 2.3. Partitions related with domination

Now, we consider vertex partitions related with the concept of domination. First, we determine the $\mathscr{P}$-energy of $G(p,k)$ with respect to the domatic partition. It can be observed that the $\mathscr{P}$-matrix of $G(p,k)$ corresponding to a domatic partition is positive semi-definite. Hence the value of $E_{\mathscr{P}(d)}(G(p,k))$ can be determined using the approach followed in Theorem 1.

**Theorem 8.** *For the generalized Petersen graph $G(p,k)$ having domatic partition $\mathscr{P}(d)$,*

$$E_{\mathscr{P}(d)}(G(p,k)) = p^2. \tag{13}$$

In the next theorem, we consider total domatic partition of the vertex set of $G(p,k)$ and obtain the corresponding $\mathscr{P}$-energy in the special case when $k = 1$.

**Theorem 9.** *For the generalised Petersen graph $G(p,1)$ with a total domatic partition $\mathscr{P}(td) = \left\{ \bigcup_{i=0}^{p-1} \{u_i\}, \bigcup_{i=0}^{p-1} \{v_i\} \right\}$,*

$$E_{\mathscr{P}(td)}(G(p,1)) = \begin{cases} 14 + \sum_{j=0}^{1}\sum_{s=1}^{p-1}\Bigg| p + (-1)^{s+1} + (-1)^j \\ + 4cos\left(\frac{2s\pi}{p}\right) - 2\sum_{r=2}^{p/2-1} cos\left(\frac{2sr\pi}{p}\right)\Bigg|, & \textit{when } p \textit{ is even} \\ 14 + \sum_{j=0}^{1}\sum_{s=1}^{p-1}\Bigg| p + (-1)^j \\ + 4cos\left(\frac{2s\pi}{p}\right) - 2\sum_{r=2}^{(p-1)/2} cos\left(\frac{2sr\pi}{p}\right)\Bigg|, & \textit{when } p \textit{ is odd.} \end{cases}$$

*Proof.* The $\mathscr{P}$-matrix of $G(p,1)$ is

$$\begin{pmatrix} p & 2 & -1 & \dots & 2 & 1 & 0 & 0 & \dots & 0 \\ 2 & p & 2 & \dots & -1 & 0 & 1 & 0 & \dots & 0 \\ -1 & 2 & p & \dots & -1 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & -1 & -1 & \dots & p & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 & p & 2 & -1 & \dots & 2 \\ 0 & 1 & 0 & \dots & 0 & 2 & p & 2 & \dots & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 & 2 & p & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 2 & -1 & -1 & \dots & p \end{pmatrix}_{n\times n}.$$

It is a block circulant matrix with circulant blocks. Therefore by Theorem 4 in [20], its eigenvalues are given by Equations (14) and (15).
If $p$ is even,

$$\lambda_{j,s} = \begin{cases} 8 & \text{for } j=0, s=0, \\ 6 & \text{for } j=1, s=0, \\ p + (-1)^{s+1} + (-1)^j \\ + 4cos\left(\frac{2s\pi}{p}\right) - 2\sum_{r=2}^{p/2-1} cos\left(\frac{2sr\pi}{p}\right) & \text{for } j=0,1 \text{ and} \\ & 1 \le s \le p-1. \end{cases} \tag{14}$$

If $p$ is odd,

$$\lambda_{j,s} = \begin{cases} 8 & \text{for } j=0, s=0, \\ 6 & \text{for } j=1, s=0, \\ p + (-1)^j + 4cos\left(\frac{2s\pi}{p}\right) \\ - 2\sum_{r=2}^{(p-1)/2} cos\left(\frac{2sr\pi}{p}\right) & \text{for } j=0,1 \text{ and} \\ & 1 \le s \le p-1. \end{cases} \tag{15}$$

Hence, the result follows. □

**Remark 1.** It can be verified through calculation that $E_{\mathscr{P}(td)}(G(p,1)) = 2p^2$.

The $\mathscr{P}$-matrix of $G(p,k)$ for $p > 4, k > 1$ with total domatic partition is a semi-definite matrix. Therefore, it leads to the following theorem.

**Theorem 10.** *For generalized Petersen graph $G(p,k)$ such that $p > 4, k > 1$ with the total domatic partition $\mathscr{P}(td) = \left\{ \bigcup_{i=0}^{p-1}\{u_i\}, \bigcup_{i=0}^{p-1}\{v_i\} \right\}$,*

$$E_{\mathscr{P}(td)}(G(p,k)) = 2p^2. \tag{16}$$

Now, we obtain the $\mathscr{P}$-energy of $G(p,k)$ with doubly domatic partition. We omit the proof of the next theorem as it is similar to that of Theorem 1.

**Theorem 11.** *Let $G(p,k)$ be a generalized Petersen graph where $p$ is even. If $\mathscr{P}(d^2)$ is a doubly domatic partition of $G(p,k)$ such that it is not a triply domatic partition and order of each member of $\mathscr{P}$ is same, then*

$$E_{\mathscr{P}(d^2)}(G(p,k)) = 2p^2. \tag{17}$$

**Observation 12.** Combining all the results of $\mathscr{P}$-energy of $G(p,1)$ corresponding to the various vertex partitions we have considered so far, we obtain the following interesting chain of inequalities.

1. If $p$ is even and $k = 1$, then

$$\begin{aligned} &E_{\mathscr{P}_r}(G(p,k)) > E_{\mathscr{P}(i)}(G(p,k)) = E_{\mathscr{P}(d^3)}(G(p,k)) > E_{\mathscr{P}(td)}(G(p,k)) \\ &= E_{\mathscr{P}(d^2)}(G(p,k)) > E_{\mathscr{P}(d)}(G(p,k)) > E_{\mathscr{P}(pm)}(G(p,k)) > E_{\mathscr{P}_s}(G(p,k)). \end{aligned} \tag{18}$$

2. If $p$ is odd and $k = 1$, then

$$\begin{aligned} &E_{\mathscr{P}_r}(G(p,k)) > E_{\mathscr{P}(d^2)}(G(p,k)) > E_{\mathscr{P}(td)}(G(p,k)) > E_{\mathscr{P}(i)}(G(p,k)) \\ &> E_{\mathscr{P}(d)}(G(p,k)) > E_{\mathscr{P}(pm)}(G(p,k)) > E_{\mathscr{P}_s}(G(p,k)). \end{aligned} \tag{19}$$

## 3. A python program to obtain $E_{\mathscr{P}}(G(p,k))$

In this section, we present a python program to determine the values of $E_{\mathscr{P}}(G(p,k))$ for any value of $p$ and $k$, and four particular cases of partition $\mathscr{P}$ : the trivial partitions $\mathscr{P}_r$ and $\mathscr{P}_s$, independent partition $\mathscr{P}(i)$ and partition $\mathscr{P}(pm)$ with perfect matching containing all spokes in $G(p,k)$.

```
import networkx as nx
import matplotlib.pyplot as plt
import numpy as np
import numpy.linalg
def Graph(n):
G=nx.circular_ladder_graph(n)
return G
def Robust_Partition(n):
G0=Graph(n)
adj_mat = nx.adjacency_matrix(G0)
x=nx.to_numpy_matrix(G0)
mat = []
for i in range (0,2*n):
mat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
mat[i].append(j)
mat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
mat[i][j] = 2*n
else:
if (x.item((i,j)) ==1):
mat[i][j] = 2
else:
mat[i][j] = -1
P_matrix=np.array(mat)
return mat
def Shear_partition(n):
G1=Graph(n)
adj_mat = nx.adjacency_matrix(G1)
x=nx.to_numpy_matrix(G1)
smat = []
for i in range (0,2*n):
smat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
smat[i].append(j)
smat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
smat[i][j] = 1
else:
if (x.item((i,j)) ==1):
smat[i][j] = 1
else:
smat[i][j] = 0
P_s_matrix=np.array(smat)
return smat
def independent_partition(n):
G2=Graph(n)
colors_m=[]
for i in range (0,n) :
if (i %2)==0:
colors_m.append('blue')
else :
colors_m.append('red')
for i in range (n,2*n) :
if (i %2)==0:
colors_m.append('red')
else :
colors_m.append('blue')
adj_mat = nx.adjacency_matrix(G2)
y=nx.to_numpy_matrix(G2)
imat = []
for i in range (0,2*n):
imat.append([])
for i in range (0,2*n):
```

```
for j in range (0,2*n):
imat[i].append(j)
imat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
imat[i][j] = n
elif (y.item((i,j)) ==1):
imat[i][j] = 1
elif colors_m[i]==colors_m[j]:
imat[i][j] = -1
else:
imat[i][j] = 0
P_I_matrix=np.array(imat)
matrix=np.matrix(P_I_matrix)
return matrix
def spokes_partition(n):
G3=Graph(n)
colors_m=[]
for i in range (0,n) :
colors_m.append(i+1)
for i in range (n,2*n) :
colors_m.append(i-(n-1))
adj_mat = nx.adjacency_matrix(G3)
z=nx.to_numpy_matrix(G3)
pmat = []
for i in range (0,2*n):
pmat.append([])
for i in range (0,2*n):
for j in range (0,2*n):
pmat[i].append(j)
pmat[i][j]=0
for i in range (0,2*n):
for j in range (0,2*n):
if (i==j):
pmat[i][j] = 2
elif (z.item((i,j)) ==1) and colors_m[i]==colors_m[j]:
pmat[i][j] = 2
elif (z.item((i,j)) ==1) and colors_m[i]!=colors_m[j]:
pmat[i][j] = 1
else:
pmat[i][j] = 0
P_p_matrix=np.array(pmat)
matrix1=np.matrix(P_p_matrix)
return matrix1
def robust_energy(n):
a=Robust_Partition(n)
spec=np.linalg.eigvals(a)
spec_abs=np.abs(spec)
E=np.sum(spec_abs)
return E
def shear_energy(n):
b=Shear_partition(n)
spec_s=np.linalg.eigvals(b)
spec_abs_s=np.abs(spec_s)
E_s=np.sum(spec_abs_s)
return E_s
def independent_energy(n):
c=independent_partition(n)
spec_i=np.linalg.eigvals(c)
spec_abs_i=np.abs(spec_i)
E_i=np.sum(spec_abs_i)
return E_i
def spokes_energy(n):
d=spokes_partition(n)
spec_p=np.linalg.eigvals(d)
spec_abs_p=np.abs(spec_p)
E_p=np.sum(spec_abs_p)
return E_p
def plot():
```

```
rb_e = []
sh_e = []
in_e = []
sp_e = []
x_axis = []
x_range=int(48)
xmin= int(4)
xmax=int(xmin+2*x_range)
for i in range (0,x_range):
x_axis.append([])
rb_e.append([])
sh_e.append([])
in_e.append([])
sp_e.append([])
for i in range (xmin,xmax,2):
index=int((i-4)/2)
x_axis[index].append(i)
e1 = robust_energy(i)
rb_e[index].append(e1)
e2 = shear_energy(i)
sh_e[index].append(e2)
e3 = independent_energy(i)
in_e[index].append(e3)
e4 = spokes_energy(i)
sp_e[index].append(e4)
t = np.arange(xmin, xmax, 2)
plt.figure(figsize=(12,6))
plt.subplot(221)
sh_e_plot1, = plt.plot(t,sh_e,'r+')
sp_e_plot1, = plt.plot(t,sp_e,'y*')
plt.legend(handles = [sp_e_plot1,sh_e_plot1],labels =['$E_{\mathscr{P}(pm)}(G(p,k))$','$E_{\mathscr{P}_s}(G(p,k))$'])
plt.xlim(xmin,xmax)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.subplot(222)
in_e_plot, = plt.plot(t,in_e,'mo')
rb_e_plot, = plt.plot(t,rb_e,'c^')
plt.legend(handles = [rb_e_plot,in_e_plot],labels =['$E_{\mathscr{P}_r}(G(p,k))$','$E_{\mathscr{P}(i)}(G(p,k))$'])
plt.xlim(xmin,xmax)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.subplot(223)
in_e_plot, = plt.plot(t,in_e,'mo')
rb_e_plot, = plt.plot(t,rb_e,'c^')
sh_e_plot, = plt.plot(t,sh_e,'r+')
sp_e_plot, = plt.plot(t,sp_e,'y*')
plt.legend(handles = [rb_e_plot,in_e_plot,sp_e_plot,sh_e_plot],labels =['$E_{\mathscr{P}_r}(G(p,k))$','$E_{\mathscr{P}(i)}(G(p,k))$','$E_{\mathscr{P}(pm)}(G(p,k))$','$E_{\mathscr{P}_s}(G(p,k))$'])
plt.xlim(4,50)
plt.ylim(4,500)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.subplot(224)
in_e_plot, = plt.plot(t,in_e,'mo')
rb_e_plot, = plt.plot(t,rb_e,'c^')
sh_e_plot, = plt.plot(t,sh_e,'r+')
sp_e_plot, = plt.plot(t,sp_e,'y*')
plt.legend(handles = [rb_e_plot,in_e_plot,sp_e_plot,sh_e_plot],labels =['$E_{\mathscr{P}_r}(G(p,k))$','$E_{\mathscr{P}(i)}(G(p,k))$','$E_{\mathscr{P}(pm)}(G(p,k))$','$E_{\mathscr{P}_s}(G(p,k))$'])
plt.xlim(xmin,xmax)
plt.xlabel('p')
plt.ylabel(' $\mathscr{P}$-energy ')
plt.savefig("Plot5.png")
plt.show()
if __name__ == "__main__":
plot()
```

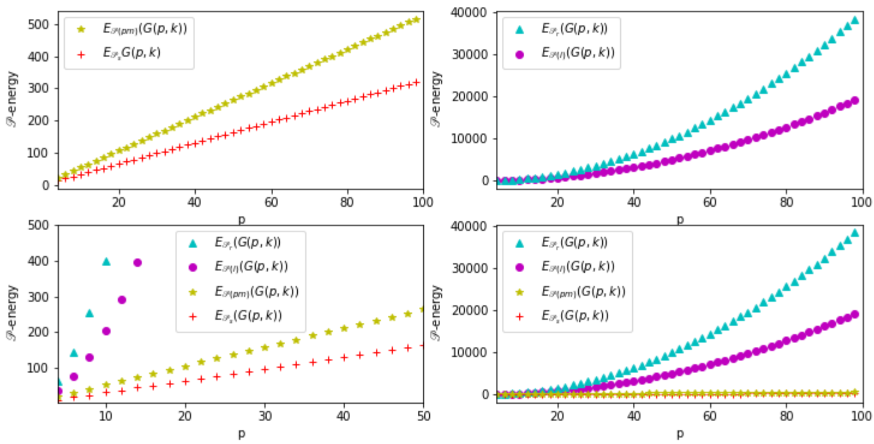

**Figure 1.** **The comparison of $\mathscr{P}$-energy of $G(p,k)$ for different vertex partitions $\mathscr{P}$**

**Remark 2.** Inference from Figure 1, leads us to the formulation of the following chain of inequalities for $G(p,k)$.

$$E_{\mathscr{P}_r}(G(p,k)) > E_{\mathscr{P}(i)}(G(p,k)) > E_{\mathscr{P}(pm)}(G(p,k)) > E_{\mathscr{P}_s}(G(p,k)).$$

## 4. Conclusion

Our exploration of $\mathscr{P}$-energy of $G(p,k)$ for various vertex partitions has revealed many interesting relations among the $\mathscr{P}$-energy corresponding to various partitions. In Observation 12, we have established a chain of inequalities for $G(p,k)$ when $k=1$. It is worth examining these inequalities for other families of graphs.

It has also been found that the domatic partition for $G(p,k)$ where $p \equiv 0(\mod 4)$ gives rise to achromatic coloring. We can view the achromatic number of a graph $G$ $\chi_a(G)$ as the maximum number of independent sets in the partition of $V(G)$ such that between any two different such sets there is at least one edge [6]. In this context, we observe that $\mathscr{P}$-energy for independent partition with maximum cardinality $\chi_a(G(p,k))$ is less than $\mathscr{P}$-energy for independent partition with minimum cardinality $\chi(G(p,k))$ where $\chi(G(p,k))$ is chromatic number of $G(p,k)$. Our study of $\mathscr{P}$-energy of common classes graphs also revealed that, $\mathscr{P}$-energy of these graphs with independent vertex partition is lesser than that of their $\mathscr{P}$-energy with respect to domatic partition. On the other hand, when we consider generalized Petersen graphs, we have $E_{\mathscr{P}(d)}(G(p,k)) < E_{\mathscr{P}(i)}(G(p,k))$. Hence it would be interesting to explore further and characterize those classes of graphs $G$ for which $E_{\mathscr{P}(d)}(G) < E_{\mathscr{P}(i)}(G)$.

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