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Total domination in cubic Knödel graphs

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Abstract: A subset D of vertices of a graph G is a *dominating set* if for each $u \in V(G)\backslash D$, u is adjacent to some vertex $v \in D$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. A set $D \subseteq V(G)$ is a *total dominating set* if for each $u \in V(G)$, u is adjacent to some vertex $v \in D$. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. For an even integer $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, a *Knödel graph* $W_{\Delta,n}$ is a Δ-regular bipartite graph of even order n, with vertices (i,j), for $i = 1,2$ and $0 \leq j \leq \frac{n}{2} - 1$, where for every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1,j)$ and every vertex $(2,(j+2^k-1) \bmod \frac{n}{2})$, for $k = 0,1,\ldots,\Delta-1$. In this paper, we determine the total domination number in 3-regular Knödel graphs $W_{3,n}$.

Keywords: Knödel graph, domination number, total domination number, Pigeonhole Principle.

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1. Introduction

For graph theory notation and terminology not given here, we refer to [14]. Let $G = (V, E)$ denote a simple graph of order $n = |V(G)|$ and size $m = |E(G)|$. Two vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$. The *open neighborhood* of a vertex $u \in V(G)$ is denoted by $N(u) = \{v \in V(G)|uv \in E(G)\}$ and for a vertex set $S \subseteq V(G)$, $N(S) = \bigcup_{u \in S} N(u)$. The cardinality of $N(u)$ is called the *degree* of u and is denoted by $\deg(u)$, (or $\deg_G(u)$ to refer it to G). The *maximum degree* and *minimum degree* among all vertices in G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. A graph G is a *bipartite graph* if its vertex set can be divide into two disjoint sets X and Y such that each edge in $E(G)$ connects a vertex in X with a vertex in Y. A set $D \subseteq V(G)$ is a *dominating set* if for each $u \in V(G) \setminus D$, u is adjacent to some vertex $v \in D$. The *domination number*, $\gamma(G)$ of G, is the minimum cardinality of a dominating set of G. A set $D \subseteq V(G)$ is a *total dominating set* if for each $u \in V(G)$, u is adjacent to some vertex $v \in D$. The *total domination number*, $\gamma_t(G)$ of G, is the minimum cardinality of a total dominating set of G. The concept of domination theory is a widely studied concept in graph theory and for a comprehensive study see, for example [14, 15].

An interesting family of graphs namely *Knödel graphs* have been introduced about 1975 [17], and have been studied seriously by some authors since 2001, see for example [1–4, 7, 8, 10]. For an even integer $n \geq 2$ and $1 \leq \Delta \leq \lfloor \log_2 n \rfloor$, a *Knödel graph* $W_{\Delta,n}$ is a Δ-regular bipartite graph of even order n, with vertices (i, j), for $i = 1, 2$ and $0 \leq j \leq \frac{n}{2} - 1$, where for every j, $0 \leq j \leq \frac{n}{2} - 1$, there is an edge between vertex $(1, j)$ and every vertex $(2, (j + 2^k - 1) \bmod \frac{n}{2})$, for $k = 0, 1, \ldots, \Delta - 1$ (see [20]). Knödel graphs, $W_{\Delta,n}$, are one of the three important families of graphs that they have good properties in terms of broadcasting and gossiping, see for example [5, 6, 9, 11–13, 16]. It is worth-noting that any Knödel graph is a Cayley graph and so it is a vertex-transitive graph (see [3]).

Xueliang et. al. [20] studied the domination number in 3-regular Knödel graphs $W_{3,n}$. They obtained exact domination number for $W_{3,n}$. Domination critical and stable Knödel graphs are studied in [18]. Some domination parameters in Knödel graphs are studied in [19]. In this paper, we determine the total domination number in 3-regular Knödel graphs $W_{3,n}$. We will prove the following.

Theorem 1. *For each even integer $n \geq 8$,*

$$\gamma_t(W_{3,n}) = 4\left\lceil \frac{n}{10} \right\rceil - \begin{cases} 0 & n \equiv 0, 6, 8 \pmod{10} \\ 2 & n \equiv 2, 4 \pmod{10}. \end{cases}$$

In Section 2, we prove some necessary Lemmas, and in the Section 3 we prove our main result. We need the following simple observation from number theory.

Observation 2. If a, b, c, d and x are positive integers such that $x^a - x^b = x^c - x^d \neq 0$, then $a = c$ and $b = d$.

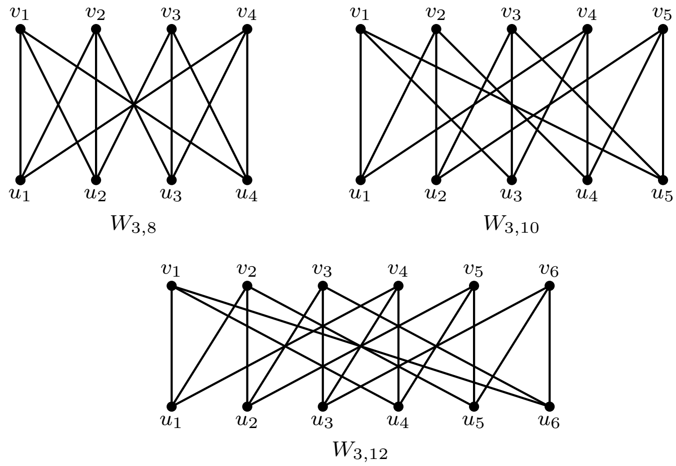

Figure 1. **New labeling of Knödel graphs $W_{3,8}, W_{3,10}$ and $W_{3,12}$.**

2. Necessary Lemmas

In this section we prove necessary lemmas we need for the proof of the main result. For simplicity, in this paper, we re-label the vertices of a Knödel graph as follows: we label $(1, i)$ by u_{i+1} for each $i = 0, 1, \ldots, \frac{n}{2} - 1$, and $(2, j)$ by v_{j+1} for $j = 0, 1, \ldots, \frac{n}{2} - 1$. Let $U = \{u_1, u_2, \cdots, u_{\frac{n}{2}}\}$ and $V = \{v_1, v_2, \ldots, v_{\frac{n}{2}}\}$. From now on, the vertex set of each Knödel graph $W_{\Delta,n}$ is $U \cup V$ such that U and V are the two partite sets of the graph. If S is a set of vertices of $W_{\Delta,n}$, then clearly, $S \cap U$ and $S \cap V$ partition S, $|S| = |S \cap U| + |S \cap V|$, $N(S \cap U) \subseteq V$ and $N(S \cap V) \subseteq U$. Note that two vertices u_i and v_j are adjacent if and only if $j \in \{i + 2^0 - 1, i + 2^1 - 1, \ldots, i + 2^{\Delta-1} - 1\}$, where the addition is taken in modulo $\frac{n}{2}$. Figure 1, shows new labeling of Knödel graphs $W_{3,8}, W_{3,10}$ and $W_{3,12}$.
For any subset $\{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$, we correspond a sequence based on the differences of the indices of u_j, $j = i_1, \ldots, i_k$, as follows.

Definition 1. For any subset $A = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ of U with $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ we define a sequence $n_1, n_2, \ldots, n_k$, namely **cyclic-sequence**, where $n_j = i_{j+1} - i_j$ for $1 \leq j \leq k-1$ and $n_k = \frac{n}{2} + i_1 - i_k$. For two vertices $u_{i_j}, u_{i_{j'}} \in A$ we define **index-distance** of u_{i_j} and $u_{i_{j'}}$ by $id(u_{i_j}, u_{i_{j'}}) = min\{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$.

Observation 3. Let $A = \{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\} \subseteq U$ be a set such that $1 \leq i_1 < i_2 < \cdots < i_k \leq \frac{n}{2}$ and let $n_1, n_2, \cdots, n_k$ be the corresponding cyclic-sequence of A. Then,
(1) $n_1 + n_2 + \cdots + n_k = \frac{n}{2}$.
(2) If $u_{i_j}, u_{i_{j'}} \in A$, then $id(u_{i_j}, u_{i_{j'}})$ equals to sum of some consecutive elements of the cyclic-sequence of A and $\frac{n}{2} - id(u_{i_j}, u_{i_{j'}})$ is sum of the remaining elements of the cyclic-sequence. Furthermore, $\{id(u_{i_j}, u_{i_{j'}}), \frac{n}{2} - id(u_{i_j}, u_{i_{j'}})\} = \{|i_j - i_{j'}|, \frac{n}{2} - |i_j - i_{j'}|\}$.

We henceforth use the notation $\mathscr{M}_\Delta = \{2^a - 2^b : 0 \le b < a < \Delta\}$ for $\Delta \ge 2$.

Lemma 1. *In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for two distinct vertices u_i and u_j, $N(u_i) \cap N(u_j) \neq \emptyset$ if and only if $id(u_i, u_j) \in \mathscr{M}_\Delta$ or $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$.*

Proof. Since $W_{\Delta,n}$ is vertex-transitive, for simplicity, we put $1 = i < j \le \frac{n}{2}$. We have $id(u_1, u_j) = \min\{j-1, \frac{n}{2} - (j-1)\}$ and so $\frac{n}{2} - id(u_1, u_j)\} = \max\{j-1, \frac{n}{2} - (j-1)\}$. Also, we have $N(u_1) = \{v_1, v_2, v_4, \dots, v_{2^{\Delta-1}}\}$ and $N(u_j) = \{v_j, v_{j+1}, v_{j+3}, \dots, v_{j+2^{\Delta-1}-1}\}$. First assume that $N(u_1) \cap N(u_j) \neq \emptyset$. Let $v_k \in N(u_1) \cap N(u_j)$. There exist two integers a and b such that $0 \le a < b \le \Delta - 1$ and $k \equiv 2^a \equiv j + 2^b - 1 (\text{mod } \frac{n}{2})$. Since $1 \le 2^a, 2^b, j \le \frac{n}{2}$, we have $1 \le j + 2^b - 1 < n$. If $1 \le j + 2^b - 1 \le \frac{n}{2}$, then $2^a = j + 2^b - 1$ and $j - 1 = 2^a - 2^b \in \mathscr{M}_\Delta$ and if $\frac{n}{2} < j + 2^b - 1 < n$, then $2^a = j + 2^b - 1 - \frac{n}{2}$ and $\frac{n}{2} - (j-1) = 2^b - 2^a \in \mathscr{M}_\Delta$. Therefore, by Observation 3, $id(u_i, u_j) \in \mathscr{M}_\Delta$ or $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$.
Conversely, suppose $id(u_1, u_j) \in \mathscr{M}_\Delta$ or $\frac{n}{2} - id(u_1, u_j) \in \mathscr{M}_\Delta$. Then $j - 1 \in \mathscr{M}_\Delta$ or $\frac{n}{2} - (j-1) \in \mathscr{M}_\Delta$. If $j - 1 \in \mathscr{M}_\Delta$, then we have $j - 1 = 2^a - 2^b$ for two integers $0 \le a, b \le \Delta - 1$. Then $2^a = j + 2^b - 1$ and $v_{2^a} \in N(u_1) \cap N(u_j)$. If $\frac{n}{2} - (j-1) \in \mathscr{M}_\Delta$, then we have $\frac{n}{2} - (j-1) = 2^c - 2^d$ for two integers $0 \le c, d \le \Delta - 1$. Now $2^c = j + 2^d - 1 - \frac{n}{2} \equiv j + 2^d - 1 \ (\text{mod } \frac{n}{2})$ and $v_{2^c} \in N(u_1) \cap N(u_j)$. Thus in each case, $N(u_i) \cap N(u_j) \neq \emptyset$. □

Lemma 2. *In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for two distinct vertices u_i and u_j, $|N(u_i) \cap N(u_j)| = 2$ if and only if $id(u_i, u_j) \in \mathscr{M}_\Delta$ and $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$.*

Proof. Without loss of generality, we assume that $1 \le i < j \le \frac{n}{2}$. Suppose that $|N(u_i) \cap N(u_j)| = 2$ and $v_k, v_{k'} \in N(u_i) \cap N(u_j)$ are two distinct vertices in V. There exist two integers a and b such that $0 \le a, b \le \Delta - 1$ and $k \equiv i + 2^a - 1 \equiv j + 2^b - 1 (\text{mod } \frac{n}{2})$. Similarly, there exist two integers a' and b' such that $0 \le a', b' \le \Delta - 1$ and $k' \equiv i + 2^{a'} - 1 \equiv j + 2^{b'} - 1 (\text{mod } \frac{n}{2})$. Now we have $j - i \equiv 2^b - 2^a \equiv 2^{b'} - 2^{a'} (\text{mod } \frac{n}{2})$. We know that $-\frac{n}{2} < 2^b - 2^a, 2^{b'} - 2^{a'} < \frac{n}{2}$. If $-\frac{n}{2} < 2^b - 2^a, 2^{b'} - 2^{a'} < 0$ or $0 < 2^b - 2^a, 2^{b'} - 2^{a'} < \frac{n}{2}$, then we have $2^b - 2^a = 2^{b'} - 2^{a'} \neq 0$. Observation 2 implies that $b = b'$ and therefore $k \equiv k' (\text{mod } \frac{n}{2})$ and $v_k = v_{k'}$, a contradiction. By symmetry, we assume that $0 < 2^b - 2^a < \frac{n}{2}$ and $-\frac{n}{2} < 2^{b'} - 2^{a'} < 0$. Since $0 < j - i < \frac{n}{2}$, we have $j - i = 2^b - 2^a$ and $\frac{n}{2} - (j - i) = 2^{a'} - 2^{b'}$ which implies that $j - i \in \mathscr{M}_\Delta$ and $\frac{n}{2} - (j - i) \in \mathscr{M}_\Delta$. Thus by Observation 3, $id(u_i, u_j) \in \mathscr{M}_\Delta$ and $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$.
Conversely, assume that $id(u_i, u_j) \in \mathscr{M}_\Delta$ and $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$ for two distinct vertices u_i and u_j. There exist two integers a and b such that $0 \le b < a \le \frac{n}{2}$ and $j - i = 2^a - 2^b$. Also there exist two integer a' and b' such that $0 \le a' < b' \le \frac{n}{2}$ and $\frac{n}{2} - (j - i) = 2^{b'} - 2^{a'}$. Now we have $i + 2^a - 1 = j + 2^b - 1$ and $i + 2^{a'} - 1 = j + 2^{b'} - 1 - \frac{n}{2} \equiv j + 2^{b'} - 1 (\text{mod } \frac{n}{2})$. We set $k = i + 2^a - 1$ and $k' = i + 2^{a'} - 1$. Then $v_k, v_{k'} \in N(u_i) \cap N(u_j)$ and $|N(u_i) \cap N(u_j)| \ge 2$. Notice that $k \not\equiv k' (\text{mod } \frac{n}{2})$, since otherwise $a = a'$ and $2^{b'} - 2^b = \frac{n}{2}$, a contradiction. Suppose that $|N(u_i) \cap N(u_j)| \ge 3$. Let $v_k, v_{k'}, v_{k''} \in N(u_i) \cap N(u_j)$ be three distinct vertices. Similar to the first part of

the proof, for v_k and $v_{k'}$, there exist two integers a'' and b'' such that $0 \leq a'', b'' \leq \Delta - 1$ and $k'' \equiv i + 2^{a''} - 1 \equiv j + 2^{b''} - 1 (\text{mod } \frac{n}{2})$ and thus $j - i \equiv 2^{a''} - 2^{b''} (\text{mod } \frac{n}{2})$. Since u_i and u_j are distinct, we have $a'' \neq b''$. If $a'' > b''$, then $j - i = 2^{a''} - 2^{b''}$ and it can be seen that $j - i = \frac{n}{2} - (2^a - 2^b) = \frac{n}{2} - (2^{a'} - 2^{b'})$ and Observation 2 implies that $a = a'$ and thus $v_k = v_{k'}$, a contradiction. If $a'' < b''$, then $j - i = \frac{n}{2} - (2^{a''} - 2^{b''})$ and it can be seen that $j - i = 2^b - 2^a = 2^{b'} - 2^{a'}$ and Observation 2 implies that $a = a'$, a contradiction. Consequently $|N(u_i) \cap N(u_j)| = 2$. □

Corollary 1. *(i) In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$, for each $1 \leq i < j \leq \frac{n}{2}$, $|N(u_i) \cap N(u_j)| = 1$ if and only if precisely one of the values $id(u_i, u_j)$ and $\frac{n}{2} - id(u_i, u_j)$ belongs to $\mathscr{M}_\Delta$.*
(ii) In the Knödel graph $W_{\Delta,n}$, there exist distinct vertices with two common neighbors if and only if $n = 2^a - 2^b + 2^c - 2^d$ and $a > b \geq 1, c > d \geq 1$.

Corollary 2. *Any three vertices in the Knödel graph $W_{\Delta,n}$ have at most one common neighbor. Indeed, any Knödel graph is a $K_{2,3}$-free graph.*

Lemma 3. *In the Knödel graph $W_{\Delta,n}$ with vertex set $U \cup V$ and $\Delta < \log_2(\frac{n}{2} + 2)$, we have:*
(i) $|N(u_i) \cap N(u_j)| \leq 1$, $1 \leq i < j \leq \frac{n}{2}$.
(ii) $|N(u_i) \cap N(u_j)| = 1$ if and only if $id(u_i, u_j) \in \mathscr{M}_\Delta$.

Proof. (i) Suppose to the contrary that $|N(u_i) \cap N(u_j)| > 1$, then by Corollary 2 we have $|N(u_i) \cap N(u_j)| = 2$. Then the Lemma 2 implies that $id(u_i, u_j) \in \mathscr{M}_\Delta$ and $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$. Thus $id(u_i, u_j) \leq 2^{\Delta-1} - 1$, $\frac{n}{2} - id(u_i, u_j) \leq 2^{\Delta-1} - 1$ and $\frac{n}{2} \leq 2^\Delta - 2$. This inequality implies that $\Delta \geq \log_2(\frac{n}{2} + 2)$, a contradiction. Hence $|N(u_i) \cap N(u_j)| \leq 1$, as desired.
(ii) Assume that $|N(u_i) \cap N(u_j)| = 1$. By Corollary 1, precisely one of the values $id(u_i, u_j)$ and $\frac{n}{2} - id(u_i, u_j)$ belongs to $\mathscr{M}_\Delta$. If $\frac{n}{2} - id(u_i, u_j) \in \mathscr{M}_\Delta$, then $\frac{n}{2} - id(u_i, u_j) \leq 2^{\Delta-1} - 1$ and so $2^\Delta - 2 - id(u_i, u_j) < 2^{\Delta-1} - 1$. Now , we have $2^{\Delta-1} - 1 < id(u_i, u_j)$ and so $\frac{n}{2} - id(u_i, u_j) < id(u_i, u_j)$, a contradiction by definition of index-distance. Therefore, $id(u_i, u_j) \in \mathscr{M}_\Delta$.
Conversely, Assume that $id(u_i, u_j) \in \mathscr{M}_\Delta$. Thus, $id(u_i, u_j) \leq 2^{\Delta-1} - 1$ and so $\frac{n}{2} - id(u_i, u_j) \geq \frac{n}{2} - 2^{\Delta-1} + 1 > 2^\Delta - 2 - 2^{\Delta-1} + 1 = 2^{\Delta-1} - 1$. Therefore, $\frac{n}{2} - id(u_i, u_j) \notin \mathscr{M}_\Delta$ and by Corollary 1 we have $|N(u_i) \cap N(u_j)| = 1$. □

Lemma 4. *Let $W_{\Delta,n}$ be a Knödel graph with vertex set $U \cup V$. For any non-empty subset $A \subseteq U$:*
(i) $\sum_{v \in N(A)} |N(v) \cap A| = \Delta|A|$.
(ii) The corresponding cyclic-sequence of A has at most $\Delta|A| - |N(A)|$ elements belonging to $\mathscr{M}_\Delta$.

Proof. Let $A \subseteq U$ be a non-emptyset.
(i) It is obvious that the induced subgraph graph $H = W_{\Delta,n}[A \cup N(A)]$ is a bipartite

graph and $|E(H)| = \sum_{u\in A} \deg_H(u) = \sum_{v\in N(A)} \deg_H(v)$. If $u \in A$, then $\deg_H(u) = \Delta$, and for $v \in N(A)$ we have $\deg_H(v) = |N(v)\cap A|$. Thus, $\sum_{u\in A} \deg_H(u) = \sum_{u\in A} \Delta = \Delta|A|$ and $\sum_{v\in N(A)} \deg_H(v) = \sum_{v\in N(A)} |N(v)\cap A|$. Consequently, $\sum_{v\in N(A)} |N(v)\cap A| = \Delta|A|$.

(ii) Suppose that $A = \{u_{i_1}, u_{i_2}, \ldots, u_{i_{|A|}}\}$, where $1 \le i_1 < i_2 < \cdots < i_{|A|} \le \frac{n}{2}$, and let $n_1, n_2, \ldots, n_{|A|}$ be the corresponding cyclic-sequence of A. For any vertex $v \in N(A)$, let $r(v) = |N(v) \cap A|$. Let $J = \{j : n_j \in \mathscr{M}_\Delta\}$ and $R = \Delta|A| - |N(A)|$. We prove that $R \ge |J|$. If $R \ge |A|$, then we have nothing to prove, since $|J| \le |A|$. Assume that $R < |A|$ and notice that by part (i),

$$R = \Delta|A| - |N(A)| = \sum_{v\in N(A)} |N(v)\cap A| - \sum_{v\in N(A)} 1 = \sum_{v\in N(A)} [r(v) - 1].$$

If $\{v \in N(A) : r(v) \ge 2\} = \emptyset$, then $R = 0$ and $J = \emptyset$, and so $R \ge |J|$. Thus assume that $\{v \in N(A) : r(v) \ge 2\} \ne \emptyset$. Then $R = \sum_{\substack{v\in N(A)\\ r(v)\ge 2}} [r(v) - 1]$.

Assume that there exists $v' \in N(A)$ such that $r(v') = |A|$. Then

$$R = r(v') - 1 + \sum_{\substack{v\in N(A)\\ r(v)\ge 2\\ v\ne v'}} [r(v) - 1] = |A| - 1 + \sum_{\substack{v\in N(A)\\ r(v)\ge 2\\ v\ne v'}} [r(v) - 1].$$

Since $R < |A|$, we obtain that $\sum_{\substack{v\in N(A)\\ r(v)\ge 2\\ v\ne v'}} [r(v) - 1] = 0$, $R = |A| - 1$, and for each $v \in N(A) \setminus \{v'\}$ we have $r(v) = 1$. Since $W_{\Delta,n}$ is vertex transitive, without loss of generality, we assume that $v' = v_{\frac{n}{2}}$.

According to the definition of a Knodel graph, there exist integers $0 \le a_{|A|} < a_{|A|-1} < \cdots < a_2 < a_1 \le \Delta - 1$ such that $i_j = \frac{n}{2} - 2^{a_j} + 1$ for each $1 \le j \le |A|$. Moreover, $n_j = i_{j+1} - i_j = 2^{a_{i_j}} - 2^{a_{i_{j+1}}} \in \mathscr{M}_\Delta$ for each $1 \le j \le |A| - 1$. Evidently, $i_{|A|} - i_{|A|-1} = n_1 + n_2 + \cdots + n_{|A|-1} = 2^{a_{|A|-1}} - 2^{a_{|A|}} \in \mathscr{M}_\Delta$ and $n_{|A|} = \frac{n}{2} - (i_{|A|} - i_{|A|-1})$. We show that $n_{|A|} \notin \mathscr{M}_\Delta$. Suppose to the contrary that $n_{|A|} \in \mathscr{M}_\Delta$. Since $n_{|A|} = \frac{n}{2} - (i_{|A|} - i_{|A|-1}) \in \mathscr{M}_\Delta$ and $i_{|A|} - i_{|A|-1} \in \mathscr{M}_\Delta$, by Observation 3, $id(u_{i_1}, u_{i_{|A|}}) \in \mathscr{M}_\Delta$ and $\frac{n}{2} - id(u_{i_1}, u_{i_{|A|}}) \in \mathscr{M}_\Delta$, and by Lemma 2, $|N(u_{i_1}) \cap N(u_{i_{|A|}})| = 2$. Now there exists $v'' \ne v_{\frac{n}{2}}$ such that $v'' \in N(u_{i_1}) \cap N(u_{i_{|A|}})$ and $r(v'') \ge 2$, a contradiction. Therefore, $n_{|A|} \notin \mathscr{M}_\Delta$. Since $n_j \in \mathscr{M}_\Delta$ for each $1 \le j \le |A| - 1$, we obtain that $|J| = |A| - 1 = R$. Thus there are at most $R = |A| - 1$ elements of the cyclic sequence of A which belong to $\mathscr{M}_\Delta$.

Next assume that $r(v) < |A|$ for any $v \in N(A)$. Let $X_v = \{j : u_{i_j}, u_{i_{j+1}} \in N(v) \cap A\}$. We prove that $J \subseteq \bigcup_{v\in N(A)} X_v$. Let $j \in J$. Then $n_j = i_{j+1} - i_j \in \mathscr{M}_\Delta$. By Observation 3, $n_j = i_{j+1} - i_j \in \{id(u_{i_j}, u_{i_{j+1}}), \frac{n}{2} - id(u_{i_j}, u_{i_{j+1}})\}$ and by Lemma 1, $|N(u_{i_j}) \cap N(u_{i_{j+1}})| \ge 1$. Let $v \in N(u_{i_j}) \cap N(u_{i_{j+1}})$. Then $u_{i_j}, u_{i_{j+1}} \in N(v) \cap A$. Therefore

$j \in X_v$ and $j \in \bigcup_{v\in N(A)} X_v$ that implies $J \subseteq \bigcup_{v\in N(A)} X_v$. Then $|J| \leq |\bigcup_{v\in N(A)} X_v|$. Observe that $X_v = \{j : u_{i_j} \in N(v)\cap A\} - \{j : u_{i_j} \in N(v)\cap A, u_{i_{j+1}} \notin N(v)\cap A\}$, and $|\{j : u_{i_j} \in N(v)\cap A\}| = |N(v)\cap A| = r(v)$. Since $N(v)\cap A \subsetneq A$, we have $\{j : u_{i_j} \in N(v)\cap A, u_{i_{j+1}} \notin N(v)\cap A\} \neq \emptyset$. Therefore $|X_v| \leq r(v)-1$. Consequently, $|J| \leq |\bigcup_{v\in N(A)} X_v| \leq \sum_{v\in N(A)} |X_v| \leq \sum_{v\in N(A)} [r(v)-1]$. □

We remark that one can define the cyclic-sequence and index-distance for any subset of V in a similar way, and thus the Observation 3, Lemmas 1 and 2 and corollaries 1 and 2 are valid for cyclic-sequence and index-distance on subsets of V as well.

3. Proof of Theorem 1

We are now ready to determine the total domination number of $W_{3,n}$. We will prove that for each even integer $n \geq 8$,

$$\gamma_t(W_{3,n}) = 4\left\lceil \frac{n}{10} \right\rceil - \begin{cases} 0 & n \equiv 0,6,8 \pmod{10} \\ 2 & n \equiv 2,4 \pmod{10}. \end{cases}$$

Clearly $n \geq 8$ is an even integer by the definition of $W_{3,n}$. We divide the proof into five cases depending on n.

Proof. We distinguish four cases.

Case 1: $n \equiv 0 \pmod{10}$. Let $n = 10t$, where $t \geq 1$. Then the set $D_1 = \{u_{5k+b}, v_{5k+b} : k = 0,1,\ldots,t-1; b = 1,2\}$ is a total dominating set for $W_{3,n}$ and thus $\gamma_t(W_{3,n}) \leq |D_1| = 4t = 4\lceil \frac{n}{10} \rceil$. We show that $\gamma_t(W_{3,n}) = 4t$. Suppose to the contrary, that $\gamma_t(W_{3,n}) < 4t$. Let D be a total dominating set with $4t-1$ elements. Then by the Pigeonhole Principle either $|D\cap U| \leq 2t-1$ or $|D\cap V| \leq 2t-1$. Without loss of generality, assume that $|D\cap U| \leq 2t-1$. Let $|D\cap U| = 2t-1-a$, where $a \geq 0$. Then $|D\cap V| = 2t+a$. Observe that $D\cap U$ dominates at most $3|D\cap U| = 6t-3-3a$ vertices of V, and so $6t-3-3a \geq 5t = |V|$, since $D\cap U$ dominates V. Clearly the inequality $6t-3-3a \geq 5t$ does not hold if $t \in \{1,2\}$, and thus this contradiction implies that $\gamma_t(W_{3,n}) = 4t = 4\lceil \frac{n}{10} \rceil$ for $t = 1,2$. From here on, assume that $t \geq 3$. By Lemma 4(ii), at most $3|D\cap U| - |N(D\cap U)| = 3(2t-1-a) - 5t = t-3-3a$ elements of the cyclic-sequence of $D\cap U$ belong to $\mathscr{M}_3 = \{1,2,3\}$. Hence, at least $(2t-1-a)-(t-3-3a) = t+2+2a$ elements of the cyclic-sequence of $D\cap U$ do not belong to $\mathscr{M}_3$ and are greater than 3. Then by Observation 3, we have

$$5t = \sum_{i=1}^{2t-1} n_i \geq 4(t+2+2a) + (t-3-3a) = 5t+5+5a,$$

a contradiction. Therefore, $\gamma_t(W_{3,n}) = 4t = 4\lceil \frac{n}{10} \rceil$.

Case 2: $n \equiv 2 \pmod{10}$. Let $n = 10t+2$, where $t \geq 1$. Then the set $D_2 = \{u_{5k+b}, v_{5k+b} : k = 0,1,\ldots,t-1; b = 1,2\} \cup \{u_{5t+1}, v_{5t+1}\}$ is a total dominating set for

$W_{3,n}$ and thus $\gamma_t(W_{3,n}) \le |D_2| = 4t+2 = 4\lceil \frac{n}{10} \rceil - 2$. We show that $\gamma_t(W_{3,n}) = 4t+2$. Suppose to the contrary, that $\gamma_t(W_{3,n}) < 4t+2$. Let D be a total dominating set with $4t+1$ elements. Then by the Pigeonhole Principle either $|D \cap U| \le 2t$ or $|D \cap V| \le 2t$. Without loss of generality, assume that $|D \cap U| \le 2t$. Let $|D \cap U| = 2t - a$, where $a \ge 0$. Then $|D \cap V| = 2t + 1 + a$. Observe that $D \cap U$ dominates at most $6t - 3a$ vertices of V and so $6t - 3a \ge 5t + 1 = |V|$, since $D \cap U$ dominates V. By Lemma 4(ii), at most $3|D \cap U| - |N(D \cap U)| = 3(2t - a) - (5t + 1) = t - 1 - 3a$ elements of the cyclic-sequence of $D \cap U$ belong to $\mathscr{M}_3$. Hence, at least $(2t - a) - (t - 1 - 3a) = t + 1 + 2a$ elements of the cyclic-sequence of $D \cap U$ do not belong to $\mathscr{M}_3$ and are greater than 3. Then by Observation 3, we have

$$5t + 1 = \sum_{i=1}^{2t-a} n_i \ge 4(t + 1 + 2a) + (t - 1 - 3a) = 5t + 3 + 5a,$$

a contradiction. Therefore, $\gamma_t(W_{3,n}) = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$.

Case 3: $n \equiv 4 \pmod{10}$. Let $n = 10t + 4$, where $t \ge 1$. Then the set $D_3 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \ldots, t-1; b = 1, 2\} \cup \{u_{5t+1}, v_{5t-1}\}$ is a total dominating set for $W_{3,n}$ and thus $\gamma_t(W_{3,n}) \le |D_3| = 4t+2 = 4\lceil \frac{n}{10} \rceil - 2$. We show that $\gamma_t(W_{3,n}) = 4t+2$. Suppose to the contrary, that $\gamma_t(W_{3,n}) < 4t+2$. Let D be a total dominating set with $4t+1$ elements. Then by the Pigeonhole Principle either $|D \cap U| \le 2t$ or $|D \cap V| \le 2t$. Without loss of generality, assume that $|D \cap U| \le 2t$. Let $|D \cap U| = 2t - a$, where $a \ge 0$. Then $|D \cap V| = 2t + 1 + a$. Observe that $D \cap U$ dominates at most $6t - 3a$ vertices of V, and so $6t - 3a \ge 5t + 2 = |V|$, since $D \cap U$ dominates V. Clearly the inequality $6t - 3a \ge 5t + 2$ does not hold if $t = 1$, and thus this contradiction implies that $\gamma_t(W_{3,n}) = 4t+2 = 4\lceil \frac{n}{10} \rceil - 2$ for $t = 1$. From here on, assume that $t \ge 2$. By Lemma 4, at most $3|D \cap U| - |N(D \cap U)| = 3(2t - a) - (5t + 2) = t - 2 - 3a$ elements of the cyclic-sequence of $D \cap U$ belong to $\mathscr{M}_3$. Hence, at least $(2t - a) - (t - 2 - 3a) = t + 2 + 2a$ elements of the cyclic-sequence of $D \cap U$ do not belong to $\mathscr{M}_3$ and are greater than 3. Then by Observation 3, we have

$$5t + 2 = \sum_{i=1}^{2t-a} n_i \ge 4(t + 2 + 2a) + (t - 2 - 3a) = 5t + 6 + 5a,$$

a contradiction. Therefore, $\gamma_t(W_{3,n}) = 4t + 2 = 4\lceil \frac{n}{10} \rceil - 2$.

Case 4: $n \equiv 6 \pmod{10}$. Let $n = 10t + 6$, where $t \ge 1$. Then the set $D_4 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \ldots, t; b = 1, 2\}$ is a total dominating set for $W_{3,n}$ and thus $\gamma_t(W_{3,n}) \le |D_4| = 4t + 4 = 4\lceil \frac{n}{10} \rceil$. We show that $\gamma_t(W_{3,n}) = 4t + 4$. Suppose to the contrary, that $\gamma_t(W_{3,n}) < 4t+4$. Let D be a total dominating set with $4t+3$ elements. Then by the Pigeonhole Principle either $|D \cap U| \le 2t+1$ or $|D \cap V| \le 2t+1$. Without loss of generality, assume that $|D \cap U| \le 2t+1$. Let $|D \cap U| = 2t+1-a$, where $a \ge 0$. Then $|D \cap V| = 2t+2+a$. Observe that $D \cap U$ dominates at most $6t+3-3a$ vertices of V, and so $6t+3-3a \ge 5t+3 = |V|$, since $D \cap U$ dominates V. By Lemma 4, at most $3|D \cap U| - |N(D \cap U)| = 3(2t+1-a) - (5t+3) = t - 3a$ elements of the cyclic-sequence of $D \cap U$ belong to $\mathscr{M}_3$. Hence, at least $(2t + 1 - a) - (t - 3a) = t + 1 + 2a$ elements of the cyclic-sequence of $D \cap U$ do not belong to $\mathscr{M}_3$ and are greater than 3. Then by Observation 3, we have

$$5t + 3 = \sum_{i=1}^{2t+1-a} n_i \geq 4(t + 2 + 2a) + (t - 3a) = 5t + 8 + 5a,$$

a contradiction. Therefore, $\gamma_t(W_{3,n}) = 4t + 4 = 4\lceil \frac{n}{10} \rceil$.

Case 5: $n \equiv 8 \pmod{10}$. Let $n = 10t + 8$, where $t \geq 0$. Then the set $D_5 = \{u_{5k+b}, v_{5k+b} : k = 0, 1, \ldots, t; b = 1, 2\}$ is a total dominating set for $W_{3,n}$ and thus $\gamma_t(W_{3,n}) \leq |D_5| = 4t + 4 = 4\lceil \frac{n}{10} \rceil$. We show that $\gamma_t(W_{3,n}) = 4t + 4$. Suppose to the contrary, that $\gamma_t(W_{3,n}) < 4t+4$. Let D be a total dominating set with $4t+3$ elements. Then by the Pigeonhole Principle either $|D \cap U| \leq 2t+1$ or $|D \cap V| \leq 2t+1$. Without loss of generality, assume that $|D \cap U| \leq 2t + 1$. Let $|D \cap U| = 2t + 1 - a$ and $a \geq 0$. Then $|D \cap V| = 2t+2+a$. Observe that $D \cap U$ dominates at most $6t+3-3a$ vertices of V, and so $6t + 3 - 3a \geq 5t + 4 = |V|$, since $D \cap U$ dominates V. By Lemma 4, at most $3|D \cap U| - |N(D \cap U)| = 3(2t+1-a) - (5t+4) = t-1-3a$ elements of the cyclic-sequence of $D \cap U$ belong to $\mathscr{M}_3$. Hence, at least $(2t+1-a) - (t-1-3a) = t+2+2a$ elements of the cyclic-sequence of $D \cap U$ do not belong to $\mathscr{M}_3$ and are greater than 3. Then by Observation 3, we have

$$5t + 4 = \sum_{i=1}^{2t+1-a} n_i \geq 4(t + 2 + 2a) + (t - 1 - 3a) = 5t + 7 + 5a,$$

a contradiction. Therefore $\gamma_t(W_{3,n}) = 4t + 2 = 4\lceil \frac{n}{10} \rceil$. □

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