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Outer independent Roman domination number of trees

Nasrin Dehgardi [1]*, Mustapha Chellali[2]

[1]Department of Mathematics and Computer Science, Sirjan University of Technology
Sirjan, I.R. Iran
n.dehgardi@sirjantech.ac.ir

[2]LAMDA-RO Laboratory, Department of Mathematics, University of Blida, B.P. 270
Blida, Algeria
m_chellali@yahoo.com

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Abstract: A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. An RDF f is called an outer independent Roman dominating function (OIRDF) if the set of vertices assigned a 0 under f is an independent set. The weight of an OIRDF is the sum of its function values over all vertices, and the outer independent Roman domination number $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. In this paper, we show that if T is a tree of order $n \geq 3$ with $s(T)$ support vertices, then $\gamma_{oiR}(T) \leq \min\{\frac{5n}{6}, \frac{3n+s(T)}{4}\}$. Moreover, we characterize the tress attaining each bound.

Keywords: Outer independent Roman dominating function, outer independent Roman domination number, tree.

AMS Subject classification: 05C69

1. Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* $\deg_G(v)$ of a vertex v is the cardinality of its open neighborhood. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex*. A vertex adjacent to two or more leaves is called a *strong*

* *Corresponding Author*

support vertex. The set of leaves adjacent to a support vertex v is denoted by L_v. For $r, s \geq 1$, a *double star* $DS_{r,s}$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. The *distance* $d(u, v)$ between two vertices u and v in a graph G is the length of a shortest (u, v)-path in G, and the *diameter,* $\operatorname{diam}(G)$, of a graph G is the greatest distance between two vertices of G. For a vertex v in a (rooted) tree T, let $C(v)$ and $D(v)$ denote the set of children and descendants of v, respectively and let $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. The *depth* of v denoted by $\operatorname{depth}(v)$ is the largest distance from v to a vertex in $D(v)$.

A *Roman dominating function* (RDF) of graph G is a function f from V to the set $\{0, 1, 2\}$ such that each vertex $v \in V$ with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The *weight* of an RDF f is the value $\omega(f) = \sum_{v\in V} f(v)$. For an RDF f, let $V_i = \{v \in V \mid f(v) = i\}$ for $i = 0, 1, 2$. Since these three sets determine f, we can equivalently write $f = (V_0, V_1, V_2)$. Introduced by Cockayne et al. [9] in 2004, the Roman domination is now well studied, where several variations have been defined. For more on Roman domination, we refer the reader to the book chapters [4, 6] and surveys [5, 7, 8].

An *outer independent Roman dominating function* (OIRDF) on a graph G is a Roman dominating function with the additional property that the set V_0 is independent, that is no two vertices in V_0 are adjacent. The *outer independent Roman domination number* $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF of G. A $\gamma_{oiR}(G)$-function is an OIRDF of G with weight $\gamma_{oiR}(G)$. Outer independent Roman domination was introduced by Abdollahzadeh Ahangar et al. in [1] in 2017, and studied recently in [2, 3, 10–12].

In this paper, we provide two upper bounds on the outer independent Roman domination number of trees in terms of the order and number of support vertices, and we characterize the trees attaining each bound. More precisely, we shall prove:

Theorem 1. If T is a tree of order $n \geq 3$, then

$$\gamma_{oiR}(T) \leq \frac{5n}{6}, \tag{1}$$

with equality if and only if $T \in \mathcal{T}$ (The family $\mathcal{T}$ is defined in Section 2).

Theorem 2. If T is a tree of order $n \geq 3$ with $s(T)$ support vertices, then

$$\gamma_{oiR}(T) \leq \frac{3n + s(T)}{4}, \tag{2}$$

with equality if and only if $T \in \mathcal{F}$ (The family $\mathcal{F}$ is defined in Section 3).

2. Proof of Theorem 1

In this section, we prove Theorem 1. For the purpose of characterizing the trees attaining the upper bound in Theorem 1, we introduce the family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let T_1 be a P_6, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by operation $\mathcal{O}$ defined below.

Operation $\mathcal{O}$: If v is a vertex of T_i which is neither a support nor a leaf, then $\mathcal{O}$ adds a path P_6 by joining v to the third vertex of P_6.

From the way in which a tree $T \in \mathcal{T}$ is constructed we make the following observation.

Observation 3. Let T be a tree of $\mathcal{T}$. Then

(i) every support vertex has degree two.

(ii) every vertex which is neither a support vertex nor a leaf is adjacent to a support vertex.

Lemma 1. If T_i is a tree with $\gamma_{oiR}(T_i) = \frac{5n(T_i)}{6}$ and T_{i+1} is obtained from T_i by Operation $\mathcal{O}$, then $\gamma_{oiR}(T_{i+1}) = \frac{5n(T_{i+1})}{6}$.

Proof. Let us denote in order by $u_1, u_2, u_3, u_4, u_5, u_6$ the vertices of the added path P_6 attached at $v \in V(T_i)$ by u_3v. If f is a $\gamma_{oiR}(T_i)$-function, then f can be extended to an OIRDF of T_{i+1} by assigning a 2 to u_3, u_5, a 1 to u_1 and a 0 to u_4 and u_6. Hence $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 5$.
Now let f be a $\gamma_{oiR}(T_{i+1})$-function. Clearly, $\sum_{i=1}^{6} f(u_i) \geq 5$. If $f(u_3) \neq 2$ or $f(v) \neq 0$, then the function of f, restricted to T_i is an OIRDF of T_i, and we deduce from the assumption that $\gamma_{oiR}(T_{i+1}) - 5 \geq \omega(f|_{T_i}) \geq \gamma_{oiR}(T_i)$. Now let $f(u_3) = 2$ and $f(v) = 0$. By Observation 3, let w be the support vertex adjacent to v in T_i and w' the leaf neighbor of w. Clearly $f(w) \neq 0$, since $f(v) = 0$, and $f(w) + f(w') \geq 2$. Therefore, we can assume that $f(w) = 2$ and $f(w') = 0$, and thus as previously, $f|_{T_i}$ is an OIRDF of T_i yielding $\gamma_{oiR}(T_{i+1}) - 5 \geq \gamma_{oiR}(T_i)$. Hence $\gamma_{oiR}(T_{i+1}) = \gamma_{oiR}(T_i) + 5$, and thus $\gamma_{oiR}(T_{i+1}) = \frac{5n(T_{i+1})}{6}$. □

Proposition 1. If $T \in \mathcal{T}$, then $\gamma_{oiR}(T) = \frac{5n}{6}$.

Proof. Let $T \in \mathcal{T}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 = P_6$, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by Operation $\mathcal{O}$ for $i \in \{1, 2, \ldots, k-1\}$. We use an induction on the number of operations applied to construct T. If $k = 1$, then $T = P_6$ and clearly $\gamma_{oiR}(P_6) = 5$. Let $k \geq 2$, and assume the property is true for all trees of $\mathcal{T}$ constructed with $k-1 \geq 0$ operations. Let $T = T_k$ and $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{oiR}(T') = \frac{5n(T')}{6}$. Since T is obtained from T' by operation $\mathcal{O}$, we conclude from Lemma 1 that $\gamma_{oiR}(T) = \frac{5n(T)}{6}$.
□

Proposition 2. If T is a tree of order $n \geq 3$, then $\gamma_{oiR}(T) \leq \frac{5n}{6}$, with equality only if $T \in \mathcal{T}$.

Proof. The proof is by induction on n. If $n \in \{3, 4, 5\}$, then T is either a star, a double star or a path P_5. Clearly, for stars $\gamma_{oiR}(T) = 2$ and for double stars $\gamma_{oiR}(T) = 4$, while for paths P_5 we have $\gamma_{oiR}(P_5) = 4$. In all cases, $\gamma_{oiR}(T) < \frac{5n}{6}$. Let $n = 6$ and $\mathrm{diam}(T) \geq 4$. If $\mathrm{diam}(T) = 4$, then T is obtained from a path P_5 by adding a new vertex adjacent to either a support vertex or the center vertex of P_5. In any case, $\gamma_{oiR}(T) = 4 < \frac{5n}{6}$. If $\mathrm{diam}(T) = 5$, then $T = P_6$, where $\gamma_{oiR}(T) = 5 = \frac{5n}{6}$ and $P_6 \in \mathcal{T}$. Thus let $n \geq 7$, and assume that every tree T' of order $3 \leq n' < n$ satisfies $\gamma_{oiR}(T') \leq \frac{5n'}{6}$ with equality only if $T' \in \mathcal{T}$. Let T be a tree of order n and diameter at least four.

Let $v_1 v_2 \ldots v_d$, with $d \geq 5$, be a diametral path in T such that $\deg_T(v_2)$ is as large as possible. Root T at v_d.

We first assume that $\deg_T(v_2) \geq 3$. Let $T' = T - T_{v_2}$. Since any $\gamma_{oiR}(T)$-function can be extended to an OIRDF of T by assigning a 2 to v_2 and a 0 to its leaves, $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2$. By the induction hypothesis, we have

$$\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2 \leq \frac{5(n - \deg_T(v_2))}{6} + 2 < \frac{5n}{6}.$$

Hence let $\deg_T(v_2) = 2$. By the choice of the diametral path, we may assume that any child of v_3 with depth 1 is of degree 2.

Assume now that $\deg_T(v_3) \geq 3$, and let v_3 have s children with depth 1 and r children with depth 0. Let $T' = T - T_{v_3}$, and let g be a $\gamma_{oiR}(T')$-function. Define a function h on $V(T)$ by $h(v_3) = 2$, $h(x) = 1$ if x is a leaf in T_{v_3} not adjacent to v_3, $h(x) = g(x)$ for $x \in V(T')$ and $h(x) = 0$ otherwise. Obviously, h is an OIRDF of T yielding $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + s + 2$. Since $r + s \geq 2$, we deduce from the induction hypothesis that

$$\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + s + 2 \leq \frac{5(n - 2s - r - 1)}{6} + s + 2 \leq \frac{5n}{6} - \frac{4s + 5r - 7}{6} < \frac{5n}{6}.$$

In the sequel, we can assume that $\deg_T(v_3) = 2$. Considering the above argument, we can assume that any child of v_4 with depth 2, is of degree 2. Let v_4 have k children with depth 2, say $v_3 = y_1, \ldots, y_k$, s children with depth 1, say $z_1, \ldots, z_s$ and r leaves, say $x_1, \ldots, x_r$, where $s, r \geq 0$. For the sake of simplicity, any pendant path of length three containing v_4 will be denoted by $v_4 y_j y'_j y''_j$ for each $j \in \{1, ..., k\}$. Also, if $\deg_T(z_i) \geq 3$ for some i, then using a similar proof as in above when $\deg_T(v_2) \geq 3$, we can see that $\gamma_{oiR}(T) < \frac{5n}{6}$. Henceforth, we assume that $\deg_T(z_i) = 2$ for each i, if any, and z'_i its unique leaf neighbor. Consider the following cases.

Case 1. $s + r = 0$.

Let $T' = T - T_{v_4}$. Note that since $n \geq 7$, T' has order $n' \geq 3$. Clearly, if f is a

$\gamma_{oiR}(T')$-function, then the function h defined by $h(v_4) = 1$, $h(y'_j) = 2$ for $1 \le j \le k$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 2k + 1$. We conclude from the induction hypothesis that

$$\begin{array}{rcl} \gamma_{oiR}(T) & \le & \gamma_{oiR}(T') + 2k + 1 \\ & \le & \frac{5(n-3k-1)}{6} + 2k + 1 \\ & = & \frac{5n}{6} - \frac{3k-1}{6} < \frac{5n}{6}. \end{array}$$

Case 2. $s + r \ge 1$.
Let $T' = T - T_{v_4}$. Clearly, $n' \ge 3$. If f is a $\gamma_{oiR}(T')$-function, then the function h defined by $h(v_4) = h(y'_j) = 2$ for every $j \in \{1, ..., k\}$, $h(z'_i) = 1$ for every $i \in \{1, ..., s\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T. Therefore, $\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 2k + s + 2$. We conclude from the induction hypothesis that

$$\begin{array}{rcl} \gamma_{oiR}(T) & \le & \gamma_{oiR}(T') + 2k + s + 2 \\ & \le & \frac{5(n-3k-2s-r-1)}{6} + 2k + s + 2 \\ & = & \frac{5n}{6} - \frac{3k+4s+5r-7}{6} \\ & \le & \frac{5n}{6} \end{array}.$$

Further, if $\gamma_{oiR}(T) = \frac{5n}{6}$, then we have equality throughout this inequality chain. In particular, $k = s = 1$, $r = 0$, and $\gamma_{oiR}(T') = \frac{5n'}{6}$. By the induction hypothesis on T', we obtain that $T' \in \mathcal{T}$. Next we shall show that v_5 is neither a support vertex nor a leaf in T'. Assume first that v_5 is a support vertex, and let w be a leaf neighbor of v_5. Let T'' be the tree obtained from T by removing vertices w, v_1, v_2, z_1, z'_1. Clearly, there is $\gamma_{oiR}(T'')$-function f such that $\{f(v_3), f(v_4), f(v_5)\} \cap \{2\} \ne \emptyset$ and thus accordingly $\gamma_{oiR}(T) \le \gamma_{oiR}(T'') + 4$. By the induction hypothesis, we deduce that

$$\gamma_{oiR}(T) \le \gamma_{oiR}(T'') + 4 \le \frac{5(n-5)}{6} + 4 < \frac{5n}{6},$$

a contradiction. Hence v_5 is not a support vertex. Now, assume that v_5 is a leaf of T'. Since $T' \in \mathcal{T}$, we must have $\deg_{T'}(v_6) = 2$ (by Observation 3). Let $T'' = T - T_{v_6}$. Clearly, if f is a $\gamma_{oiR}(T'')$-function, then the function h defined by $h(v_4) = h(v_2) = 2$, $h(v_6) = h(z'_1) = 1$, $h(u) = f(u)$ for $u \in V(T'')$ and $h(x) = 0$ otherwise, is an OIRDF of T. Therefore, $\gamma_{oiR}(T) \le \gamma_{oiR}(T'') + 6$, and by the induction hypothesis we obtain

$$\gamma_{oiR}(T) \le \gamma_{oiR}(T'') + 6 \le \frac{5(n-8)}{6} + 6 < \frac{5n}{6},$$

a contradiction. Thus v_5 is not a leaf too. Since T is obtained from T' by Operation $\mathcal{O}$, we conclude that $T \in \mathcal{T}$. This complete the proof. □

According to Propositions 1 and 2, we have proven Theorem 1.

3. Proof of Theorem 2

In this section, we prove Theorem 2. For the purpose of characterizing the trees attaining the upper bound in Theorem 2, we define the set $W_T = \{v \in V(T) \mid$ there is no $\gamma_{oiR}(T)$-function f such that $f(w) = 1$ for every $w \in N[v]\}$, and we introduced the family $\mathcal{F}$ of trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ of trees such that $T_1 = P_6$, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations for $i \in \{1, \ldots, k-1\}$. Let $s(T)$ denote the number of support vertices of a tree T.

Operation $\mathcal{O}_1$: If $v \in W_{T_i}$, then $\mathcal{O}_1$ adds a path P_6 by joining v to the third vertex of P_6.

Operation $\mathcal{O}_2$: If v is a leaf of T_i, then $\mathcal{O}_2$ adds a path P_4 by joining one of its leaves to v.

From the way in which a tree $T \in \mathcal{F}$ is constructed, every support vertex of T is adjacent to exactly one leaf.

Lemma 2. If T_i is a tree with $4\gamma_{oiR}(T_i) = 3n(T_i) + s(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_1$, then $4\gamma_{oiR}(T_{i+1}) = 3n(T_{i+1}) + s(T_{i+1})$.

Proof. To prove that $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 5$, let f be a $\gamma_{oiR}(T_i)$-function and define $g : V(T_{i+1}) \to \{0,1,2\}$ by $g(u_1) = 1, g(u_3) = g(u_5) = 2$, $g(u_2) = g(u_4) = g(u_6) = 0$, and $g(y) = f(y)$ otherwise. Obviously, g is an OIRDF of T_{i+1} and so $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 5$.
Now let f be a $\gamma_{oiR}(T_{i+1})$-function. Clearly, we have $\sum_{i=1}^{6} f(u_i) \geq 5$. If $f(u_3) \neq 2$ or $f(v) \neq 0$, then the function of f, restricted to T_i is an OIRDF of T_i and we deduce from the assumption that $\gamma_{oiR}(T_{i+1}) - 5 \geq \omega(f|_{T_i}) \geq \gamma_{oiR}(T_i)$. Now let $f(u_3) = 2$ and $f(v) = 0$. If there is $w \in N(v) - \{u_3\}$ such that $f(w) = 2$, then the function of f, restricted to T_i is an OIRDF of T_i and we deduce from the assumption that $\gamma_{oiR}(T_{i+1}) - 5 \geq \omega(f|_{T_i}) \geq \gamma_{oiR}(T_i)$. Hence let $f(w) = 1$ for every $w \in N(v) - \{u_3\}$. Then the function $g : V(T_i) \to \{0,1,2\}$ defined by $g(v) = 1$ and $g(x) = f(x)$ for $x \in V(T_i) - \{u\}$, is an OIRDF of T_i of weight at most $\omega(f) - 4$. Since $v \in W_{T_i}$, we deduce that $\omega(g) \geq \gamma_{oiR}(T_i) + 1$ yielding $\gamma_{oiR}(T_{i+1}) \geq \gamma_{oiR}(T_i) + 5$. It follows that $\gamma_{oiR}(T_{i+1}) = \gamma_{oiR}(T_i) + 5$. Now the result follows from the assumption $\gamma_{oiR}(T_i) = \frac{3n(T_i)+s(T_i)}{4}$. □

Lemma 3. If T_i is a tree with $4\gamma_{oiR}(T_i) = 3n(T_i) + s(T_i)$ and T_{i+1} is a tree obtained from T_i by Operation $\mathcal{O}_2$, then $4\gamma_{oiR}(T_{i+1}) = 3n(T_{i+1}) + s(T_{i+1})$.

Proof. Let $P_4 : u_1u_2u_3u_4$ be the added path attached at a leaf $v \in V(T_i)$ by u_1v. Let f be a $\gamma_{oiR}(T_i)$-function and define the function g on $V(T_{i+1})$ by $g(u_1) = 1, g(u_3) = 2$, $g(u_2) = g(u_4) = 0$, and $g(y) = f(y)$ otherwise. Obviously, g is an OIRDF of T_{i+1} and so $\gamma_{oiR}(T_{i+1}) \leq \gamma_{oiR}(T_i) + 3$.

Now let f be a $\gamma_{oiR}(T_{i+1})$-function. Clearly, $\sum_{i=1}^{4} f(u_i) \geq 3$. Now, if $f(u_1) \neq 2$ or $f(v) \neq 0$, then the function of f, restricted to T_i is an OIRDF of T_i and thus $\gamma_{oiR}(T_{i+1}) - 3 \geq \omega(f|_{T_i}) \geq \gamma_{oiR}(T_i)$. Hence, assume that $f(u_1) = 2$ and $f(v) = 0$. It follows that $f(u_3) = 2$ and $f(u_2) = f(u_4) = 0$. In this case, the function g defined on $V(T_i)$ by $g(v) = 1$ and $h(u) = f(u)$ for $u \in V(T_i) - \{v\}$, is an OIRDF of T_i implying that $\gamma_{oiR}(T_i) \leq \gamma_{oiR}(T_{i+1}) - 3$. In any case, $\gamma_{oiR}(T_{i+1}) = \gamma_{oiR}(T_i) + 3$. The result easily follows from the facts that $n(T_{i+1}) = n(T_i) + 4$, $s(T_{i+1}) = s(T_i)$ and $4\gamma_{oiR}(T_i) = 3n(T_i) + s(T_i)$. □

Proposition 3. If $T \in \mathcal{F}$, then $\gamma_{oiR}(T) = \frac{3n+s(T)}{4}$.

Proof. Let $T \in \mathcal{F}$. Then there exists a sequence of trees $T_1, T_2, \ldots, T_k$ $(k \geq 1)$ such that $T_1 = P_6$, and if $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the aforementioned operations for $i \in \{1, 2, \ldots, k-1\}$. We use an induction on the number of operations applied to construct T. If $k = 1$, then $T = P_6$ and clearly $\gamma_{oiR}(P_6) = \frac{3n+s(T)}{4} = 5$. Let $k \geq 2$, and assume the property is true for all trees of $\mathcal{F}$ constructed with $k-1 \geq 0$ operations. Let $T = T_k$ and $T' = T_{k-1}$. By the induction hypothesis, we have $\gamma_{oiR}(T') = \frac{3n'+s(T')}{4}$. Since $T = T_k$ is obtained from T' by Operations $\mathcal{O}_1$ or $\mathcal{O}_2$, we deduce from Lemmas 2, 3 that $\gamma_{oiR}(T) = \frac{3n+s(T)}{4}$. □

Proposition 4. If T is a tree of order $n \geq 3$ with s(T) support vertices, then $\gamma_{oiR}(T) \leq \frac{3n+s(T)}{4}$, with equality only if $T \in \mathcal{F}$.

Proof. We use an induction on n. Let $n \in \{3, 4, 5\}$. If $\operatorname{diam}(T) = 2$, then T is a star and thus $\gamma_{oiR}(T) = 2 < \frac{3n+s(T)}{4}$. If $\operatorname{diam}(T) = 3$, then T is a double star $DS_{r,s}$, where $\min\{r, s\} = 1$ and thus $\gamma_{oiR}(T) = 3 < \frac{3n+s(T)}{4}$. If $\operatorname{diam}(T) = 4$, then $T = P_5$ and clearly $\gamma_{oiR}(T) = 4 < \frac{3n+s(T)}{4}$. Now, let $n = 6$. As above, if $\operatorname{diam}(T) \in \{2, 3, 4\}$, then one can see that $\gamma_{oiR}(T) < \frac{3n+s(T)}{4}$. If $\operatorname{diam}(T) = 5$, then $T = P_6$ and clearly $\gamma_{oiR}(T) = 4 = \frac{3n+s(T)}{4}$. Thus let $n \geq 7$, and assume that every tree T' of order $3 \leq n' < n$ with $s(T')$ support vertices satisfies $\gamma_{oiR}(T') \leq \frac{3n'+s(T')}{4}$ with equality only if $T' \in \mathcal{F}$. Let T be a tree of order n with $s(T)$ support vertices. Since stars and double stars satisfies the result, we may assume that $\operatorname{diam}(T) \geq 4$.
Let $v_1 v_2 \ldots v_d$, with $d \geq 5$, be a diametral path in T such that $\deg(v_2)$ is as large as possible. Root T at v_d, and consider the following situations.

Case 1. $\deg_T(v_2) \geq 3$.
Assume first that $\deg_T(v_3) \geq 3$, and let $T' = T - T_{v_2}$. Clearly, $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2$

and $s(T) = s(T') + 1$. By the induction hypothesis, we obtain

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 2\\ &\leq \frac{3(n - \deg_T(v_2)) + s(T) - 1}{4} + 2\\ &\leq \frac{3n + s(T)}{4} - \frac{3\deg_T(v_2) - 7}{4}\\ &< \frac{3n + s(T)}{4}.\end{aligned}$$

Now, assume that $\deg_T(v_3) = 2$ and $\deg_T(v_4) \geq 3$. Let $T' = T - T_{v_3}$. Then $s(T) = s(T') + 1$ and any $\gamma_{oiR}(T')$-function can be extended to an OIRDF of T by assigning a 1 to v_3, a 2 to v_2 and a 0 to leaves in L_{v_2}. Hence $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 3\\ &\leq \frac{3(n - \deg_T(v_2) - 1) + s(T) - 1}{4} + 3\\ &\leq \frac{3n + s(T)}{4} - \frac{3\deg_T(v_2) - 8}{4}\\ &< \frac{3n + s(T)}{4}.\end{aligned}$$

Finally, assume that $\deg_T(v_3) = \deg_T(v_4) = 2$, and let $T' = T - T_{v_4}$. Note that if $n' = 1$, then $\gamma_{oiR}(T) = 4 < \frac{3n+s(T)}{4}$, and if $n' = 2$, then $\gamma_{oiR}(T) = 5 < \frac{3n+s(T)}{4}$. Thus we can assume that $n' \geq 3$. Since any $\gamma_{oiR}(T')$-function can be extended to an OIRDF of T by assigning a 1 to v_4, a 2 to v_2 and a 0 to any vertex in $L_{v_2} \cup \{v_3\}$, we obtain that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 3\\ &\leq \frac{3(n - \deg_T(v_2) - 2) + s(T)}{4} + 3\\ &\leq \frac{3n + s(T)}{4} - \frac{3\deg_T(v_2) - 6}{4}\\ &< \frac{3n + s(T)}{4}.\end{aligned}$$

Case 2. $\deg_T(v_2) = 2$.
By the choice of the diametral path, we may assume that any child of v_3 with depth 1 is of degree 2. Consider the following subcases.

Subcase 2.1. $\deg_T(v_3) \geq 3$.
Let v_3 have r children with depth 1, say $z_1 = v_2, \ldots, z_r$ and ℓ children with depth 0, say $x_1, \ldots, x_\ell$. Let z'_i denote the leaf neighbor of z_i for $1 \leq i \leq r$.

Assume first that $\ell \geq 1$, and let $T' = T - T_{v_3}$. Note that $n' \geq 3$, since $n \geq 7$. If f is a $\gamma_{oiR}(T')$-function, then the function h defined by $h(v_3) = 2$, $h(z'_i) = 1$ for every $i \in \{1, \ldots, r\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T. It follows that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + r + 2$, and by the induction hypothesis we obtain

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + r + 2\\ &\leq \frac{3(n - 2r - \ell - 1) + s(T) - r}{4} + r + 2\\ &= \frac{3n + s(T)}{4} - \frac{3r + 3\ell - 5}{4}\\ &< \frac{3n + s(T)}{4}.\end{aligned}$$

Assume now that $\ell = 0$. Since $\deg_T(v_3) \geq 3$, we have $r \geq 2$. If $\deg_T(v_4) \geq 3$ or $r \geq 3$, then let $T' = T - T_{v_3}$. Note that if $n' = 2$, then T is a subdivided star centered at v_3, where $\gamma_{oiR}(T) = \deg_T(v_3) + 2 < \frac{3n+s(T)}{4}$. Hence we can assume that $n' \geq 3$. Clearly, if f is a $\gamma_{oiR}(T')$-function, then the function h defined by $h(v_3) = 2$, $h(z'_i) = 1$ for every $i \in \{1, \ldots, r\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + r + 2$. Let $j = 0$ if $\deg_T(v_4) \geq 3$ and $j = 1$ $\deg_T(v_4) = 2$. Note that if $j = 1$, then $r \geq 3$. Also, $S(T') \leq S(T) - r + j$. Now, by the induction hypothesis we have,

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + r + 2\\ &\leq \frac{3(n - 2r - 1) + s(T) - r + j}{4} + r + 2\\ &= \frac{3n + s(T)}{4} - \frac{3r - 5 - j}{4}\\ &< \frac{3n + s(T)}{4}.\end{aligned}$$

Now, assume that $\deg_T(v_4) = 2$ and $r = 2$. Let $T' = T - \{v_1, v_2, z'_2\}$, and let f be a $\gamma_{oiR}(T')$-function such that $f(v_3)$ is maximum. If $f(v_3) + f(z_2) = 1$, then we must have $f(v_3) = 0$, $f(z_2) = 1$ and thus $f(v_4) = 2$. Now, if $f(v_5) \neq 0$, then assigning z_2, v_3 and v_4, the values $0, 2$ and 0 instead of $1, 0, 2$ respectively, provides a $\gamma_{oiR}(T')$-function g with $g(v_3) > f(v_3)$, contradicting our choice of f. Thus $f(v_5) = 0$. Then assigning z_2, v_3, v_4 and v_5, the values $0, 2, 0$ and 1 instead of $1, 0, 2, 0$ respectively, provides a $\gamma_{oiR}(T')$-function g' with $g'(v_3) > f(v_3)$, a contradiction too. Therefore $f(v_3) + f(z_2) = 2$, and thus it is easy to see that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2$. By the induction hypothesis we obtain,

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 2 \\ &\leq \frac{3(n-3)+s(T)-1}{4} + 2 \\ &= \frac{3n+s(T)}{4} - \frac{2}{4} \\ &< \frac{3n+s(T)}{4}.\end{aligned}$$

Subcase 2.2. $\deg_T(v_3) = 2$ and $\deg_T(v_4) \geq 3$.
According to Subcase 1 and the choice of the diametral path, we may assume that any child of v_4 with depth 2 is of degree 2. Let v_4 have k children with depth 2, say $y_1 = v_3, \ldots, y_k$, r children with depth 1, say $z_1, \ldots, z_r$ and ℓ children with depth 0, say $x_1, \ldots, x_\ell$. For any y_j, let y'_j the child of y_j and y''_j the child of y'_j. Now, if $\deg(z_i) \geq 3$ for some i, then let T' be the tree obtained from T by removing z_i and its leaves. One can easily see that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2$. Using the induction and the fact that $s(T') = s(T) - 1$, we obtain $\gamma_{oiR}(T) < \frac{3n+s(T)}{4}$. Henceforth, we assume that $\deg(z_i) = 2$ for each i, if any. Also, for any z_i, let z'_i be the unique leaf neighbor of z_i. Consider the following situations.
Assume first that $\ell \geq 1$. If $\deg_T(v_5) \geq 3$, then let $T' = T - T_{v_4}$. Clearly, if f is a $\gamma_{oiR}(T')$-function, then the function h defined on $V(T)$ by $h(v_4) = h(y'_j) = 2$ for every $j \in \{1, \ldots, k\}$, $h(z'_i) = 1$ for every $i \in \{1, \ldots, r\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T, implying that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2k + r + 2$. We conclude from the induction hypothesis that

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 2k + r + 2 \\ &\leq \frac{3(n-3k-2r-\ell-1)+s(T)-k-r-1}{4} + 2k + r + 2 \\ &= \frac{3n+s(T)}{4} - \frac{2k+3r+3\ell-4}{6} \\ &< \frac{3n+s(T)}{4}.\end{aligned}$$

Now assume that $\deg(v_5) = 2$. Let $T' = T - T_{v_3}$, and let f be a $\gamma_{oiR}(T')$-function f such that $f(v_4)$ is maximum. Assume that $f(v_4) = 0$. Then $\ell = 1$ for otherwise ($\ell \geq 2$) we reassign v_4 by 2 and its leaves by 0 we obtain a $\gamma_{oiR}(T')$-function g with $g(v_4) > f(v_4)$, a contradiction. In this the unique leaf neighbor x_1 must be assigned a 1. Now, if $r \neq 0$, then $f(z_i) = 2$ and $f(z'_i) = 2$ for every i, but then as above we can reassign x_1, v_4, z_i and z'_i by $0, 2, 0, 1$ respectively and we get a $\gamma_{oiR}(T')$-function g with $g(v_4) > f(v_4)$, a contradiction. Thus $r = 0$. It follows that $f(v_5) = 2$ and $f(v_6) = 0$. But then reassigning x_1, v_4, v_5 and v_6 by $0, 2, 0, 1$ respectively provides a $\gamma_{oiR}(T')$-function g with $g(v_4) > f(v_4)$, a contradiction. Therefore $f(v_4) \neq 0$. Then

f can be extended to an OIRDF of T by assigning a 0 to v_3 and v_1 and a 2 to v_2. Hence $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2$. By the induction hypothesis we have,

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 2\\ &\leq \frac{3(n-3)+s(T)-1}{4} + 2\\ &\leq \frac{3n+s(T)}{4} - \frac{2}{4}\\ &< \frac{3n+s(T)}{4}.\end{aligned}$$

In the sequel, we assume that $\ell = 0$. Assume now that $r + k \geq 3$, and let $T' = T - T_{v_4}$. First let $r \geq 1$. If $n' = 2$, then $\gamma_{oiR}(T) = 2k + r + 3 < \frac{3n+s(T)}{4}$. Thus let $n' \geq 3$. Clearly, if f is a $\gamma_{oiR}(T')$-function, then the function h defined on $V(T)$ by $h(v_4) = h(y'_j) = 2$ for every $j \in \{1, \ldots, k\}$, $h(z'_i) = 1$ for every $i \in \{1, \ldots, r\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2k + r + 2$. We conclude from the induction hypothesis that

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 2k + r + 2\\ &\leq \frac{3(n-3k-2r-1)+s(T)-k-r+1}{4} + 2k + r + 2\\ &= \frac{3n+s(T)}{4} - \frac{2k+3r-6}{6}\\ &< \frac{3n+s(T)}{4}.\end{aligned}$$

Now let $r = 0$. If f is a $\gamma_{oiR}(T')$-function, then the function h defined on $V(T)$ by $h(v_4) = 1$, $h(y'_j) = 2$ for every $j \in \{1, \ldots, k\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 2k + 1$. We conclude from the induction hypothesis that

$$\begin{aligned}\gamma_{oiR}(T) &\leq \gamma_{oiR}(T') + 2k + 1\\ &\leq \frac{3(n-3k-1)+s(T)-k+1}{4} + 2k + 1\\ &= \frac{3n+s(T)}{4} - \frac{2k-2}{6}\\ &< \frac{3n+s(T)}{4}.\end{aligned}$$

Finally, assume that $r + k = 2$. First let $r = 0$, $k = 2$ and $T' = T - T_{v_4}$. If f is a $\gamma_{oiR}(T')$-function, then the function h defined on $V(T)$ by $h(v_4) = 1$, $h(y'_j) = 2$ for $j \in \{1, 2\}$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T

implying that $\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 5$. By the induction hypothesis we obtain

$$\begin{aligned}\gamma_{oiR}(T) &\le \gamma_{oiR}(T') + 5 \le \frac{3(n-7)+s(T)-1}{4} + 5 \\ &= \frac{3n+s(T)}{4} - \frac{2}{4} \\ &< \frac{3n+s(T)}{4}.\end{aligned}$$

Now, let $r = k = 1$. Suppose that $\deg_T(v_5) = 2$ and let $T' = T - \{v_1, v_2, z_1, z_1'\}$. Let f be a $\gamma_{oiR}(T')$-function such that $f(v_4)$ is maximum. Clearly, if $f(v_4) = 1$, then $f(v_3) = 1$ and so we can reassign v_3 and v_4 by 0 and 2, respectively which provides a $\gamma_{oiR}(T')$-function g with $g(v_4) > f(v_4)$, a contradiction. Now, if $f(v_4) = 0$, then we must have $f(v_3) = 1$, $f(v_5) = 2$ and $f(v_6) = 0$. But then assigning v_3, v_4, v_5 and v_6 the values $0, 2, 0$ and 1 provides a $\gamma_{oiR}(T')$-function g with $g(v_4) > f(v_4)$, a contradiction. Hence $f(v_4) = 2$, and thus $f(v_3) = 0$. Then f can be extended to an OIRDF function on T by assigning a 0 to v_1, z_1, a 2 to v_2 and a 1 to z_1'. By the inductive hypothesis, we obtain

$$\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 3 \le \frac{3(n-4)+s(T)-1}{4} + 3 < \frac{3n+s(T)}{4}.$$

For the next, we can assume that $\deg_T(v_5) \ge 3$. Let $T' = T - T_{v_4}$. Clearly, if f is a $\gamma_{oiR}(T')$-function, then the function h defined by $h(v_4) = h(v_2) = 2$, $h(z_1') = 1$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 5$. We conclude from the induction hypothesis that

$$\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 5 \le \frac{3(n-6)+s(T)-2}{4} + 5 = \frac{3n+s(T)}{4}.$$

Further if $\gamma_{oiR}(T) = \frac{3n+s(T)}{4}$, then we have equality throughout this inequality chain. In particular, $\gamma_{oiR}(T') = \frac{3n'+s(T')}{4}$ and thus $T' \in \mathcal{F}$. Since $\deg_T(v_5) \ge 3$, v_5 is not a leaf of T'. Next we shall show $v_5 \in W_{T'}$. Suppose to the contrary that $v_5 \notin W_{T'}$. Then there is $\gamma_{oiR}(T')$-function f such that $f(w) = 1$ for every $w \in N[v_5]$. Then the function h defined by $h(v_4) = h(v_2) = 2$, $h(v_5) = 0$, $h(z_1') = 1$, $h(u) = f(u)$ for $u \in V(T') - \{v_5\}$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 4$. We conclude from the induction hypothesis that

$$\gamma_{oiR}(T) \le \gamma_{oiR}(T') + 4 \le \frac{3(n-6)+s(T)-2}{4} + 4 < \frac{3n+s(T)}{4},$$

which is a contradiction. Thus $v_5 \in W_{T'}$. Now $T \in \mathcal{F}$ because it can be obtained from T' by Operation $\mathcal{O}_1$.

Subcase 2.3. $\deg_T(v_3) = 2$ and $\deg_T(v_4) = 2$.

Let $T' = T - T_{v_4}$. If f is a $\gamma_{oiR}(T')$-function, then the function h defined on $V(T)$ by $h(v_4) = 1$, $h(v_2) = 2$, $h(u) = f(u)$ for $u \in V(T')$ and $h(x) = 0$ otherwise, is an OIRDF of T implying that $\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3$. If $\deg_T(v_5) \geq 3$, then by the induction hypothesis we can see that

$$\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3 \leq \frac{3(n-4) + s(T) - 1}{4} + 3 < \frac{3n + s(T)}{4}.$$

If $\deg_T(v_5) = 2$, then by the induction hypothesis we have

$$\gamma_{oiR}(T) \leq \gamma_{oiR}(T') + 3 \leq \frac{3(n-4) + s(T)}{4} + 3 = \frac{3n + s(T)}{4}.$$

Further if $\gamma_{oiR}(T) = \frac{3n+s(T)}{4}$, then we have equality throughout this inequality chain. In particular, $\gamma_{oiR}(T') = \frac{3n'+s(T')}{4}$. By the induction hypothesis, $T' \in \mathcal{F}$. Since v_5 is a leaf of T', and T can be obtained from T' by Operation $\mathcal{O}_2$, we deduce that $T \in \mathcal{F}$. This complete the proof. □

According to Propositions 3 and 4, we have proven Theorem 2.

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