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The annihilator-inclusion ideal graph of a commutative ring

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Abstract: Let R be a commutative ring with non-zero identity. The annihilator-inclusion ideal graph of R, denoted by ξ_R, is a graph whose vertex set is the of all non-zero proper ideals of R and two distinct vertices I and J are adjacent if and only if either $\mathrm{Ann}(I) \subseteq J$ or $\mathrm{Ann}(J) \subseteq I$. The purpose of this paper is to provide some basic properties of the graph ξ_R. In particular, shows that ξ_R is a connected graph with diameter at most three, and has girth 3 or ∞. Furthermore, is determined all isomorphic classes of non-local Artinian rings whose annihilator-inclusion ideal graphs have genus zero or one.

Keywords: Annihilator-inclusion ideal graph, Artinian ring, planar graph; genus

AMS Subject classification: 05C25, 05C10

1. Introduction

Let G be a simple graph with the vertex set $V(G)$ and edge set $E(G)$. For every vertex $v \in V(G)$, the *degree* of a vertex v is defined as $d_G(v) = |\{u \in V(G) \mid uv \in E(G)\}|$. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest uv-path in G. The greatest distance between any pair of vertices u and v in G is the *diameter* of G and denoted by $\mathrm{diam}(G)$. If a graph G contains one vertex adjacent to all other vertices and with no extra edge, then G is called a *star*

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graph. The *girth* of a graph G, denoted by $g(G)$, is the length of its shortest cycle. The girth of a graph with no cycle is defined ∞.

A simple graph is said to be planar if it can be drawn in the plane or on the surface of a sphere. It is known that $K_{3,3}$ and K_5 are not planar and can be drawn without crossings on the surface of a torus. The torus can be thought of as a sphere with one handle. More generally, a surface is said to be of genus g if it is topologically homeomorphic to a sphere with g handles. Thus the genus of a sphere is 0 and the one of torus is one. A graph can be drawn without crossings on the surface of genus g, but not on one of genus $g-1$, is called a graph of genus g. We write $\lambda(G)$ for the genus of a graph G. Therefore $\lambda(K_{3,3}) = \lambda(K_5) = 1$. A well-known fact is that if G is a connected graph of genus g, with n vertices, m edges and f faces, then $n - m + f = 2 - 2g$. For terminology and notation not defined here, the reader is referred to [16].

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the last two decades, leading to many interesting results and questions. In ring theory, the structure of a ring R is closely tied to behavior ideals more than elements, and so it is deserving to define a graph with vertex set as ideals instead of elements. There are many papers on assigning a graph to a ring. The old one is the zero divisor graph $\Gamma(R)$ (see for instance [4, 5]). The vertex set of this graph is $Z(R) \setminus (0)$ and two distinct vertices v_1 and v_2 are adjacent if and only if $v_1v_2 = 0$ and some of them to mention annihilating [7, 8], co-annihilating [1], essential ideal graph [2, 3] and co-maximal graph [10] of commutative rings. Several authors studied about various properties of these graphs including diameter, planarity and genus [13–15].

Here we propose a new graph associated to a commutative ring which we call annihilator-inclusion ideal graph. The annihilator-inclusion ideal graph, denoted by ξ_R, is the (undirected) graph with vertices $\mathbb{I}^*(R)$, and two distinct vertices I and J are adjacent if and only if either $\operatorname{Ann}(I) \subseteq J$ or $\operatorname{Ann}(J) \subseteq I$.

Throughout this paper, all rings are assumed to be commutative rings with identity that are not integral domains. We denote the collection of all non-zero proper ideals of R by $\mathbb{I}^*(R)$. If X is either an element or a subset of R, then the *annihilator* of X is defined as $\operatorname{Ann}(X) = \{r \in R \mid rX = 0\}$. By $\operatorname{Max}(R)$ we denote the set of all maximal ideals. Furthermore, a ring R is a local ring if R has a unique maximal ideal.

In this paper we initiate the study of the annihilator-inclusion ideal graph and we investigate its basic properties. In particular, we characterize all rings whose annihilator-inclusion ideal graphs have genus 0 or 1.

We make use of the following results.

Observation 1. Let $(R, \mathfrak{m})$ be a local ring. If $\dim(\mathfrak{m}/\mathfrak{m}^2) \geq 2$, then R has at least three distinct non-trivial ideals I, J and K such that $I, J, K \neq \mathfrak{m}^i$ for every i.

Proof. By hypothesis, $\dim(\mathfrak{m}/\mathfrak{m}^2) \geq 2$. Hence, it is possible to find $x_1, x_2 \in \mathfrak{m}$ such that $\{x_1 + \mathfrak{m}^2, x_2 + \mathfrak{m}^2\}$ is linearly independent over $R/\mathfrak{m}$. Then the ideals

$I = Rx_1, J = Rx_2$, and $K = R(x_1 + x_2)$ are distinct non-trivial ideals of R such that $I, J, K \notin \{\mathfrak{m}^i \mid i \in \mathbb{N}\}$. □

Observation 2. Let R be a ring and $\mathfrak{m}$ be a maximal ideal in R. If $\mathrm{Ann}(\mathfrak{m}) \neq 0$, then $\mathfrak{m} = Z(\mathrm{Ann}(\mathfrak{m}))$.

Proof. Since $\mathfrak{m}\mathrm{Ann}(\mathfrak{m}) = 0$, $\mathfrak{m} \subseteq Z(\mathrm{Ann}(\mathfrak{m}))$. Now, if x is an arbitrary element in $Z(\mathrm{Ann}(\mathfrak{m}))$, then there is a nonzero element $y \in \mathrm{Ann}(\mathfrak{m})$ such that $xy = 0$. If $x \notin \mathfrak{m}$, then there exists $z \in \mathfrak{m}$ such that $rx + z = 1$, for some $r \in R$, and so $y = 0$, a contradiction. Thus $Z(\mathrm{Ann}(\mathfrak{m})) \subseteq \mathfrak{m}$ and so, $\mathfrak{m} = Z(\mathrm{Ann}(\mathfrak{m}))$. □

Theorem A. *([14]* **Lemma 2.6**) Let (R, m) be a local ring. If $\dim(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = 1$ and for some positive integer t, $\mathfrak{m}^t = (0)$, then the set of all non-trivial ideals of R is the set $\{\mathfrak{m}^i \mid 1 \leq i < t\}$.

Observation 3. Let R be a ring. Then the subgraph $\xi_R[\mathrm{Max}(R)]$ is a clique.

Proof. If R is local we are done. Assume R is not local. If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two distinct maximal ideals of R, then it follows from $\mathfrak{m}_1\mathrm{Ann}(\mathfrak{m}_1) = (0) \subseteq \mathfrak{m}_2$ that $\mathrm{Ann}(\mathfrak{m}_1) \subseteq \mathfrak{m}_2$. So $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent. This implies that the subgraph induced by $\mathrm{Max}(R)$ is a clique. □

Observation 4. Let R be a ring. Then ξ_R is finite if and only if the degree of each vertex of ξ_R is finite .

Proof. If ξ_R is finite, then obviously the degree of each vertex of ξ_R is finite. Suppose that the degree of each vertex of ξ_R is finite. Since the subgraph induced by $\mathrm{Max}(R)$ is a clique, $|\mathrm{Max}(R)| < \infty$. Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ and define $X_i = \{(0) \neq I \lhd R \mid \mathrm{Ann}(I) \subseteq \mathfrak{m}_i\}$ for each $1 \leq i \leq n$. Then $\mathbb{I}(R) = X_1 \cup X_2 \cup \ldots \cup X_n$. Since $\deg \mathfrak{m}_i < \infty$, we have $|X_i| < \infty$ for each i and hence ξ_R is finite. □

2. Properties of the annihilator-inclusion ideal graphs

In this section we first show that the annihilator-inclusion ideal graph of a commutative ring is connected with diameter at most 3 and girth 3 or ∞, and then we classify all commutative rings whose annihilator-inclusion ideal graphs are stars or cycles. Our first theorem shows that ξ_R is a connected graph with $\mathrm{diam}(\xi_R) \leq 3$.

Theorem 5. *Let R be a ring. Then ξ_R is connected and* $\mathrm{diam}(\xi_R) \leq 3$.

Proof. If R is a local ring with the maximal ideal $\mathfrak{m}$, then $\mathrm{Ann}(I) \subseteq \mathfrak{m}$ for every non-zero proper ideal I of R and so $\mathfrak{m}$ is adjacent to all non-zero proper ideals of R. This implies that ξ_R is connected and $\mathrm{diam}\xi_R \leq 2$.

Assume R is not a local ring. If I, J are two vertices of ξ_R, then there are two maximal ideals $\mathfrak{m}$ and $\mathfrak{m}'$ such that $\mathrm{Ann}(I) \subseteq \mathfrak{m}$ and $\mathrm{Ann}(J) \subseteq \mathfrak{m}'$. Hence $d(I, J) \leq 3$ and so ξ_R is a connected graph with diameter at most 3 and the proof is complete. □

Next we show that the girth of ξ_R is either 3 or ∞. We need the followings lemmas.

Lemma 1. Let R be a ring. Then one of the following conditions is fulfilled:

(a) For each maximal ideal $\mathfrak{m}$ of R, $\mathrm{Ann}(\mathfrak{m}) \subseteq \mathfrak{m}$.

(b) $R \cong F \times S$, where F is a field and S is a commutative ring with identity.

Proof. Let $\mathfrak{m}$ be a maximal ideal of R such that $\mathrm{Ann}(\mathfrak{m}) \nsubseteq \mathfrak{m}$. Then $\mathfrak{m}+\mathrm{Ann}(\mathfrak{m}) = R$ and so $a + b = 1$ for some $a \in \mathrm{Ann}(\mathfrak{m})$ and $b \in \mathfrak{m}$. If $r \in \mathfrak{m} \cap \mathrm{Ann}(\mathfrak{m})$, then we have $r = ra + rb = 0$. It follows that $R \cong \frac{R}{\mathfrak{m}} \times \frac{R}{\mathrm{Ann}(\mathfrak{m})}$ and the proof is complete. □

Lemma 2. Let R be a commutative ring with identity such that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$, $\mathrm{Ann}(\mathfrak{m}_1) \neq 0$ and $\mathrm{Ann}(\mathfrak{m}_2) \neq 0$. Then one of the following conditions is fulfilled.

(a) $R \cong F \times S$, where F is a field and S is a commutative local ring with identity.

(b) $\mathrm{Ann}(J(R)) \neq \mathfrak{m}_i$, for each $i = 1, 2$.

Proof. If $\mathrm{Ann}(\mathfrak{m}_1) \nsubseteq \mathfrak{m}_1$ or $\mathrm{Ann}(\mathfrak{m}_2) \nsubseteq \mathfrak{m}_2$, then by Lemma 1 we are done. Suppose that $\mathrm{Ann}(\mathfrak{m}_1) \subseteq \mathfrak{m}_1$ and $\mathrm{Ann}(\mathfrak{m}_2) \subseteq \mathfrak{m}_2$. Note that if $\mathrm{Ann}(\mathfrak{m}_1) = \mathfrak{m}_1$ (the case $\mathrm{Ann}(\mathfrak{m}_2) = \mathfrak{m}_2$ is similar), then $\mathfrak{m}_1 = \mathfrak{m}_2$, a contradiction. Thus $\mathrm{Ann}(\mathfrak{m}_1) \subsetneq \mathfrak{m}_1$ and $\mathrm{Ann}(\mathfrak{m}_2) \subsetneq \mathfrak{m}_2$. Clearly, $\mathfrak{m}_1\mathfrak{m}_2 \neq (0)$, for otherwise $\mathfrak{m}_1 \subseteq \mathrm{Ann}(\mathfrak{m}_2) \subseteq \mathfrak{m}_2$ which is impossible. It follows that $J(R) \neq (0)$. Assume, to the contrary, that $\mathrm{Ann}(J(R)) = \mathfrak{m}_1$ (the case $\mathrm{Ann}(J(R)) = \mathfrak{m}_2$ is similar). By Observation 2, $Z(\mathrm{Ann}(\mathfrak{m}_2)) = \mathfrak{m}_2$. Since $\mathrm{Ann}(\mathfrak{m}_2) \subseteq J(R)$, we have $\mathfrak{m}_1\mathrm{Ann}(\mathfrak{m}_2) = (0)$. Let $0 \neq r \in \mathrm{Ann}(\mathfrak{m}_2)$ and $x \in \mathfrak{m}_1 - \mathfrak{m}_2$. Then $xr = 0$ and this implies $x \in Z(\mathrm{Ann}(\mathfrak{m}_2)) = \mathfrak{m}_2$, a contradiction. Therefore, $\mathrm{Ann}(J(R)) \neq \mathfrak{m}_1$ and $\mathrm{Ann}(J(R)) \neq \mathfrak{m}_2$ and so (b) holds. □

Theorem 6. Let R be a ring. Then $\mathrm{girth}(\xi_R) = 3$ or ∞.

Proof. If $|\mathrm{Max}(R)| \geq 3$, then we have $\mathrm{girth}(\xi_R) = 3$ by Observation 3. If $|\mathrm{Max}(R)| = 1$, then $(R, \mathfrak{m})$ is a local ring and clearly $\mathfrak{m}$ is adjacent to all other vertices. This implies that $\mathrm{girth}(\xi_R) = 3$ or ∞. Assume that $|\mathrm{Max}(R)| = 2$ and let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. If $J(R) = (0)$, then $R \simeq F_1 \times F_2$, where F_1, F_2 are fields. Then clearly $\xi_R \cong K_2$ and so $\mathrm{girth}(\xi_R) = \infty$. Suppose that $J(R) \neq (0)$. If $\mathrm{Ann}(\mathfrak{m}_1) = 0$ or $\mathrm{Ann}(\mathfrak{m}_2) = 0$ then $\mathfrak{m}_1$ or $\mathfrak{m}_2$ is a universal vertex and so $\mathrm{girth}(\xi_R) = 3$ or ∞. Assume that $\mathrm{Ann}(\mathfrak{m}_1) \neq (0)$ and $\mathrm{Ann}(\mathfrak{m}_2) \neq (0)$. By lemma 2, we distinguish two cases.

Case 1. $R \cong F \times S$, where F is a field and S is commutative local ring with identity. Let $\mathfrak{m}$ be the maximal ideal of S. If S has exactly one non-zero proper ideal, then clearly $\xi_R \cong P_4$ and so $\mathrm{girth}(\xi_R) = \infty$. Let I be a non-zero proper ideal of S different from $\mathfrak{m}$, then the subgraph induced by $\{F \times \mathfrak{m}, F \times I, (0) \times S\}$ is a triangle and this implies that girth$(\xi_R) = 3$.

Case 2. $\mathrm{Ann}(J(R)) \neq \mathfrak{m}_i$, for $i = 1, 2$.
Since $\mathrm{Ann}(\mathfrak{m}_i) \subseteq \mathrm{Ann}(J(R))$ for $i = 1, 2$, $\mathrm{Ann}(J(R))$ is adjacent to $\mathfrak{m}_1, \mathfrak{m}_2$. It follows from observation 3 that the subgraph induced by $\{\mathrm{Ann}(J(R)), \mathfrak{m}_1, \mathfrak{m}_2\}$ is a triangle and so girth$(\xi_R) = 3$. This completes the proof. □

Corollary 1. Let R be a ring. Then ξ_R is a cycle if and only if $(R, \mathfrak{m})$ is a local ring such that $\mathbb{I}^*(R) = \{\mathfrak{m}^3, \mathfrak{m}^2, \mathfrak{m}\}$.

Proof. Let ξ_R be a cycle of order $n \geq 3$. By Theorem 6, we have $n = 3$ and so R has exactly three non-trivial ideals. Thus R is Artinian. This implies that $R = R_1 \times \cdots \times R_s$ where $(R_i, \mathfrak{m}_i)$ is an Artinian local ring for each i. Since R has exactly three non-trivial ideals, we have $s = 1$ and $\dim(\frac{\mathfrak{m}_1}{\mathfrak{m}_1^2}) = 1$. It follows from Theorem A that $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the non-trivial ideals of R and the proof is complete. □

Next, we classify all rings R whose annihilator-inclusion ideal graph ξ_R is a star.

Theorem 7. Let R be a ring. Then ξ_R is a star if and only if one of the following statements hold.

(a) $R \cong F_1 \times F_2$, where F_1, F_2 are fields.

(b) $(R, \mathfrak{m})$ is a local ring such that $\mathrm{Ann}(I) \in \{I, \mathfrak{m}\}$ for each non-zero proper ideal I of R.

Proof. If $R \cong F_1 \times F_2$ where F_1, F_2 are fields, then $\xi_R \cong K_2$ and we are done. Let (b) hold. Then clearly $\mathfrak{m}$ is adjacent to all vertices in ξ_R. Let I, J be two distinct ideals of R different from $\mathfrak{m}$. We show that I and J are not adjacent. If $\mathrm{Ann}(I) = \mathfrak{m}$ and $\mathrm{Ann}(J) = \mathfrak{m}$, then clearly $\mathrm{Ann}(J) \not\subseteq I$ and $\mathrm{Ann}(I) \not\subseteq J$ and so $IJ \notin E(\xi_R)$. Assume $\mathrm{Ann}(I) = I$ and $\mathrm{Ann}(J) = \mathfrak{m}$. Then clearly $\mathrm{Ann}(J) \not\subseteq I$. If $\mathrm{Ann}(I) \subseteq J$, then we have $\mathfrak{m} = \mathrm{Ann}(J) \subseteq \mathrm{Ann}(I) = I$, a contradiction. Therefore $\mathrm{Ann}(I) \not\subseteq J$ and so I and J are not adjacent. Now let $\mathrm{Ann}(I) = I$, $\mathrm{Ann}(J) = J$. If $I = \mathrm{Ann}(I) \subseteq J$, then $J = \mathrm{Ann}(J) \subseteq \mathrm{Ann}(I) = I$ and we have $I = J$, a contradiction. Thus $\mathrm{Ann}(I) \not\subseteq J$. Similarly, $\mathrm{Ann}(J) \not\subseteq I$ and this implies that I and J are not adjacent.
Conversely, let ξ_R be a star. Since the subgraph induced by $\mathrm{Max}(R)$ is a clique, we conclude that $|\mathrm{Max}(R)| \leq 2$. We consider two cases:

Case 1. $|\mathrm{Max}(R)| = 2$.
Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. We claim that $J(R) = (0)$. Suppose, to the contrary, that $J(R) \neq (0)$. Since ξ_R is a star and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are adjacent in ξ_R, we may assume without loss of generality that $\mathfrak{m}_1$ is the central vertex of ξ_R. If $\mathrm{Ann}(\mathfrak{m}_2) = (0)$, then $\mathfrak{m}_1, \mathfrak{m}_2, J(R)$ is a triangle, a contradiction. Assume that $\mathrm{Ann}(\mathfrak{m}_2) \neq (0)$. If $\mathrm{Ann}(\mathfrak{m}_2) = \mathfrak{m}_2$, then $\mathfrak{m}_2^2 = (0)$ which implies that $\mathfrak{m}_2 \subseteq \mathfrak{m}_1$, a contradiction. Thus $\mathrm{Ann}(\mathfrak{m}_2) \neq \mathfrak{m}_2$. Obviously, $\mathfrak{m}_2$ is adjacent to $\mathrm{Ann}(\mathfrak{m}_2)$ in ξ_R. If $\mathfrak{m}_1 \neq \mathrm{Ann}(\mathfrak{m}_2)$, then

$\mathfrak{m}_1, \mathfrak{m}_2, \mathrm{Ann}(\mathfrak{m}_2)$ is a triangle which is a contradiction. Let $\mathfrak{m}_1 = \mathrm{Ann}(\mathfrak{m}_2)$. Then $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathfrak{m}_1.\mathfrak{m}_2 = 0$, a contradiction again. Thus $J(R) = (0)$ and $R \cong F_1 \times F_2$ where F_1 and F_2 are fields.

Case 2. $\mathrm{Max}(R) = \{\mathfrak{m}\}$.
Clearly, $\mathfrak{m}$ is the central vertex of ξ_R. If R has exactly two non-zero proper ideals, then we are done. Henceforth, we suppose that R has at least three non-zero proper ideals. Let I be a non-zero proper ideal of R different from $\mathfrak{m}$. If $\mathrm{Ann}(I) = (0)$, then I is adjacent to all vertices which leads to a contradiction. So $\mathrm{Ann}(I) \neq (0)$. If $\mathrm{Ann}(I) \neq \mathfrak{m}$ and $\mathrm{Ann}(I) \neq I$, then the subgraph induced by the vertices $I, \mathrm{Ann}(I), \mathfrak{m}$ is a triangle, a contradiction. Therefore, $\mathrm{Ann}(I) \in \{\mathfrak{m}, I\}$ and the proof is complete. □

Corollary 2. Let R be a ring. Then ξ_R is a tree if and only if ξ_R is a star or P_4.

Proof. One side is clear. Suppose that ξ_R is a tree. We conclude from Observation 3 that $|\mathrm{Max}(R)| \leq 2$. If $(R, \mathfrak{m})$ is a local ring, then $\mathfrak{m}$ is adjacent to all other vertices and so ξ_R is a star. Suppose that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. By the same arguments as in the proof of Theorem 6, we have $\xi_R \cong K_2$ if $J(R) = (0)$ and $\xi_R \cong P_4$ if $J(R) \neq (0)$. □

3. Classification of Artinian rings whose annihilator-inclusion ideal graph has genus at most one

In this section we characterize all Artinian non-local rings whose annihilator-inclusion ideal graph has genus at most one. We make use of the following Theorems in this section.

Theorem B. *(Kuratowski [9]) A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$.*

Theorem C. *([16]) Let G be a graph of order n and size m. Then $m \leq 3(n - 2 + 2\lambda(G))$.*

The proof of the following results can be found in Ringel and Youngs [12]; Ringel [11], respectively.

Theorem D. *For $n \geq 3$, $\lambda(K_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$. In particular, $\lambda(K_n) = 1$ if $n = 5, 6, 7$.*

Theorem E. *For $m, n \geq 2$, $\lambda(K_{m,n}) = \lceil \frac{(m-2)(n-2)}{4} \rceil$.*

Lemma 3. Let $n \geq 2$ and $R \simeq F_1 \times F_2 \times \cdots \times F_n$, where F_i is a field for each i. Then ξ_R is planar if and only if $n = 2, 3$.

Proof. If $n = 2$ then $\xi_R \simeq K_2$ and so ξ_R is planar. If $n = 3$, then Figure 1, shows that ξ_R is planar.
Conversely, let ξ_R be planar. If $n \geq 4$, then the subgraph induced by $\{(0) \times F_2 \times \cdots \times F_n, (0) \times (0) \times F_3 \times \cdots \times F_n, F_1 \times (0) \times F_3 \times \cdots \times F_n, F_1 \times F_2 \times (0) \times \cdots \times (0), F_1 \times F_2 \times F_3 \times (0) \times \cdots \times (0), F_1 \times F_2 \times (0) \times F_4 \times \cdots \times F_n)\}$ contains $K_{3,3}$ whose bipartite sets are $X = \{(0) \times F_2 \times \cdots \times F_n, (0) \times (0) \times F_3 \times \cdots \times F_n, F_1 \times (0) \times F_3 \times \cdots \times F_n\}$ and $Y = \{F_1 \times F_2 \times (0) \times \cdots \times (0), F_1 \times F_2 \times F_3 \times (0) \times \cdots \times (0), F_1 \times F_2 \times (0) \times F_4 \times \cdots \times F_n)\}$, a contradiction. Hence $n \leq 3$ and the proof is complete. □

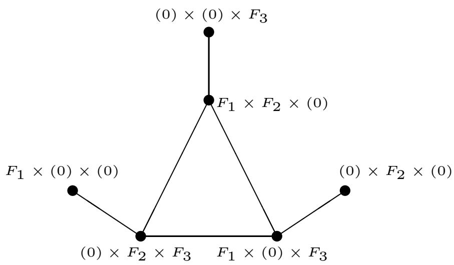

Figure 1. $\xi_{F_1 \times F_2 \times F_3}$

Theorem 8. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then ξ_R is planar if and only if $n = 2$ and $|\mathbb{I}^*(R_1)| + |\mathbb{I}^*(R_2)| \leq 3$.*

Proof. If $n = 2$ and $|\mathbb{I}^*(R_1)| + |\mathbb{I}^*(R_2)| \leq 3$, then ξ_R is one of the graphs illustrated in Figures 2 or 3 respectively with vertex sets $\{v_1 = (0) \times R_2, v_2 = R_1 \times \mathfrak{m}_2, v_3 = (0) \times J, v_4 = (0) \times \mathfrak{m}_2, v_5 = R_1 \times J, v_6 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_7 = \mathfrak{m}_1 \times R_2, v_8 = \mathfrak{m}_1 \times (0)\}$ and $\{v_1 = R_1 \times (0), v_2 = \mathfrak{m}_1 \times R_2, v_3 = \mathfrak{m}_1 \times (0), v_4 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_5 = (0) \times \mathfrak{m}_2, v_6 = R_1 \times \mathfrak{m}_2, v_7 = (0) \times R_2\}$ respectively and hence is planar. Conversely, let ξ_R be a planar graph. If $n \geq 4$, then it can be shown as in the proof of Lemma 3 that ξ_R contains $K_{3,3}$ as a subgraph and this is in contraction to the assumption that ξ_R is planar. Hence $n \leq 3$. If $n = 3$, then the subgraph induced by

$$\{R_1 \times (0) \times \mathfrak{m}_3, R_1 \times (0) \times R_3, R_1 \times \mathfrak{m}_2 \times R_3, R_1 \times R_2 \times \mathfrak{m}_3, (0) \times R_2 \times R_3, (0) \times R_2 \times \mathfrak{m}_3\}$$

contains $K_{3,3}$ whose bipartite sets are $X = \{R_1 \times (0) \times \mathfrak{m}_3, R_1 \times (0) \times R_3, R_1 \times \mathfrak{m}_2 \times R_3\}$ and $Y = \{R_1 \times R_2 \times \mathfrak{m}_3, (0) \times R_2 \times R_3, (0) \times R_2 \times \mathfrak{m}_3\}$ which is a contradiction. Suppose that $n = 2$. If $|\mathbb{I}^*(R_1)| \geq 2$ and $|\mathbb{I}^*(R_2)| \geq 2$, then let I_i be a non-zero proper ideal of R_i different from $\mathfrak{m}_i$, for $i = 1, 2$. Then the subgraph induced by $\{R_1 \times \mathfrak{m}_2, R_1 \times I_2, R_1 \times (0), \mathfrak{m}_1 \times R_2, I_1 \times R_2, (0) \times R_2\}$ contains $K_{3,3}$ whose bipartite sets are $X = \{R_1 \times \mathfrak{m}_2, R_1 \times I_2, R_1 \times (0)\}$ and $= \{\mathfrak{m}_1 \times R_2, I_1 \times R_2, (0) \times R_2\}$, a contradiction.

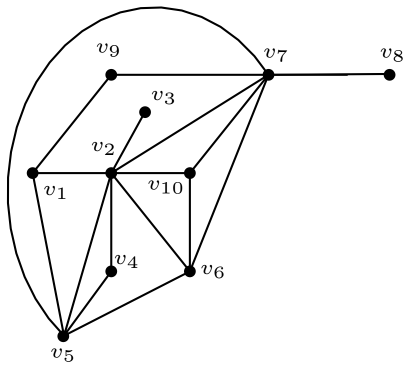

Figure 2. $|\mathbb{I}^*(R_1)| = 1, |\mathbb{I}^*(R_2)| = 2$

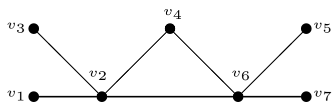

Figure 3. $|\mathbb{I}^*(R_1)| = |\mathbb{I}^*(R_2)| = 1$

Henceforth, we may assume without the loss of generality that $|\mathbb{I}^*(R_1)| = 1$. If $|\mathbb{I}^*(R_2)| \geq 3$, then let I, J be two distinct non-zero proper ideals of R_2 different from $\mathfrak{m}_3$. Then the subgraph induced by $\{R_1 \times \mathfrak{m}_2, R_1 \times J, R_1 \times I, \mathfrak{m}_1 \times \mathfrak{m}_2, (0) \times R_2, (0) \times \mathfrak{m}_2\}$ contains $K_{3,3}$ whose bipartite sets are $X = \{R_1 \times \mathfrak{m}_2, R_1 \times J, R_1 \times I\}$ and $Y = \{\mathfrak{m}_1 \times \mathfrak{m}_2, (0) \times R_2, (0) \times \mathfrak{m}_2\}$, a contradiction. Hence, $|\mathbb{I}^*(R_2)| \leq 2$. □

Theorem 9. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and each F_j is a field, $n \geq 1$ and $m \geq 1$. Then ξ_R is planar if and only if one of the following conditions hold:*

(i) $R = R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ is the only non-trivial ideal in R_1;

(ii) $R = R_1 \times F_1$ and $\{\mathfrak{m}_1, \mathfrak{m}_1^2\}$ or $\{\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3\}$ is the set of all non-zero proper ideals of R_1.

Proof. Assume that ξ_R is planar. If $m + n \geq 4$, then it can be shown as in the proof of Lemma 3 that ξ_R contains $K_{3,3}$ as a subgraph and this is a contradiction. Hence $2 \leq m+n \leq 3$. If $n = 2$ and $m = 1$, then the subgraph induced by $\{\mathfrak{m}_1 \times (0) \times F_1, R_1 \times (0) \times F_1, R_1 \times \mathfrak{m}_2 \times F_1, \mathfrak{m}_1 \times R_2 \times F_1, R_1 \times R_2 \times (0), \mathfrak{m}_1 \times R_2 \times (0)$ is $K_{3,3}$ whose bipartite sets are $X = \{\mathfrak{m}_1 \times (0) \times F_1, R_1 \times (0) \times F_1, R_1 \times \mathfrak{m}_2 \times F_1\}$ and $Y = \{\mathfrak{m}_1 \times R_2 \times F_1, R_1 \times R_2 \times (0), \mathfrak{m}_1 \times R_2 \times (0)\}$, which is a contradiction. Now suppose that $m = 2$ and $n = 1$.

If R_1 has a non-zero proper ideal I different from $\mathfrak{m}_1$, then the subgraph induced by $\{\mathfrak{m}_1 \times (0) \times F_2, R_1 \times (0) \times F_2, \mathfrak{m}_1 \times F_1 \times F_2, R_1 \times F_1 \times (0), \mathfrak{m}_1 \times F_1 \times (0), I \times F_1 \times (0)\}$ is $K_{3,3}$ whose bipartite sets are $X = \{\mathfrak{m}_1 \times (0) \times F_2, R_1 \times (0) \times F_2, \mathfrak{m}_1 \times F_1 \times F_2\}$ and $Y = \{R_1 \times F_1 \times (0), \mathfrak{m}_1 \times F_1 \times (0), I \times F_1 \times (0)\}$, a contradiction. Hence R_1 has exactly one non-zero proper ideal and so R satisfies in (i).
Now let $n = m = 1$. If $|\mathbb{I}^*(R_1)| \geq 4$, then let I, J, K be three distinct non-zero proper ideals of R_1 different $\mathfrak{m}_1$. Then the subgraph induced by $I \times F_1, J \times F_1, K \times F_1, \mathfrak{m}_1 \times F_1, \mathfrak{m}_1 \times (0), R_1 \times (0)$ is $K_{3,3}$ whose bipartite sets are $X = \{I \times F_1, J \times F_1, K \times F_1\}$ and $Y = \{\mathfrak{m}_1 \times F_1, \mathfrak{m}_1 \times (0), R_1 \times (0)\}$, a contradiction. Hence $|\mathbb{I}^*(R_1)| \leq 3$. Using by Observation 1, as above, it can be shown that $\dim_{R_1/\mathfrak{m}_1} \mathfrak{m}_1/\mathfrak{m}_1^2 = 1$. Now we conclude from Theorem A that R_1 satisfies (ii).
Conversely, if R satisfies (i), then $\{v_1 = \mathfrak{m}_1 \times F_1 \times (0), v_2 = \mathfrak{m}_1 \times (0) \times F_2, v_3 = \mathfrak{m}_1 \times (0) \times (0), v_4 = \mathfrak{m}_1 \times F_1 \times F_2, v_5 = R_1 \times (0) \times (0), v_6 = (0) \times F_1 \times F_2, v_7 = (0) \times (0) \times F_2, v_8 = (0) \times F_1 \times (0), v_9 = R_1 \times (0) \times F_2, v_{10} = R_1 \times F_1 \times (0)\}$ is the vertex set of ξ_R and so ξ_R is the graph illustrated in Figure 4.
Now let R satisfies (ii), then it is easy to verify that ξ_R is one of the graphs illustrated in Figure 5 respectively with the vertex set $\{v_1 = (0) \times F_1, v_2 = R_1 \times (0), v_3 = \mathfrak{m}_1 \times F_1, v_4 = \mathfrak{m}_1^2 \times F_1, v_5 = \mathfrak{m}_1 \times (0), v_6 = \mathfrak{m}_1^2 \times (0)\}$ and $\{v_1 = (0) \times F_1, v_2 = R_1 \times (0), v_3 = \mathfrak{m}_1 \times F_1, v_4 = \mathfrak{m}_1^2 \times F_1, v_5 = \mathfrak{m}_1 \times (0), v_6 = \mathfrak{m}_1^3 \times (0), v_7 = \mathfrak{m}_1^2 \times (0), v_8 = \mathfrak{m}_1^3 \times F_1\}$. □

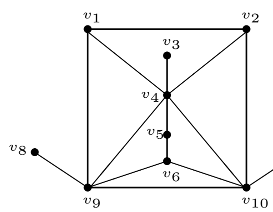

Figure 4. $\xi_{R_1 \times F_1 \times F_2}$ **with** $|\mathbb{I}^*(R_1)| = 1$

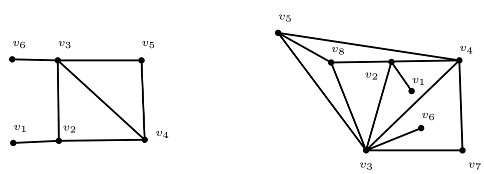

Figure 5. $\xi_{R_1 \times F_1}$ **with** $2 \leq |\mathbb{I}^*(R_1)| \leq 3$

It is well known that every commutative Artinian ring R isomorphic to the direct product of finitely many local rings. Using this, we have the following corollary which gives a characterization for ξ_R to be planar for a commutative non-local Artinian ring R.

Corollary 3. Let R be a non-local Artinian ring. Then ξ_R is planar if and only if one of the following conditions is fulfilled.

(a) $R = F_1 \times F_2$ or $R = F_1 \times F_2 \times F_3$, where each F_i is a field;

(b) $R = R_1 \times F_1 \times F_2$, where $(R_1, \mathfrak{m}_1)$ is a local ring, where $\mathfrak{m}_1$ is the only non-trivial ideal in R_1 and F_i is a field for $i = 1, 2$;

(c) $R = R_1 \times R_2$, where $(R_i, \mathfrak{m}_i)$ is a local ring for $i = 1, 2$ such that $|\mathbb{I}^*(R_1)| + |\mathbb{I}^*(R_2)| \leq 3$;

(d) $R = R_1 \times F_1$, where $(R_1, \mathfrak{m}_1)$ is a local ring such that $\{\mathfrak{m}_1, \mathfrak{m}_1^2\}$ or $\{\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3\}$ is the set of all non-zero proper ideals of R_1 and F_1 is a field.

Now, we classify all Artinian rings whose annihilator-inclusion ideal graphs have genus one. Given a connected graph G, we say that a vertex v of G is a cut vertex if $G - v$ is disconnected. A block is a maximal connected subgraph of G having no cut vertices. A result of Battle, Harary, Kodama, and Youngs states that the genus of a graph is the sum of the genus of its blocks [6]. For example, the graph in Figure 6 has two blocks, both isomorphic to $K_{3,3}$, and so has genus 2.

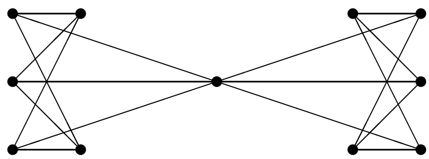

Figure 6. **A graph with two blocks, each isomorphic to** $K_{3,3}$

Proposition 1. Let $R \simeq F_1 \times F_2 \times \cdots \times F_n$ $(n \geq 3)$ where $F_1, \ldots, F_n$ are fields. Then $\lambda(\xi_R) = 1$ if and only if $n = 4$.

Proof. If $n = 4$, then the vertex set of ξ_R is $\{v_1 = F_1 \times F_2 \times (0) \times (0), v_2 = F_1 \times (0) \times F_3 \times (0), v_3 = F_1 \times (0) \times (0) \times F_4), v_4 = (0) \times F_2 \times F_3 \times (0), v_5 = (0) \times F_2 \times (0) \times F_5, v_6 = (0) \times (0) \times F_3 \times F_4, v_7 = F_1 \times F_2 \times F_3 \times (0), v_8 = F_1 \times F_2 \times (0) \times F_4, v_9 = F_1 \times (0) \times F_3 \times F_4, v_{10} = (0) \times F_2 \times F_3 \times F_4, v_{11} = F_1 \times (0) \times (0) \times (0), v_{12} = (0) \times F_2 \times (0) \times (0), v_{13} = (0) \times (0) \times F_3 \times (0), v_{14} = (0) \times (0) \times (0) \times F_4\}$ and the graph ξ_R is illustrated in Figure 7. This implies that $\lambda(\xi_R) = 1$ and the proof is complete. Conversely, let $\lambda(\xi_R) = 1$. It follows from Corollary 3 that $n \geq 4$. If $n \geq 5$, then ξ_R contains $K_{3,7}$ whose bipartite sets are

$$X = \{(0) \times F_2 \times F_3 \times \cdots \times F_n, (0) \times (0) \times F_3 \times \cdots \times F_n, F_1 \times (0) \times F_3 \times \cdots \times F_n\} \quad \text{and}$$

$Y = \{F_1 \times F_2 \times (0) \times \cdots \times (0), F_1 \times F_2 \times F_3 \times (0) \times \cdots \times (0), F_1 \times F_2 \times (0) \times F_4 \times (0) \times \cdots \times (0), F_1 \times F_2 \times (0) \times (0) \times F_5 \times \cdots \times F_n, F_1 \times F_2 \times F_3 \times (0) \times F_5 \times \cdots \times F_n, F_1 \times F_2 \times (0) \times F_4 \times \cdots \times F_n, F_1 \times F_2 \times F_3 \times F_4 \times (0) \times F_6 \times \cdots \times F_n\}$ which leads to a contradiction by Theorem E. Thus $n = 4$, and the proof is complete. □

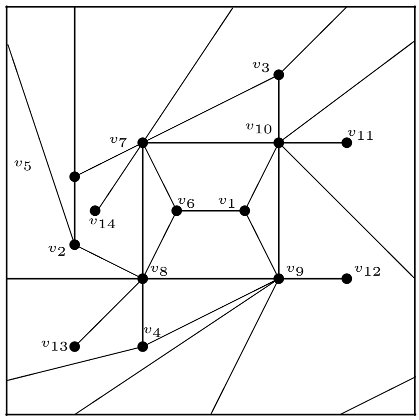

Figure 7. **toroidal embedding of** $\xi(F_1 \times F_2 \times F_3 \times F_4)$

The following results are very useful in the subsequent sections.

Theorem 10. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a commutative ring with identity where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and $n \geq 2$. Let n_i be the nilpotency of $\mathfrak{m}_i$. Then $\lambda(\xi_R) = 1$ if and only if $n = 2$ and one of the following conditions hold:*

(i) $n_1 = 2, n_2 = 3, \mathfrak{m}_1$ is the only non-trivial ideal in R_1 and $\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3$ are the only non-trivial ideals in R_2;

(ii) $n_1 = 3, n_2 = 3, \mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in R_1 and $\mathfrak{m}_2$ is the only non-trivial ideal in R_2;

(iii) $n_1 = n_2 = 2, \mathfrak{m}_1$ is the only non-trivial ideal in R_1 and R_2 has exactly three non-trivial ideal I, J and K different from $\mathfrak{m}_2$;

(iv) $n_1 = n_2 = 2, \mathfrak{m}_2$ is the only non-trivial ideal in R_2 and R_1 has exactly three non-trivial ideal I, J and K different from $\mathfrak{m}_1$.

Proof. Assume that $\lambda(\xi_R) = 1$. If $n \geq 3$, then ξ_R contains $K_{3,7}$ whose bipartite sets are $X = \{R_1 \times R_2 \times \mathfrak{m}_3 \times R_4 \times \cdots \times R_n, R_1 \times \mathfrak{m}_2 \times R_3 \times R_4 \times \cdots \times R_n, R_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times R_4 \times \cdots \times R_n\}$ and $Y = \{(0) \times R_2 \times R_3 \times R_4 \times \cdots \times R_n, \mathfrak{m}_1 \times R_2 \times R_3 \times R_4 \times \cdots \times R_n, (0) \times R_2 \times R_3 \times R_4 \times \cdots \times R_n, \mathfrak{m}_1 \times R_2 \times \mathfrak{m}_3 \times R_4 \times \cdots \times R_n, (0) \times \mathfrak{m}_2 \times R_3 \times R_4 \times \cdots \times R_n, \mathfrak{m}_1 \times \mathfrak{m}_2 \times R_3 \times R_4 \times \cdots \times R_n, (0) \times \mathfrak{m}_2 \times \mathfrak{m}_3 \times R_4 \times \cdots \times R_n\}$ which is a contradiction. Hence $n = 2$. It follows by Theorem 8, that $|\mathbb{I}^*(R_1)| + |\mathbb{I}^*(R_2)| \geq 4$.

Claim 1 $|\mathbb{I}^*(R_1)| \leq 3$ or $|\mathbb{I}^*(R_2)| \leq 3$.

Proof of claim 1. Let, to contrary, $|\mathbb{I}^*(R_1)| \geq 4$ and $|\mathbb{I}^*(R_2)| \geq 4$. Let I_1, J_1 and I_2, J_2 be non-trivial ideals in R_1 and R_2 different from $\mathfrak{m}_1$ and $\mathfrak{m}_2$ respectively. Then the subgraph induced by $\{u_1 = R_1 \times \mathfrak{m}_2, u_2 = R_1 \times J_2, u_3 = R_1 \times I_2, v_1 = J_1 \times \mathfrak{m}_2, v_2 = \mathfrak{m}_1 \times R_2, v_3 = I_1 \times R_2, v_4 = J_1 \times R_2, v_5 = (0) \times \mathfrak{m}_2, v_6 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_7 = I_1 \times \mathfrak{m}_2\}$ contains $K_{3,7}$ whose bipartite sets are $X = \{u_1, u_2, u_3\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. By Theorem E, $\lambda(\xi_R) > 1$ which is a contradiction, which proves that claim.

Claim 2 $|\mathbb{I}^*(R_1)| \neq 2$ and $|\mathbb{I}^*(R_2)| \neq 2$.

Proof of claim 2. Let, to contrary, $|\mathbb{I}^*(R_1)| = 2$ or $|\mathbb{I}^*(R_2)| = 2$. Without loss of generality suppose that $|\mathbb{I}^*(R_1)| = 2$. Then $|\mathbb{I}^*(R_2)| \geq 2$. Let $|\mathbb{I}^*(R_2)| = 3$. If $\dim \mathfrak{m}_i/\mathfrak{m}_i^2 \geq 2$, for $i = 1, 2$, then it follows from Observation 1 that there are distinct non-trivial ideals I, J, K of R_2 such that $I, J, K \notin \{\mathfrak{m}_i^j | j \in \mathbb{N}\}$. By Nakayama's lemma, we get $\mathfrak{m}_i \neq \mathfrak{m}_i^2$. This implies that $\{I, J, K, \mathfrak{m}_i, \mathfrak{m}_i^2\} \subseteq \mathbb{I}^*(R_i)$ and so, $|\mathbb{I}^*(R_i)| \geq 5$ for $i = 1, 2$. This is in contradiction to the assumption that $|\mathbb{I}^*(R_1)| = 2$ or $|\mathbb{I}^*(R_2)| = 3$. Therefore, $\dim \mathfrak{m}_i/\mathfrak{m}_i^2 = 1$ for $i = 1, 2$. Then it follows from Theorem A that $\mathbb{I}^*(R_1) = \{\mathfrak{m}_1, \mathfrak{m}_1^2\}$ and $\mathbb{I}^*(R_2) = \{\mathfrak{m}_2, \mathfrak{m}_2^2, \mathfrak{m}_2^3\}$. Then the subgraph induced by $\{u_1 = \mathfrak{m}_1 \times \mathfrak{m}_2, u_2 = \mathfrak{m}_1 \times R_2, u_3 = R_1 \times \mathfrak{m}_2, v_1 = \mathfrak{m}_1^2 \times \mathfrak{m}_2^3, v_2 = \mathfrak{m}_1^2 \times R_2, v_3 = R_1 \times \mathfrak{m}_2^2, v_4 = R_1 \times \mathfrak{m}_2^3, v_5 = \mathfrak{m}_1 \times \mathfrak{m}_2^3, v_6 = \mathfrak{m}_1 \times \mathfrak{m}_2^2, v_7 = \mathfrak{m}_1^2 \times \mathfrak{m}_2^2, v_8 = \mathfrak{m}_1^2 \times \mathfrak{m}_2\}$ contains $K_{3,8}$ whose bipartite sets are $X = \{u_1, u_2, u_3\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$. Hence by Theorem E, $\lambda(\xi_R) > 1$, a contradiction. Let $|\mathbb{I}^*(R_2)| \geq 4$. Consider I, J and K are non-trivial ideals different from $\mathfrak{m}_2$ in R_2, Then the subgraph induced by $\{u_1 = R_1 \times I, u_2 = R_1 \times J, u_3 = R_1 \times K, u_4 = R_1 \times \mathfrak{m}_2, v_1 = (0) \times \mathfrak{m}_2, v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_3 = \mathfrak{m}_1^2 \times \mathfrak{m}_2, v_4 = (0) \times R_2, v_5 = \mathfrak{m}_1^2 \times R_2\}$ contains $K_{4,5}$ whose bipartite sets are $X = \{u_1, u_2, u_3\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5\}$. Hence by Theorem E, $\lambda(\xi_R) > 1$, a contradiction.

Now let $|\mathbb{I}^*(R_2)| = 2$, then it can be shown as in Claim 2 that $\mathfrak{m}_1, \mathfrak{m}_1^2$ and $\mathfrak{m}_2, \mathfrak{m}_2^2$ are the only non-trivial ideals in R_1 and R_2 respectively. Consider the subgraph G of ξ_R induced by the non-trivial ideals $u_1 = \mathfrak{m}_1 \times R_2, u_2 = R_1 \times \mathfrak{m}_2, u_3 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_1 = \mathfrak{m}_1^2 \times \mathfrak{m}_2, v_2 = \mathfrak{m}_1^2 \times R_2, v_3 = R_1 \times \mathfrak{m}_2^2, v_4 = \mathfrak{m}_1^2 \times \mathfrak{m}_2^2, v_5 = \mathfrak{m}_1 \times \mathfrak{m}_2^2, x_1 = (0) \times R_2, x_2 = R_1 \times (0), x_3 = \mathfrak{m}_1 \times (0), x_4 = (0) \times \mathfrak{m}_2$. Let $G' = (G - \{x_3, x_4\}) - \{u_1u_2, u_1u_3, u_2u_3, v_1v_3, v_1v_5, v_2v_3, v_2v_5\}$ and $G'' = G' - \{x_1, x_2\}$. Then $G'' \cong K_{3,5}$ and so $\lambda(G'') = 1$. Since $\lambda(G'') \leq \lambda(G') \leq \lambda(G)$ and $\lambda(G) = 1$, $\lambda(G') = 1$. Note that $|V(G)| = 10$ and $|E(G)| = 20$. Then by Euler's formula, there are 10 faces when drawing G' on a torus. Fix a representation of G' and let $\{F_1', \ldots, F_{10}'\}$ be of faces of G' corresponding to the representation. Let $\{F_1'', \ldots, F_r''\}$ be the set of faces of G'' obtained by deleting x_1, x_2 and all the edges incident with x_1, x_2 from the representation of G'. Notice that $G'' \cong K_{3,5}$. From the fact that $n - m + f = 2 - 2g$, $K_{3,5}$ has 7 faces, six with 4 boundary edges and one with 6 boundary edges. So $r = 7$. Moreover, for every i, each boundary of F'' cannot have consecutive repetition of a single edge. Therefore in $K_{3,5}$, the only way to have a closed walk of length 6 without consecutive repetition of single edge is to have 6-cycle. Then in $K_{3,5}$, all faces boundaries are 4-cycle but with one 6-cycle. We may assume that the boundary of F''_7 is 6. Now $\{F'_1, \ldots, F'_{10}\}$ can be recovered by inserting x_1, x_2 onto the representation corresponds to $\{F''_1, \ldots, F''_7\}$. Note that $x_1x_2 \in E(G')$. Hence x_1, x_2

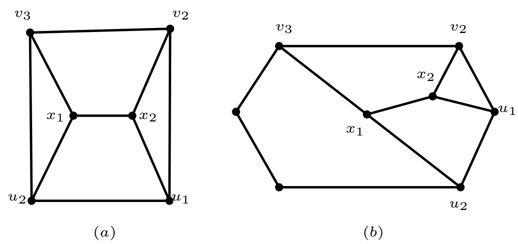

Figure 8.

should be inserted to the same face say F''_m of G'' to avoid crossing. Also note that $x_1v_3, x_1u_2, x_2u_1, x_2v_2 \in E(G')$ and therefore v_3, u_2, u_1, v_2 are the boundary vertices of F''_m. Consider the following edges of G: $e_1 = x_1v_3, e_2 = x_1u_2, e_3 = x_2u_1, e_4 = x_2v_2, e_5 = x_1x_2, e_6 = v_3u_1, e_7 = v_2u_2$. After inserting x_1, x_2 and e_i, $i = 1$ to 5 into the face $F''_m, m \neq 7$, we obtain Fig 8 (a) as above. Then the edge e_6 can be inserted into the face F''_7. But there is no other face with v_2 and u_2 as the boundary vertices and so there is no way to insert the edge e_7 without crossing in the embedding if G. After inserting x_1, x_2 and e_i, $i = 1$ to 5 into the face F''_7, we obtain Fig 8 (b) as above. Then the edge e_5 can be inserted in to the face F''_m where $m \neq 7$. But there is no other face with v_2 and u_2 as the boundary vertices and so there is no way to insert the edge e_7 without crossing in the embedding of G. Hence we conclude that $\lambda(\xi_R) > 1$, a contraction.

Now without loss of generality, suppose that $|\mathbb{I}^*(R_1)| \leq 3$. Then $|\mathbb{I}^*(R_1)| = 1$ or $|\mathbb{I}^*(R_1)| = 3$. We consider two cases.

Case 1 $|\mathbb{I}^*(R_1)| = 1$. If $|\mathbb{I}^*(R_2)| \geq 5$, then let I, J, K and L are non-trivial ideals different from $\mathfrak{m}_2$ in R_2. Then the subgraph induced by $\{u_1 = R_1 \times I, u_2 = R_1 \times J, u_3 = R_1 \times K, u_4 = R_1 \times L, v_1 = (0) \times \mathfrak{m}_2, v_2 = R_1 \times \mathfrak{m}_2, v_3 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_4 = \mathfrak{m}_1 \times R_2, v_5 = (0) \times R_2\}$ contains $K_{4,5}$ whose bipartite sets are $X = \{u_1, u_2, u_3, u_4\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5\}$, By Theorem E, $\gamma(\xi_R) > 1$ which is a contradiction. Therefore $3 \leq |\mathbb{I}^*(R_2)| \leq 4$.

Let $|\mathbb{I}^*(R_2)| = 4$. We claim that $\mathfrak{m}_2^2 = 0$. Suppose to the contrary that $\mathfrak{m}_2^2 \neq 0$. By Nakayama's lemma $\mathfrak{m}_2 \neq \mathfrak{m}_2^2$. If $\dim(\mathfrak{m}_2/\mathfrak{m}_2^2) \geq 2$, then it follows from Observation 1 that there are distinct non-trivial ideals I, J, K of R_2 such that $I, J, K \notin \{\mathfrak{m}^i | i \in \mathbb{N}\}$. This implies that $\{I, J, K\mathfrak{m}_2, \mathfrak{m}_2^2\} \subseteq \mathbb{I}^*(R_2)$ and so $|\mathbb{I}^*(R_2)| \geq 5$, a contradiction with the assumption that $|\mathbb{I}^*(R_2)| = 4$. Therefore, $\dim(\mathfrak{m}_2/\mathfrak{m}_2^2) = 1$. Then it follows from

Theorem A that $\mathbb{I}^*(R_2) = \{\mathfrak{m}^i \mid i \in \{1, 2, 3, 4\}\}$. Then the subgraph induced by $\{u_1 = \mathfrak{m}_1 \times \mathfrak{m}_2^2, u_2 = R_1 \times \mathfrak{m}_2^2, u_3 = R_1 \times m_2^3, u_4 = R_1 \times \mathfrak{m}_2^4, v_1 = (0) \times \mathfrak{m}_2, v_2 = \mathfrak{m}_1 \times m_2, v_3 = \mathfrak{m}_1 \times R_2, v_4 = (0) \times R_2, v_5 = R_1 \times \mathfrak{m}_2, w_1 = R_1 \times \mathfrak{m}_2^3, w_2 = R_1 \times \mathfrak{m}_2^2\}$ contains a subdivision of $K_{4,5}$ (see Figure 9). By Theorem E, $\gamma(\xi_R) > 1$ which is a contradiction. Then $\mathfrak{m}_2^2 = 0$ and R_2 has exactly three non-trivial ideal I, J and K different from $\mathfrak{m}_2$ and so R satisfies in (iv). Moreover if $|\mathbb{I}^*(R_2)| = 3$, then R satisfies in (i), as desired.

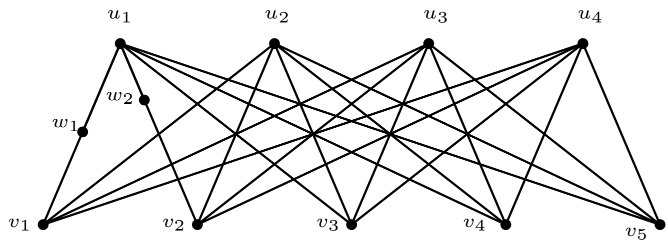

Figure 9. **A subdivision of** $K_{4,5}$

Case 2 $|\mathbb{I}^*(R_1)| = 3$. If $|\mathbb{I}^*(R_2)| \geq 3$, then let I and J are non-trivial ideals of R_2 different from $\mathfrak{m}_2$. Then the sub graph induced by $\{u_1 = R_1 \times \mathfrak{m}_2, u_2 = R_1 \times I, u_3 = R_1 \times J, v_1 = \mathfrak{m}_1^2 \times \mathfrak{m}_2, v_2 = \mathfrak{m}_1^3 \times \mathfrak{m}_2, v_3 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_4 = \mathfrak{m}_1^2 \times R_2, v_5 = \mathfrak{m}_1 \times R_2, v_6 = \mathfrak{m}_1^3 \times R_2, v_7 = (0) \times R_2\}$ contains $K_{3,7}$ whose bipartite sets are $X = \{u_1, u_2, u_3\}$ and $Y = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. By Theorem E $\lambda(\xi_R) > 1$, a contradiction. Hence $|\mathbb{I}^*(R_2)| = 1$. Therefore R satisfies (ii).
Conversely if R satisfies (i) or (iii), then it is easy to verify that ξ_R is the graphs illustrated in Figures 10 and Figure 11 with vertex sets $\{v_1 = R_1 \times \mathfrak{m}_2, v_2 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_3 = \mathfrak{m}_1 \times \mathfrak{m}_2^2, v_4 = \mathfrak{m}_1 \times \mathfrak{m}_2^3, v_5 = \mathfrak{m}_1 \times R_2, v_6 = R_1 \times \mathfrak{m}_2^2, v_7 = (0) \times \mathfrak{m}_2, v_8 = R_1 \times \mathfrak{m}_2^3, v_9 = (0) \times R_2, v_{10} = R_1 \times (0), v_{11} = (0) \times \mathfrak{m}_2^2, v_{12} = \mathfrak{m}_1 \times (0), v_{13} = (0) \times \mathfrak{m}_2^3\}$ and $\{v_1 = R_1 \times \mathfrak{m}_2, v_2 = R_1 \times J, v_3 = R_1 \times I, v_4 = R_1 \times K, v_5 = \mathfrak{m}_1 \times R_1, v_6 = \mathfrak{m}_1 \times I, v_7 = \mathfrak{m}_1 \times K, v_8 = \mathfrak{m}_1 \times J, v_9 = \mathfrak{m}_1 \times \mathfrak{m}_2, v_{10} = R_1 \times (0), v_{11} = (0) \times R_2, v_{12} = (0) \times \mathfrak{m}_2, v_{13} = (0) \times I, v_{14} = (0) \times K, v_{15} = (0) \times J, v_{16} = \mathfrak{m}_1 \times (0)\}$ respectively. □

The following result is very useful in the subsequent sections.

Theorem 11. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring with $\mathfrak{m}_i \neq 0$ and each F_j is a field, $n \geq 1, m \geq 1$ and $n + m \geq 3$. Then $\lambda(\xi_R) = 1$ if and only if one of the following conditions hold:*

(i) $R = R_1 \times F_1 \times F_2$ and $\mathfrak{m}_1$ and $\mathfrak{m}_1^2$ are the only non-trivial ideals in R_1;

(ii) $R = R_1 \times R_2 \times F_1$ and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are the only non-trivial ideals in R_1 and R_2 respectively.

Proof. Let $\lambda(\xi_R) = 1$. If $n + m \geq 5$, then an argument similar to that describe in the proof of 1 shows that ξ_R contains $K_{3,7}$ which is a contradiction. Thus $m + n \leq 4$.

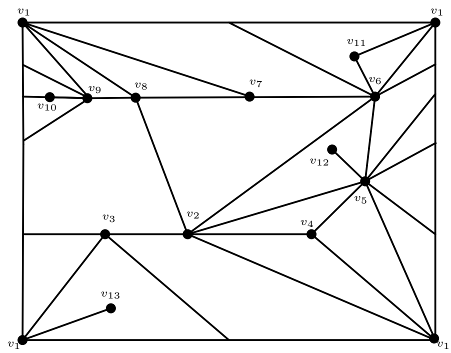

Figure 10. **Torus embedding of $\xi_{R_1\times R_2}$ with $n_1 = 2, n_2 = 4$**

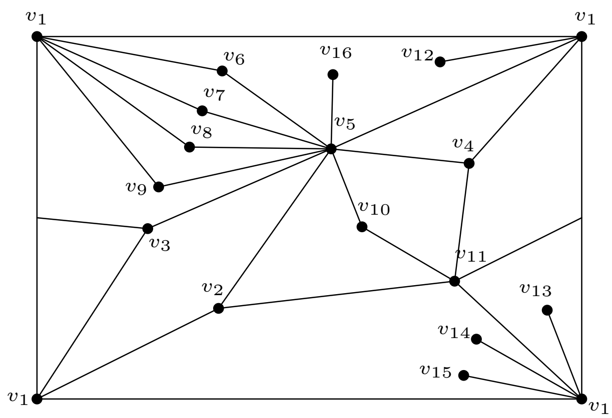

Figure 11. Torus embedding of $\xi_{R_1\times R_2}$ with $\mathfrak{m}_1$ is the only non-trivial ideals in R_1 and I, J, K are the only non-trivial ideal in R_2 different from $\mathfrak{m}_2$

Let $n + m = 4$ and assume without loss of generality, that $n = 1$ and $m = 3$. Then ξ_R contains a subdivision of $K_{4,5}$ whose vertices are $u_1 = (0) \times F_1 \times F_2 \times F_3, u_2 = \mathfrak{m}_1 \times F_1 \times F_2 \times F_3, u_3 = R_1 \times F_1 \times F_2 \times (0), u_4 = \mathfrak{m}_1 \times F_1 \times F_2 \times (0), v_1 = \mathfrak{m}_1 \times F_1 \times (0) \times F_3, v_2 = \mathfrak{m}_1 \times (0) \times F_2 \times F_3, v_3 = R_1 \times F_1 \times (0) \times F_3, v_4 = R_1 \times (0) \times F_2 \times F_3, v_5 = R_1 \times (0) \times (0) \times F_3, w_1 = R_1 \times F_1 \times F_2 \times (0), w_2 = R_1 \times F_1 \times (0) \times (0)\}$ (see Figure 9) implying that $\lambda(\xi_R) \geq 2$ by Theorem E, a contradiction.
Hence $n + m \leq 3$. Consider two cases.

Case 1 $n = 1, m = 2$.
If R_1 has at least three ideals I, J and K different from $\mathfrak{m}_1$, then ξ_R contains $K_{3,7}$

whose bipartite sets are $X = \{\mathfrak{m}_1 \times (0) \times F_2, R_1 \times (0) \times F_2, \mathfrak{m}_1 \times F_1 \times F_2\}$ and $Y = \{R_1 \times F_1 \times (0), \mathfrak{m}_1 \times F_1 \times (0), I \times F_1 \times (0), J \times F_1 \times (0), K \times F_1 \times (0), I \times F_1 \times F_2, J \times F_1 \times F_2\}$ which is a contradiction. If R_1 has exactly three non-trivial ideal, then it can be shown as in the proof of Theorem 10 that $\mathfrak{m}_1, \mathfrak{m}_1^2, \mathfrak{m}_1^3$ are the only non-trivial ideals in R_1 and $\mathfrak{m}_1^4 = (0)$. Consider $S = \{u_1 = R_1 \times (0) \times (0), u_2 = R_1 \times (0) \times F_2, u_3 = \mathfrak{m}_1 \times (0) \times (0), v_1 = \mathfrak{m}_1^2 \times F_1 \times F_2, v_2 = \mathfrak{m}_1 \times F_1 \times F_2, v_3 = \mathfrak{m}_1^3 \times F_1 \times F_2, w_1 = \mathfrak{m}_1 \times (0) \times F_2, w_2 = \mathfrak{m}_1^2 \times (0) \times F_2, z_1 = \mathfrak{m}_1 \times F_1 \times (0), z_2 = R_1 \times F_1 \times (0), z_3 = \mathfrak{m}_1^2 \times F_1 \times (0)\}$. Then the subgraph induced by S in ξ_R contains two block, both isomorphic to $K_{3,3}$ as in Figure 12. Then $\lambda(S) = \lambda(K_{3,3}) + \lambda(K_{3,3}) \geq 2$ and it implies that $\lambda(\xi_R) \geq 2$, a contraction. Thus R_1 has at most two non-trivial ideal. We conclude from Theorem 11 (part i) that R_1 has exactly two non-trivial ideal and so R satisfies (i).

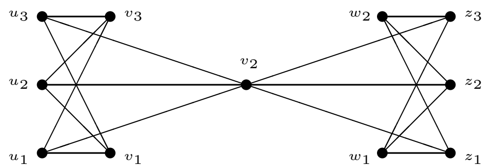

Figure 12. H

Case 2 $n = 2, m = 1$. If R_i has an ideal I different from $\mathfrak{m}_i$ for some $i \in \{1, 2\}$, say $i = 1$, then ξ_R contains $K_{4,5}$ whose bipartite sets are $X = \{R_1 \times R_2 \times (0), \mathfrak{m}_1 \times R_2 \times (0), R_1 \times \mathfrak{m}_2 \times (0), \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0)\}$ and $Y = \{\mathfrak{m}_1 \times R_2 \times F_1, R_1 \times \mathfrak{m}_2 \times F_1, \mathfrak{m}_1 \times \mathfrak{m}_2 \times F_1, I \times R_2 \times F_1, I \times \mathfrak{m}_2 \times F_1\}$ that leads to a contradiction by Theorem E. Hence R satisfies (ii).

Conversely, if R satisfies (i), then the vertex set of ξ_R is $\{v_1 = \mathfrak{m}_1 \times F_1 \times F_2, v_2 = \mathfrak{m}_1^2 \times F_1 \times F_2, v_3 = R_1 \times (0) \times F_2, v_4 = \mathfrak{m}_1 \times (0) \times F_2, v_5 = R_1 \times F_1 \times (0), v_6 = \mathfrak{m}_1 \times F_1 \times (0), v_7 = \mathfrak{m}_1 \times (0) \times (0), v_8 = R_1 \times (0) \times (0), v_9 = \mathfrak{m}_1^2 \times (0) \times F_2, v_{10} = \mathfrak{m}_1^2 \times F_1 \times (0), v_{11} = \mathfrak{m}_1^2 \times (0) \times (0), v_{12} = (0) \times F_1 \times F_2, v_{13} = (0) \times (0) \times F_2, v_{14} = (0) \times F_1 \times (0)\}$ and Figure 13 implies that $\lambda(\xi_R) = 1$.

If R satisfies (ii), then the vertex set of ξ_R is $V(\xi_R) = \{v_1 = \mathfrak{m}_1 \times R_2 \times F_1, v_2 = R_1 \times \mathfrak{m}_2 \times F_1, v_3 = R_1 \times R_2 \times (0), v_4 = (0) \times R_2 \times F_1, v_5 = R_1 \times (0) \times F_1, v_6 = \mathfrak{m}_1 \times (0) \times F_1, v_7 = (0) \times \mathfrak{m}_2 \times F_1, v_8 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times (0), v_9 = \mathfrak{m}_1 \times \mathfrak{m}_2 \times F_1, v_{10} = R_1 \times \mathfrak{m}_2 \times (0), v_{11} = \mathfrak{m}_1 \times R_2 \times (0), v_{12} = \mathfrak{m}_1 \times (0) \times (0), v_{13} = (0) \times \mathfrak{m}_2 \times (0), v_{14} = (0) \times (0) \times F_1, v_{15} = (0) \times R_2 \times (0), v_{16} = R_1 \times (0) \times (0)\}$ and it follows from Figure 14 that $\lambda(\xi_R) = 1$ and the proof is the complete. □

We have the following corollary which gives a characterization for ξ_R to be has genus one for a commutative non-local Artinian ring R with $R \ncong F \times R_1$.

Corollary 4. Let R be a non-local Artinian ring and $R \ncong F \times R_1$ where F is a field and R_1 is a ring . Then $\lambda(\xi_R) = 1$ if and only if one of the following conditions is fulfilled:

(a) $R = F_1 \times F_2 \times F_3 \times F_4$, where F_i is a field for $i = 1, 2, 3, 4$.

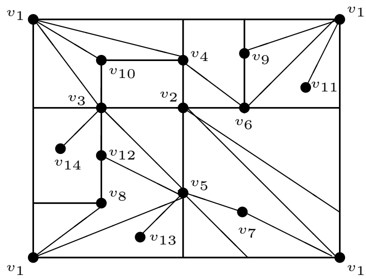

Figure 13. $\xi(F_1 \times F_2 \times R_3)$

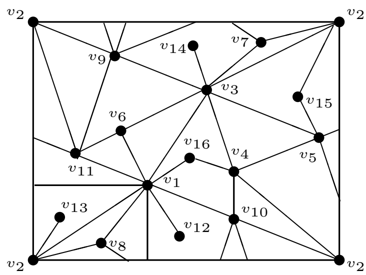

Figure 14. $\xi(F_1 \times R_2 \times R_3)$

(b) $R = F_1 \times F_2 \times R_3$, where F_i is a field for $i = 1, 2$ and $|\mathbb{I}(R_3)| = \{\mathfrak{m}_3, \mathfrak{m}_3^2\}$.

(c) $R = F_1 \times R_2 \times R_3$, where F_1 is a field and $|\mathbb{I}(R_1)| = |\mathbb{I}(R_2)| = 1$.

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