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*Research Article*

# Some remarks on the sum of the inverse values of the normalized signless Laplacian eigenvalues of graphs

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**Abstract:** Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with $n$ vertices, $m$ edges and a sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(v_i)$. Let $A = (a_{ij})_{n\times n}$ and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be the adjacency and the diagonal degree matrix of $G$, respectively. Denote by $\mathcal{L}^+(G) = D^{-1/2}(D + A)D^{-1/2}$ the normalized signless Laplacian matrix of graph $G$. The eigenvalues of matrix $\mathcal{L}^+(G)$, $2 = \gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, are normalized signless Laplacian eigenvalues of $G$. In this paper some bounds for the sum $K^+(G) = \sum_{i=1}^{n} \frac{1}{\gamma_i^+}$ are considered.

**Keywords:** Normalized Laplacian; eigenvalues; bipartite graphs

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## 1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple connected graph with $n$ vertices, $m$ edges and a sequence of vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(v_i)$. If vertices $v_i$ and $v_j$ are adjacent in $G$, we write $i \sim j$.
Let $A = (a_{ij})_{n\times n}$ and $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ be the adjacency and the diagonal degree matrix of $G$, respectively. Then $L = D - A$ is the Laplacian matrix of $G$. The normalized Laplacian is defined as $\mathcal{L} = D^{-1/2}LD^{-1/2} = I - D^{-1/2}AD^{-1/2} = I - R$. Here $R = D^{-1/2}AD^{-1/2}$ is the Randić matrix [3, 8]. Further, denote by $L^+ = D + A$

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and $\mathcal{L}^+ = D^{-1/2}L^+D^{-1/2} = I + D^{-1/2}AD^{-1/2} = I + R$ signless Laplacian and normalized signless Laplacian matrix, respectively. For more information on these matrices one can refer to [7, 8].

Eigenvalues of matrix $\mathcal{L}$, $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_{n-1}^- \geq \gamma_n^- = 0$, are normalized Laplacian eigenvalues of $G$. Some well known properties of these eigenvalues are [22]

$$\sum_{i=1}^{n-1} \gamma_i^- = n \quad \text{and} \quad \sum_{i=1}^{n-1} (\gamma_i^-)^2 = n + 2R_{-1}(G)\,,$$

where

$$R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}\,,$$

is the vertex–degree–based topological index known as the general Randić index $R_{-1}(G)$ [19] (see also [5, 10]).

Kemeny constant [13] represents the expected number of steps needed by a random walker to reach an arbitrary node from some arbitrary starting node, with the starting and ending nodes having been selected according to the equilibrium distribution of the Markov chain. It is known that this constant can be studied through the use of the characteristic polynomial of the normalized Laplacian matrix [4] (see also [14, 17]). In that case it is defined as

$$K(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i^-}.$$

The eigenvalues of matrix $\mathcal{L}^+$, $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+ \geq 0$, are normalized signless Laplacian eigenvalues of $G$. The following identities are valid for them [6]:

$$\sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and} \quad \sum_{i=1}^{n} \left(\gamma_i^+\right)^2 = n + 2R_{-1}(G)\,.$$

By analogy with Kemeny's constant, we introduce "signless Kemeny's" constant. Since for the connected non–bipartite graphs $\gamma_i^+ > 0$, $i = 1, 2, \ldots, n$ (see [2]), it can be defined as

$$K^+(G) = \sum_{i=1}^{n} \frac{1}{\gamma_i^+}\,.$$

For the connected bipartite graphs we have $\gamma_n^+ = 0$ and $\gamma_i^+ = \gamma_i^- > 0$, $i = 1, 2, \ldots, n-1$ [1]. In that case

$$K^+(G) = K(G) = \sum_{i=1}^{n-1} \frac{1}{\gamma_i^+} = \sum_{i=1}^{n-1} \frac{1}{\gamma_i^-}\,.$$

In this paper we obtain some upper and lower bounds for $K^+(G)$. Since for the connected bipartite graphs $K^+(G) = K(G)$, a number of bounds for the Kemeny's constant are obtained as well.

## 2. Preliminaries

In this section we recall some results from the literature that will be used here after.

**Lemma 1.** [12] *For any connected graph $G$, the largest normalized signless Laplacian eigenvalue is*

$$\gamma_1^+ = 2\,.$$

**Lemma 2.** [12] *Let $G$ be a graph of order $n \geq 2$ with no isolated vertices. Then*

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$$

*if and only if $G \cong K_n$.*

**Lemma 3.** [1] *If $G$ is a bipartite graph, then the eigenvalues of $\mathcal{L}$ and $\mathcal{L}^+$ coincide.*

**Lemma 4.** [2] *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then, $\gamma_i^+ > 0$, for $i = 1, 2, \ldots, n$.*

**Lemma 5.** [2] *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$\gamma_n^+ \leq \frac{n - 2R_{-1}(G)}{n} \leq \frac{\Delta - 1}{\Delta} \leq \frac{n-2}{n-1}\,,$$

*with equality if and only if $G \cong K_n$.*

**Lemma 6.** [20] *Let $G$ ($\ncong K_{p,q}$) be a connected bipartite graph with bipartition $V = X \cup Y$, $p = |X| > 1$ and $q = |Y| > 1$. Then*

$$\gamma_2^- \geq 1 + \frac{1}{\sqrt{pq}} > \gamma_2^-(K_{p,q})\,.$$

*The first equality holds if and only if $G \cong K_{p,q} - e$.*

**Lemma 7.** [15] *Let $G$ be a connected graph of order $n$. Then $\gamma_2^- \geq 1$, the equality holds if and only if $G$ is a complete bipartite graph.*

**Lemma 8.** [9] *Let $G$ be a connected graph with $n > 2$ vertices. Then $\gamma_2^- = \gamma_3^- = \cdots = \gamma_{n-1}^-$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

**Lemma 9.** [1, 8] *If $G$ is a connected bipartite graph with $n$ vertices, $m$ edges and $t(G)$ spanning trees, then*

$$\prod_{i=1}^{n-1} \gamma_i^- = \prod_{i=1}^{n-1} \gamma_i^+ = \frac{2mt(G)}{\prod_{i=1}^{n} d_i}.$$

*If $G$ is a connected non–bipartite graph with $n$ vertices, then*

$$\prod_{i=1}^{n} \gamma_i^+ = \frac{2t(G \times K_2)}{t(G)\prod_{i=1}^{n} d_i}.$$

**Lemma 10.** [11] *For $a_1, a_2, \ldots, a_n \geq 0$ and $p_1, p_2, \ldots, p_n \geq 0$ such that $\sum_{i=1}^{n} p_i = 1$,*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq n\lambda \left( \frac{1}{n} \sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{1/n} \right),$$

*where $\lambda = \min\{p_1, p_2, \ldots, p_n\}$. Moreover, the equality holds if and only if $a_1 = a_2 = \cdots = a_n$.*

## 3. Main results

In the next theorem we determine a lower bound for $K^+(G)$ in terms of parameters $n$, $\alpha$ and $\beta$, where $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$ and $\gamma_n^+ \leq \beta \leq \frac{n-2}{n-1}$.

**Theorem 1.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then for any $\alpha$ and $\beta$, $\gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$ and $\gamma_n^+ \leq \beta \leq \frac{n-2}{n-1}$, holds*

$$K^+(G) \geq \frac{1}{2} + \max\left\{ \frac{1}{\alpha} + \frac{(n-2)^2}{n-2-\alpha}, \ \frac{1}{\beta} + \frac{(n-2)^2}{n-2-\beta} \right\}. \tag{1}$$

*Equality holds if and only if $\alpha = \gamma_2^+$ and $\gamma_3^+ = \cdots = \gamma_n^+$, or $\beta = \gamma_n^+$ and $\gamma_2^+ = \cdots \gamma_{n-1}^+$.*

*Proof.* By the arithmetic–harmonic mean inequality, AM–HM (see e.g. [18]), we have

$$\sum_{i=3}^{n} \gamma_i^+ \sum_{i=3}^{n} \frac{1}{\gamma_i^+} \geq (n-2)^2,$$

i.e.

$$\sum_{i=1}^{n} \frac{1}{\gamma_i^+} \geq \frac{1}{\gamma_1^+} + \frac{1}{\gamma_2^+} + \frac{(n-2)^2}{n-\gamma_1^+-\gamma_2^+},$$

and according to Lemma 1 it follows

$$\sum_{i=1}^{n} \frac{1}{\gamma_i^+} \geq \frac{1}{2} + \frac{1}{\gamma_2^+} + \frac{(n-2)^2}{n-2-\gamma_2^+}. \tag{2}$$

Consider the function $f(x)$ defined as

$$f(x) = \frac{1}{x} + \frac{(n-2)^2}{n-2-x}\,. \tag{3}$$

It is easy to see that $f$ is increasing for $x \geq \frac{n-2}{n-1}$, therefore for any $\alpha$, $x = \gamma_2^+ \geq \alpha \geq \frac{n-2}{n-1}$, holds $f(\gamma_2^+) \geq f(\alpha)$, that is

$$\frac{1}{\gamma_2^+} + \frac{(n-2)^2}{n-2-\gamma_2^+} \geq \frac{1}{\alpha} + \frac{(n-2)^2}{n-2-\alpha}\,. \tag{4}$$

From the above and (2) we get

$$K^+(G) \geq \frac{1}{2} + \frac{1}{\alpha} + \frac{(n-2)^2}{n-2-\alpha}\,. \tag{5}$$

Similarly, by AM–HM inequality, we obtain

$$\sum_{i=2}^{n-1} \gamma_i^+ \sum_{i=2}^{n-1} \frac{1}{\gamma_i^+} \geq (n-2)^2\,,$$

that is

$$\sum_{i=1}^{n} \frac{1}{\gamma_i^+} \geq \frac{1}{2} + \frac{1}{\gamma_n^+} + \frac{(n-2)^2}{n-2-\gamma_n^+}\,. \tag{6}$$

The function $f(x)$, defined by (3), is decreasing for $x \leq \frac{n-2}{n-1}$, therefore for any $\beta$, $x = \gamma_n^+ \leq \beta \leq \frac{n-2}{n-1}$, holds $f(\gamma_n^+) \geq f(\beta)$, i.e.

$$\frac{1}{\gamma_n^+} + \frac{(n-2)^2}{n-2-\gamma_n^+} \geq \frac{1}{\beta} + \frac{(n-2)^2}{n-2-\beta}\,. \tag{7}$$

From (6) and (7) it follows

$$K^+(G) \geq \frac{1}{2} + \frac{1}{\beta} + \frac{(n-2)^2}{n-2-\beta}\,. \tag{8}$$

The inequality (1) follows according to (5) and (8).
Equality in (2) holds if and only if $\gamma_3^+ = \cdots = \gamma_n^+$. Equality in (4) holds if and only if $\gamma_2^+ = \alpha$. Similarly, equality in in (8) holds if and only if $\beta = \gamma_n^+$ and $\gamma_2^+ = \cdots = \gamma_{n-1}^+$. These imply that equality in (1) holds if and only if $\alpha = \gamma_2^+$ and $\gamma_3^+ = \cdots = \gamma_n^+$, or $\beta = \gamma_n^+$ and $\gamma_2^+ = \cdots \gamma_{n-1}^+$. $\square$

**Remark 1.** If $\alpha = \beta = \frac{n-2}{n-1}$, then by Lemma 2, equality in (1) holds if and only if $G \cong K_n$.

**Corollary 1.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$K^+(G) \geq \frac{1}{2} + \max\left\{\frac{1}{\gamma_2^+} + \frac{(n-2)^2}{n-2-\gamma_2^+},\ \frac{1}{\gamma_n^+} + \frac{(n-2)^2}{n-2-\gamma_n^+}\right\}.$$

*Equality holds if and only if $\gamma_3^+ = \cdots = \gamma_n^+$, or $\gamma_2^+ = \cdots = \gamma_{n-1}^+$.*

**Corollary 2.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then for any $s$, $\gamma_2^+ \geq s \geq \gamma_n^+$, holds*

$$K^+(G) \geq \frac{1}{2} + \frac{1}{s} + \frac{(n-2)^2}{n-2-s}.$$

*Equality holds if and only if $s = \gamma_2^+$ and $\gamma_3^+ = \cdots = \gamma_n^+$, or $s = \gamma_n^+$ and $\gamma_2^+ = \cdots = \gamma_{n-1}^+$.*

*Proof.* For any $s$, $\gamma_2^+ \geq s \geq \gamma_n^+$, it holds that $\gamma_2^+ \geq s \geq \frac{n-2}{n-1}$ or $\frac{n-2}{n-1} \geq s \geq \gamma_n^+$. Therefore, by Theorem 1 we obtain the required result. □

According to Lemma 1 the following identities hold

$$\sum_{i=2}^{n} \gamma_i^+ = n-2 \quad \text{and} \quad \sum_{i=2}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G) - 4,$$

from which it follows

$$\gamma_2^+ \sum_{i=2}^{n} \gamma_i^+ \geq \sum_{i=2}^{n} (\gamma_i^+)^2 = n + 2R_{-1}(G) - 4,$$

i.e.

$$\gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n-2}.$$

One can easily show that the following is valid

$$\gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n-2} \geq \frac{n-2}{n-1}.$$

From the above and Lemma 5 we have that

$$\gamma_2^+ \geq \frac{n + 2R_{-1}(G) - 4}{n-2} \geq \frac{n-2}{n-1} \geq \frac{\Delta - 1}{\Delta} \geq \frac{n - 2R_{-1}(G)}{n} \geq \gamma_n^+,$$

with equality holding if and only if $G \cong K_n$, thus we have the following corollaries of Theorem 1, i.e. Corollary 2.

**Corollary 3.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$K^+(G) \geq \frac{1}{2} + \frac{n-2}{n+2R_{-1}(G)-4} + \frac{(n-2)^3}{n^2-5n+8-2R_{-1}(G)}\,.$$

*Equality holds if and only if $G \cong K_n$.*

**Corollary 4.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$K^+(G) \geq \frac{n(2n-3)}{2(n-2)}\,.$$

*Equality holds if and only if $G \cong K_n$.*

**Corollary 5.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$K^+(G) \geq \frac{1}{2} + \frac{\Delta}{\Delta-1} + \frac{\Delta(n-2)^2}{(n-3)\Delta+1}\,. \tag{9}$$

*Equality holds if and only if $G \cong K_n$.*

**Remark 2.** In [2] one more general inequality is proven, which special case is the inequality (9).

Considering the proof techniques as in Theorem 1, we obtain the following result for connected bipartite graphs.

**Theorem 2.** *Let $G$ be a connected bipartite graph with $n \geq 2$ vertices. Then for any $\alpha$ and $\beta$, $\gamma_2^+ = \gamma_2^- \geq \alpha \geq 1$ and $\gamma_{n-1}^+ = \gamma_{n-1}^- \leq \beta \leq 1$, holds*

$$K^+(G) = K(G) \geq \frac{1}{2} + \max\left\{\frac{1}{\alpha} + \frac{(n-3)^2}{n-2-\alpha},\ \frac{1}{\beta} + \frac{(n-3)^2}{n-2-\beta}\right\}. \tag{10}$$

*Equality holds if and only if $\alpha = \gamma_2^+$ and $\gamma_3^+ = \cdots = \gamma_{n-1}^+$, or $\beta = \gamma_{n-1}^+$ and $\gamma_2^+ = \cdots = \gamma_{n-2}^+$.*

**Remark 3.** If $\alpha = \beta = 1$, equality in (10) holds if and only if $G \cong K_{p,q}$, $p+q=n$.

**Corollary 6.** *Let $G$ be a connected bipartite graph with $n \geq 2$ vertices. Then*

$$K^+(G) = K(G) \geq \frac{1}{2} + n - 2\,. \tag{11}$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

**Remark 4.** The inequality (11) was proven in [21].

**Corollary 7.** *Let $G$ be a connected bipartite graph with $n \geq 2$ vertices. Then*

$$K^{+}(G) = K(G) \geq \frac{1}{2} + \max\left\{ \frac{1}{\gamma_2^{+}} + \frac{(n-3)^2}{n-2-\gamma_2^{+}},\ \frac{1}{\gamma_{n-1}^{+}} + \frac{(n-3)^2}{n-2-\gamma_{n-1}^{+}} \right\}.$$

*Equality holds if and only if $\gamma_3^{+} = \cdots = \gamma_{n-1}^{+}$, or $\gamma_2^{+} = \cdots = \gamma_{n-2}^{+}$.*

From Lemma 6 we get the following corollary of Theorem 2.

**Corollary 8.** *Let $G$ ($\ncong K_{p,q}$) be a connected bipartite graph with bipartition $V = X \cup Y$, $p = |X| \geq 2$, $q = |Y| \geq 2$. Then*

$$K^{+}(G) = K(G) \geq \frac{1}{2} + \frac{\sqrt{pq}}{1+\sqrt{pq}} + \frac{\sqrt{pq}(n-3)^2}{(n-3)\sqrt{pq}-1}.$$

**Theorem 3.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$\begin{aligned} K^{+}(G) \geq{}& \frac{1}{2} + \frac{n-2}{n+2R_{-1}(G)-4} + \\ &+2(n-2)\left( \left( \frac{t(G)\prod_{i=1}^{n} d_i}{t(G\times K_2)} \right)^{\frac{2n-3}{2(n-1)(n-2)}} \left( \frac{n+2R_{-1}(G)-4}{n-2} \right)^{\frac{1}{2(n-2)}} - \right. \\ &\left. -\frac{1}{2}\left( \frac{t(G)\prod_{i=1}^{n} d_i}{t(G\times K_2)} \right)^{\frac{1}{n-1}} \right). \end{aligned} \tag{12}$$

*Equality holds if and only if $G \cong K_n$.*

*Proof.* Taking $a_i = \frac{1}{\gamma_i^{+}}$, $i = 2, \ldots, n$ and $p_2 = \frac{1}{2(n-1)}$ and $p_i = \frac{2n-3}{2(n-1)(n-2)}$, $i = 3, \ldots, n$ in Lemma 10, we obtain that

$$\begin{aligned} &\frac{1}{2(n-1)}\frac{1}{\gamma_2^{+}} + \frac{2n-3}{2(n-1)(n-2)}\sum_{i=3}^{n}\frac{1}{\gamma_i^{+}} - \left(\frac{1}{\gamma_2^{+}}\right)^{\frac{1}{2(n-1)}} \prod_{i=3}^{n}\left(\frac{1}{\gamma_i^{+}}\right)^{\frac{2n-3}{2(n-1)(n-2)}} \\ \geq{}& \frac{1}{2(n-1)}\sum_{i=2}^{n}\frac{1}{\gamma_i^{+}} - \frac{1}{2}\prod_{i=2}^{n}\left(\frac{1}{\gamma_i^{+}}\right)^{\frac{1}{n-1}}. \end{aligned}$$

From the above and Lemmas 1 and 9, we have

$$\begin{aligned} &\frac{1}{2(n-1)}\frac{1}{\gamma_2^{+}} + \frac{2n-3}{2(n-1)(n-2)}\left(K^{+}(G) - \frac{1}{2} - \frac{1}{\gamma_2^{+}}\right) - \\ &-\left(\gamma_2^{+}\right)^{\frac{1}{2(n-2)}}\left(\frac{t(G)\prod_{i=1}^{n} d_i}{t(G\times K_2)}\right)^{\frac{2n-3}{2(n-1)(n-2)}} \\ \geq{}& \frac{1}{2(n-1)}\left(K^{+}(G) - \frac{1}{2}\right) - \frac{1}{2}\left(\frac{t(G)\prod_{i=1}^{n} d_i}{t(G\times K_2)}\right)^{\frac{1}{n-1}}, \end{aligned}$$

i.e.,

$$K^{+}(G)-\frac{1}{2} \geq \frac{1}{\gamma_2^+}.+2(n-2)\left(\frac{t(G)\prod_{i=1}^n d_i}{t(G\times K_2)}\right)^{\frac{2n-3}{2(n-1)(n-2)}}\left(\gamma_2^+\right)^{\frac{1}{2(n-2)}}-$$
$$-(n-2)\left(\frac{t(G)\prod_{i=1}^n d_i}{t(G\times K_2)}\right)^{\frac{1}{n-1}} \tag{13}$$

We now consider the function $f(x)$, defined by

$$f(x)=\frac{1}{x}+2(n-2)\left(\frac{t(G)\prod_{i=1}^n d_i}{t(G\times K_2)}\right)^{\frac{2n-3}{2(n-1)(n-2)}}(x)^{\frac{1}{2(n-2)}}.$$

Note that

$$f'(x)=\frac{1}{x^2}\left(\left(x^{n-1}\frac{t(G)\prod_{i=1}^n d_i}{t(G\times K_2)}\right)^{\frac{2n-3}{2(n-1)(n-2)}}-1\right).$$

It is elementary to see that $f$ is increasing for $x \geq \left(\frac{t(G\times K_2)}{t(G)\prod_{i=1}^n d_i}\right)^{\frac{1}{n-1}}$. Further note that

$$\gamma_2^+ \geq \frac{n+2R_{-1}(G)-4}{n-2} \geq \frac{n-2}{n-1}=\frac{\sum_{i=2}^n \gamma_i^+}{n-1} \geq \left(\prod_{i=2}^n \gamma_i^+\right)^{\frac{1}{n-1}}=\left(\frac{t(G\times K_2)}{t(G)\prod_{i=1}^n d_i}\right)^{\frac{1}{n-1}}.$$

Then

$$\begin{aligned} f\left(\gamma_2^+\right) &\geq f\left(\frac{n+2R_{-1}(G)-4}{n-2}\right) \\ &= \frac{n-2}{n+2R_{-1}(G)-4}+ \\ &+ 2(n-2)\left(\frac{t(G)\prod_{i=1}^n d_i}{t(G\times K_2)}\right)^{\frac{2n-3}{2(n-1)(n-2)}}\left(\frac{n+2R_{-1}(G)-4}{n-2}\right)^{\frac{1}{2(n-2)}}. \end{aligned}$$

Considering this with (13), we arrive at the lower bound (12). The equality in (12) holds if and only if

$$\gamma_2^+=\frac{n+2R_{-1}(G)-4}{n-2} \text{ and } \gamma_2^+=\gamma_3^+=\cdots=\gamma_n^+.$$

By Lemma 2, we get that $G \cong K_n$. From the above and Lemma 2, we also get

$$\gamma_2^+=\frac{n+2R_{-1}(G)-4}{n-2}=\frac{n-2}{n-1},$$

i.e.,

$$2R_{-1}(G)=\frac{n}{n-1}.$$

This also verify that $G \cong K_n$. Therefore the equality in (12) holds if and only if $G \cong K_n$. $\square$

Considering the similar method in Theorem 3 with Lemmas 1, 3 and 7– 10, we have:

**Theorem 4.** *Let $G$ be a connected bipartite graph with $n \geq 3$ vertices and $m$ edges. Then*

$$K^{+}(G) = K(G) \geq \frac{3}{2} + 2(n-3)\left[\left(\frac{\prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{2n-5}{2(n-2)(n-3)}} - \frac{1}{2}\left(\frac{\prod_{i=1}^{n} d_i}{mt(G)}\right)^{\frac{1}{n-2}}\right]. \tag{14}$$

*Equality holds if and only if $G \cong K_{p,q}$, $p+q=n$.*

In the following theorem we determine an upper bound for invariant $K^{+}(G)$ in terms of parameters $k_1$ and $k_2$, $k_1 \geq \gamma_2^{+}$ and $0 < k_2 \leq \gamma_n^{+}$.

**Theorem 5.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then for any $k_1$ and $k_2$, $k_1 \geq \gamma_2^{+}$ and $0 < k_2 \leq \gamma_n^{+}$, holds*

$$K^{+}(G) \leq \frac{(n-1)(k_1+k_2)-n+2}{k_1 k_2} + \frac{1}{2}. \tag{15}$$

*When $k_1 = k_2 = \frac{n-2}{n-1}$, equality holds if and only if $G \cong K_n$.*

*Proof.* For every $i$, $i = 2, 3, \ldots, n$, the following is valid

$$(\gamma_2^{+} - \gamma_i^{+})(\gamma_n^{+} - \gamma_i^{+}) \leq 0,$$

that is

$$\gamma_i^{+} + \frac{\gamma_2^{+}\gamma_n^{+}}{\gamma_i^{+}} \leq \gamma_2^{+} + \gamma_n^{+}.$$

Summing up the above inequality over $i$, $i = 2, 3, \ldots, n$, gives

$$\sum_{i=2}^{n} \gamma_i^{+} + \gamma_2^{+}\gamma_n^{+} \sum_{i=2}^{n} \frac{1}{\gamma_i^{+}} \leq (\gamma_2^{+} + \gamma_n^{+}) \sum_{i=2}^{n} 1,$$

i.e.

$$\sum_{i=1}^{n} \frac{1}{\gamma_i^{+}} \leq \frac{1}{2} + \frac{(n-1)(\gamma_2^{+} + \gamma_n^{+}) - n + 2}{\gamma_2^{+}\gamma_n^{+}}. \tag{16}$$

Since the function

$$f(x) = \frac{(n-1)(x + \gamma_n^{+}) - n + 2}{x}$$

is monotone increasing for $x > 0$, therefore for $x = \gamma_2^{+} \leq k_1$ we have $f(\gamma_2^{+}) \leq f(k_1)$, and from (16) it follows

$$K^{+}(G) \leq \frac{(n-1)(k_1 + \gamma_n^{+}) - n + 2}{k_1 \gamma_n^{+}} + \frac{1}{2}. \tag{17}$$

Now, the function

$$g(x) = \frac{(n-1)(k_1+x)-n+2}{x}$$

is monotone decreasing for $x > 0$, therefore for $x = \gamma_n^+ \geq k_2 > 0$ holds

$$\frac{(n-1)(k_1+\gamma_n^+)-n+2}{\gamma_n^+} \leq \frac{(n-1)(k_1+k_2)-n+2}{k_2}\,.$$

From the above and (17) we arrive at (15).
Equality in (16) holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$, for $i = 3, \dots, n-1$. This means that for some $t$, $1 \leq t \leq n-1$, holds $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_t^+$ and $\gamma_{t+1}^+ = \cdots = \gamma_n^+$. Let $k_1 = \gamma_2^+ = \frac{n-2}{n-1}$. Then

$$(t-1)\frac{n-2}{n-1} + (n-t)\gamma_n^+ = n-2,$$

from which it follows $\gamma_2^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$, therefore the equality in (15) holds if and only if $k_1 = \frac{n-2}{n-1}$ and $G \cong K_n$. In a similar way we obtain that the equality in (15) holds if and only if $k_2 = \gamma_n^+ = \frac{n-2}{n-1}$ and $G \cong K_n$. This implies that when $k_1 = k_2 = \frac{n-2}{n-1}$, equality in (15) holds if and only if $G \cong K_n$. □

**Corollary 9.** *Let $G$ be a connected non–bipartite graph with $n \geq 3$ vertices. Then*

$$K^+(G) \leq \frac{1}{2} + \frac{(n-1)(\gamma_2^+ + \gamma_n^+) - n + 2}{\gamma_2^+\gamma_n^+}\,.$$

*Equality holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_n^+\}$ for $i = 3, \dots, n-1$.*

If $G$ is a connected bipartite graph, we have the following results.

**Theorem 6.** *Let $G$ be a connected bipartite graph with $n \geq 2$ vertices. Then for any $k_1$ and $k_2$, $k_1 \geq \gamma_2^+ = \gamma_2^-$ and $0 < k_2 \leq \gamma_{n-1}^+ = \gamma_{n-1}^-$, holds*

$$K^+(G) = K(G) \leq \frac{1}{2} + \frac{(n-2)(k_1+k_2-1)}{k_1k_2}\,.$$

*When $k_1 = k_2 = 1$, equality holds if and only if $G \cong K_{p,q}$, $p + q = n$.*

**Corollary 10.** *Let $G$ be a connected bipartite graph with $n \geq 2$ vertices. Then*

$$K^+(G) = K(G) \leq \frac{1}{2} + \frac{(n-2)(\gamma_2^+ + \gamma_{n-1}^+ - 1)}{\gamma_2^+\gamma_{n-1}^+}\,.$$

*Equality holds if and only if $\gamma_i^+ \in \{\gamma_2^+, \gamma_{n-1}^+\}$ for $i = 3, \dots, n-1$.*

**Corollary 11.** *Let $G$ be a connected bipartite graph with $n \geq 2$ vertices. Then for any $k$, $0 < k \leq \gamma_{n-1}^{+} = \gamma_{n-1}^{-}$, holds*

$$K^{+}(G) = K(G) \leq \frac{(n-1)k+n-2}{2k}\,. \tag{18}$$

*If $k = 1$, equality holds if and only $G \cong K_{p,q}$, $p+q=n$.*

**Remark 5.** Let $G$ be a connected graph with $n \geq 2$ vertices. In [16] it is proven that for any $k$, $0 < k \leq \gamma_{n-1}^{-}$, holds

$$K(G) \leq \frac{(n-1)k+n-2}{2k}\,, \tag{19}$$

with equality holding if and only if $k = \frac{n}{n-1}$ and $G \cong K_n$, or $k = 1$ and $G \cong K_{p,q}$, $p+q=n$, or $k = \frac{3}{n-1}$ and $G \cong C_n$. It is easy to see that when $G$ is a connected bipartite graph, the inequalities (18) and (19) coincide.

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