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Research Article

Stirling number of the fourth kind and Lucky partitions of a finite set

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Abstract: The concept of Lucky k-polynomials and in particular Lucky χ-polynomials was recently introduced. This paper introduces Stirling number of the fourth kind and Lucky partitions of a finite set in order to determine either the Lucky k- or Lucky χ-polynomial of a graph. The integer partitions influence Stirling partitions of the second kind.

Keywords: Stirling number of the second kind, Stirling number of the fourth kind, Lucky partition, Bell number

AMS Subject classification: 05C30, 03D20, 03E02, 03E20

1. Introduction

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ be a finite n-element set. A set of subsets of X, say, $P = \{Y_1, Y_2, Y_3, \ldots, Y_k\}$ is called a partition of set X if $x_i \in Y_j$ then $x_i \notin Y_l$, $l \neq j$ and $X = \bigcup\limits_{i=1}^{k} Y_i$.

The number of partitions of a finite set X is given by the well known Bell numbers. By convention the *Bell partition* of an empty set $\emptyset$ is given by $\{\emptyset\}$ and the corresponding Bell number is defined as $B_0 = 1$. For the non-empty set $X = \{x_1\}$, the Bell

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partition is given by $\{\{x_1\}\}$ hence, $B_1 = 1$. Numerous recurrence relations have been documented for Bell numbers, and we recall one of these [11], i. e.,

$$B_n = \sum_{k=0}^{n-1} B_k \binom{n-1}{k}. \tag{1}$$

Several upper bound asymptotic formulas are known for Bell numbers. We recall an important upper bound [2], i. e.,

$$B_n < \left(\frac{0.792n}{\ln(n+1)}\right)^n,$$

for all positive integers n. A closed formula which provides a sense of the growth rate of Bell numbers follows from the use of the Lambert $W(n)$ function where the expressions, B and P_i, Q_i are known in $W(n)$. Recall that

$$B_{n+h} = \frac{(n+h)!}{W(n)^{n+h}} \times \frac{\exp(e^{(W(n)} - 1)}{(2\pi B)^{\frac{1}{2}}}$$

uniformly for $h = O(\ln(n))$ as $n \to \infty$ [3, 10]. It is obvious from the growth rate of the Bell numbers that generating the Bell partitions manually is firstly, difficult and secondly, very inefficient for research purposes.
The Lah numbers are the coefficients obtained in expressing rising factorials in terms of falling factorials. They also correspond to the coefficients of the nth derivative of $e^{\frac{1}{x}}$. Lah numbers are related to Stirling numbers and also called, *Stirling number of the third kind* [4].

2. Stirling Partition of the Fourth Kind of a Finite Set

Throughout the rest of this paper a set refers to a finite non-empty set. It is known that the elements of a set are unordered and two sets (possibly distinct) of equal cardinality are said to be equipotent (or, of equal size). Two equipotent sets A, B have the relation $A \sim B$. The cardinality of a set $X = \{x_1, x_2, x_3, \ldots, x_n\}$ is denoted by $|X|$ or $|\{x_1, x_2, x_3, \ldots, x_n\}|$. Hence, $|X| = n$. For a set X and a partition $P = \{Y_1, Y_2, Y_3, \ldots, Y_k\}$ a total of $k + 2$ cardinalities are of importance. These are $|X| = n$, $|P| = |\{Y_1, Y_2, Y_3, \ldots, Y_k\}| = k$ and $|Y_i|$, $i = 1, 2, 3, \ldots, k$.
Consider two partitions $P_1 = \{Y_1, Y_2, Y_3, \ldots, Y_k\}$ and $P_2 = \{Z_1, Z_2, Z_3, \ldots, Z_t\}$ of the set X. The partitions are said to be congruent, denoted by $P_1 \cong P_2$ if and only if $P_1 \sim P_2$ (or, $|P_1| = k = t = |P_2|$) and a for each subset Y_i there is a distinct Z_j such that, $|Y_i| = |Z_j|$. Put differently, there must be a bijection, $P_1 \mapsto P_{2_{|Y_i|=|Z_j|}}$.
An important specialization on the Stirling partition of the second kind is introduced in this paper. *Stirling partition of the fourth kind of a set X* is a partition into k

non-empty subsets, and the cardinality of the subsets are prescribed as, $\ell_1, \ell_2, \ldots, \ell_k$, $\sum\limits_{i=1}^{k} \ell_i = n$. It means that all partitions have the partition form,

$$\{\{\ell_1\text{-element}\}, \{\ell_2\text{-element}\}, \{\ell_3\text{-element}\}, \ldots, \{\ell_k\text{-element}\}\}.$$

The fundamental difference between Lah numbers and Stirling numbers of the fourth kind is as follows. Whereas Lah numbers count the number of ways a set of n elements can be partitioned into k non-empty linearly order subsets, Stirling numbers do the same to unordered subsets. Hence, a specific Lah number is an upper-bound for the corresponding Stirling number of the fourth kind.

Observe that the Stirling partition of the fourth kind can be clustered, as it forms $P_i(n,k)$ having an equal number of subsets. These clusters can be derived from the number of integer partitions of n into exactly k parts denoted by $p_k(n)$. A well-known recursive formula exists [1], i. e.,

$$p_k(n) = p_k(n-k) + p_{k-1}(n-1).$$

Let the *Stirling number of the fourth kind* denoted by, $S(P_i(n,k))$, $i = 1, 2, 3, \ldots, p_k(n)$, be the number of partitions corresponding to a *Stirling of the fourth kind partition form.* We can derive the following important result.

Proposition 1. *For the set $X = \{x_1, x_2, x_3, \ldots, x_n\}$, let the Stirling partition form of the fourth kind be, $P_1(n,1), P_1(n,2), \ldots, P_{p_2(n)}(n,2), P_1(n,3), \ldots, P_{p_3(n)}(n,3) \ldots P_1(n,n)$. Then*

$$S(n,k) = \sum_{i=1}^{p_k(n)} S(P_i(n,k)).$$

Proof. Since $S(n,k)$ yields the exact number of distinct partitions of X with each partition having exactly k non-empty subsets, it follows that

$$S(n,k) \geq \sum_{i=1}^{p_k(n)} S(P_i(n,k)).$$

Since, $p(n)$ yields the exact number of distinct integer partitions of n, the partition form cluster C_k is exact and yields all possible partition forms, each contains exactly k non-empty subsets, i. e., $p_k(n)$ such partition forms. Therefore,

$$S(n,k) \leq \sum_{i=1}^{p_k(n)} S(P_i(n,k)).$$

The proof is thus complete. $\square$

3. Lucky Partitions

The concept of Lucky k-polynomials (and in particular Lucky χ-polynomials) was discussed in [5, 7]. These polynomials relate to the notion of chromatic completion of graphs [6, 8]. An important subclass of the Stirling partitions of the fourth kind is called the Lucky partitions. These partitions are necessary for determining the Lucky k- and the Lucky χ-polynomials. From Lucky's theorem or an optimal near-completion k-partition, it follows that for $2 \leq k \leq n$ the Lucky partitions have the form,

$$\underbrace{\{\{\lfloor\frac{n}{k}\rfloor - \text{element}\}, \{\lfloor\frac{n}{k}\rfloor - \text{element}\}, \dots, \{\lfloor\frac{n}{k}\rfloor - \text{element}\},}_{(k-r)-\text{subsets}}$$
$$\underbrace{\{\lceil\frac{n}{k}\rceil - \text{element}\}, \{\lceil\frac{n}{k}\rceil - \text{element}\}, \dots, \{\lceil\frac{n}{k}\rceil - \text{element}\}\}.}_{(r\geq 0)-\text{subsets}}$$

Example 1: For $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $k = 3$, the Lucky partition form $\{\{2 - \text{element}\}, \{2 - \text{element}\}, \{1 - \text{element}\}\}$ is defined.
For the set $\{x_1\}$, we obtain the Lucky partition $\{\{x_1\}\}$.
For the set $\{x_1, x_2\}$, we obtain the Lucky partitions $\{\{x_1, x_2\}\}$, $\{\{x_1\}, \{x_2\}\}$.
For the set $\{x_1, x_2, x_3\}$, we obtain the Lucky partitions,
$\{\{x_1, x_2\}, \{x_3\}\}$, $\{\{x_1, x_3\}, \{x_2\}\}$, $\{\{x_1\}, \{x_2, x_3\}\}$, $\{\{x_1\}, \{x_2\}, \{x_3\}\}$.
For the set $\{v_1, v_2, v_3, v_4\}$, we obtain the Lucky partitions,
$\{\{x_1, x_2\}, \{x_3, x_4\}\}$, $\{\{x_1, x_2\}, \{x_3\}, \{x_4\}\}$, $\{\{x_1, x_3\}, \{x_2, x_4\}\}$, $\{\{x_1, x_3\}, \{x_2\}, \{x_4\}\}$, $\{\{x_1, x_4\}, \{x_2, x_3\}\}$, $\{\{x_1\}, \{x_2, x_3\}, \{x_4\}\}$, $\{\{x_1, x_4\}, \{x_2\}, \{x_3\}\}$, $\{\{x_1\}, \{x_2, x_4\}, \{x_3\}\}$, $\{\{x_1\}, \{x_2\}, \{x_3, x_4\}\}$.
For the set $\{v_1, v_2, v_3, v_4, v_5\}$, we obtain the Lucky partitions,
$\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5\}\}$, $\{\{x_1, x_2\}, \{x_3, x_5\}, \{x_4\}\}$, $\{\{x_1, x_2\}, \{x_3\}, \{x_4, x_5\}\}$, $\{\{x_1, x_3\}, \{x_2, x_4\}, \{x_5\}\}$, $\{\{x_1, x_3\}, \{x_2, x_5\}, \{x_4\}\}$, $\{\{x_1, x_3\}, \{x_2\}, \{x_4, x_5\}\}$, $\{\{x_1, x_4\}, \{x_2, x_3\}, \{x_5\}\}$, $\{\{x_1, x_5\}, \{x_2, x_3\}, \{x_4\}\}$, $\{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5\}\}$, $\{\{x_1, x_4\}, \{x_2, x_5\}, \{x_3\}\}$, $\{\{x_1, x_4\}, \{x_2\}, \{x_3, x_5\}\}$, $\{\{x_1, x_5\}, \{x_2, x_4\}, \{x_3\}\}$, $\{\{x_1\}, \{x_2, x_4\}, \{x_3, x_5\}\}$, $\{\{x_1, x_5\}, \{x_2\}, \{x_3, x_4\}\}$, $\{\{x_1\}, \{x_2, x_5\}, \{x_3, x_4\}\}$.
Note that 31 iterative Lucky partitions (including the final 15 Lucky partitions) were generated instead of generating 75 iterative Bell partitions (including the 52 corresponding Bell partitions) to select from. It is expected (if not obvious) that for large n, a heuristic method to generate Lucky partitions would be noticeably more efficient.
The number of Lucky partitions corresponding to a Lucky partition form is called the Lucky partition number and is denoted $L(n, k)$. Note that, $L(n, k) = S(P_i(n, k))$, $\forall\ n, k \in \mathbb{N}$. Table 1 depicts the Lucky partition numbers for $n = 1, 2, 3, 4, 5, 6$.
A closed formula for the entries in Table 1 was announced by Dillon Lareau [9]. Hence, we have,

$$L(n, k) = \frac{n!}{A!B!(\lceil\frac{n}{k}\rceil!)^A(\lfloor\frac{n}{k}\rfloor!)^B},$$

where $A = n \mod k$ and $B = k - A$. Lareau referred to it as "the number of ways of dividing n labeled items into k unlabeled boxes as evenly as possible". The

n	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
1	1	-	-	-	-	-
2	1	1	-	-	-	-
3	1	3	1	-	-	-
4	1	3	6	1	-	-
5	1	10	15	10	1	-
6	1	10	15	45	15	1

Table 1. **Lucky partition numbers for n = 1, 2, 3, 4, 5 and 6**

numbers were not named and only in the context of finding Lucky k-polynomials did the name Lucky partition numbers arise, independently. In the graph theory context the vertex partitions correspond to those for a null graph (edgeless). For graphs with edges certain forbidden vertex partitions must be eliminated resulting in a decrease in the graph specific Lucky partition number. Untangling the closed formula into various recurrence relations for the Lucky partition numbers is viewed of importance for researchers to develop the much needed algorithm to generate graph specific Lucky partitions. Without such a coded algorithm, this research avenue cannot be furthered.

Theorem 1. *Let $n, k, t \in \mathbb{N}$.*
(a) If $k = 1$ or $k = n$ then, $L(n, k) = 1$,
(b) If $2 \leq k \leq n-1$ and $n = tk$, $t \geq 2$ then, $L(n, k) = L(n-1, k)$,
(c) If $2 \leq k \leq n-1$ and $\frac{n}{k} = \lfloor \frac{n}{k} \rfloor + r$, $0 \leq r \leq k-2$ then, $L(n+1, k) = (k-r)L(n, k) + S(P_i(n, k))$, for corresponding $P_i(n, k)$ in which:
(i) If $r = 0$, then $P_i(n, k)$ has exactly one, $\{\lceil \frac{n}{k} \rceil + 1\text{-element}\}$ subset,
(ii) If $1 \leq r \leq (k-2)$, then $P_i(n, k)$ has exactly $r+1$, $\{\lceil \frac{n}{k} \rceil - \text{element}\}$ subsets.

Proof. (a) Since $p_1(n) = p_n(n) = 1$, the result holds for $k = 1$ or $k = n$.
(b) Let $2 \leq k \leq n-1$ and $n = tk$, $t \geq 2$. Since the Lucky partition form for $L_{n-1,k}$ is

$$\underbrace{\{\{\lfloor \frac{n-1}{k} \rfloor - \text{element}\}}_{1-\text{subset}}, \underbrace{\{\frac{n}{k} - \text{element}\}, \{\frac{n}{k} - \text{element}\}, \ldots, \{\frac{n}{k} - \text{element}\}\}}_{(k-1)-\text{subsets}}$$

and since all partitions are distinct, the result holds.
(c) Let $2 \leq k \leq n-1$ and $\frac{n}{k} = \lfloor \frac{n}{k} \rfloor + r$, $0 \leq r \leq k-2$.
Case $c(i)$: Let $r = 0$. The first term follows by adding x_{n+1} to any of the k, Lucky partition subsets for n. Furthermore, consider $P_i(n, k)$ which has the partition form,

$$\underbrace{\{\{(\frac{n}{k} + 1) - \text{element}\}}_{1-\text{subset}}, \underbrace{\{\frac{n}{k} - \text{element}\}, \{\frac{n}{k} - \text{element}\}, \ldots, \{\frac{n}{k} - \text{element}\}}_{(k-2)-\text{subsets}},$$
$$\underbrace{\{(\frac{n}{k} - 1) - \text{element}\}\}}_{1-\text{subset}} \text{ and all partitions are distinct.}$$

Since, $\{\underbrace{\{(\frac{n}{k}+1)-\text{element}\}}_{1-\text{subset}}, \underbrace{\{\frac{n}{k}-\text{element}\},\{\frac{n}{k}-\text{element}\},\ldots,\{\frac{n}{k}-\text{element}\}}_{(k-2)-\text{subsets}},$

$\underbrace{\{(\frac{n}{k}-1)-\text{element}\}}_{1-\text{subset}}\cup\{x_{n+1}\}\}$ is the Lucky partition form for $n+1$ and all partitions are distinct, the second term follows. Therefore, $c(i)$ follows.

Case $c(ii)$: Let $1 \leq r \leq k-2$. The first term follows by adding x_{n+1} to any of the $r-k$, Lucky partition subsets for n. Furthermore, as $P_i(n,k)$ has the partition form

$$\{\underbrace{\{\lfloor\frac{n}{k}\rfloor-\text{element}\},\{\lfloor\frac{n}{k}\rfloor-\text{element}\},\ldots,\{\lfloor\frac{n}{k}\rfloor-\text{element}\}}_{(k-r-2)-subsets},$$

$$\underbrace{\{\lceil\frac{n}{k}\rceil-\text{element}\},\{\lceil\frac{n}{k}\rceil-\text{element}\},\ldots,\{\lceil\frac{n}{k}\rceil-\text{element}\}}_{(r+1\geq 0)-\text{subsets}},\underbrace{\{(\lfloor\frac{n}{k}\rfloor-1)-\text{element}\}}_{1-subset}\}$$

and all partitions are distinct.

Since, $\{\underbrace{\{\lfloor\frac{n}{k}\rfloor-\text{element}\},\{\lfloor\frac{n}{k}\rfloor-\text{element}\},\ldots,\{\lfloor\frac{n}{k}\rfloor-\text{element}\}}_{(k-r-2)-\text{subsets}},\{\lceil\frac{n}{k}\rceil-\text{element}\},$

$\underbrace{\{\lceil\frac{n}{k}\rceil-\text{element}\},\ldots,\{\lceil\frac{n}{k}\rceil-\text{element}\}}_{(r+1\geq 0)-\text{subsets}},\underbrace{\{(\lfloor\frac{n}{k}\rfloor-1)-\text{element}\}}_{1-\text{subset}}\cup\{x_{n+1}\}\}$ is the Lucky partition form for $n+1$ and all partitions are distinct, the second term follows. Therefore, $c(ii)$ follows. $\square$

Recall that the falling factorial is denoted, $\lambda(\lambda-1)\cdots(\lambda-(\ell-1)) = \lambda^{(\ell)}$.

Theorem 2. *Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ such that $tk \leq n \leq (t+1)k-2$, $t \geq 2$. Then,*

$$L(n+1,k) = (k-r)L(n,k) + (k-r)^{(2)}tL(n,k).$$

Proof. Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ such that $tk \leq n \leq (t+1)k-2$, $t \geq 2$. Then $n = tk + r$, $0 \leq r \leq k-2$. Hence, the Lucky partition form is

$$\{\underbrace{\{t-\text{element}\},\{t-\text{element}\},\ldots,\{t-\text{element}\}}_{(k-r)-\text{subsets}},$$

$$\underbrace{\{(t+1)-\text{element}\},\{(t+1)-\text{element}\},\ldots,\{(t+1)-\text{element}\}}_{(0\leq r\leq k-2)-\text{subsets}}\}$$

and all partitions are distinct.

Extending to $n+1$ adds the element x_{n+1} which can be added to any of the distinct $(k-r)$, t-element subsets. Therefore, the term $(k-r)L(n,k)$ is valid. The $(k-r)$, t-element distinct subsets may also be grouped pairwise and from each an element, one at a time, may be union'ed with the other, and vice versa (or conversely) to yield an additional $(t+1)$-element distinct subset and a $(t-1)$-element distinct subset. The latter can accommodate x_{n+1} as an additional element to obtain the additional

Lucky partition forms for $n+1$. Hence, $\binom{k-r}{2}$ such distinct pairs exist. Interchanging an element is possible in $2\binom{k-r}{2}t$ ways. Therefore,

$$L(n+1,k) = (k-r)L(n,k) + 2\binom{k-r}{2}t \times L(n,k)$$

$$\text{i. e.,} \ = (k-r)L(n,k) + (k-r)^{(2)}tL(n,k).$$

□

Note that Theorem 2 is an alternative result to Theorem 1(c). The choice of application will probably depend on which is developed first, i. e., a computer coded generator for Stirling numbers of the fourth kind or for Lucky numbers.
If a Lucky partition numbers table is presented as a square matrix, the matrix will be a lower triangular matrix. This leads to alternative ways to present some results.

Proposition 2. *Let the matrix A represent the Lucky partition numbers corresponding to $n = 1, 2, 3, \ldots, t$ and $k = 1, 2, 3, \ldots, t$. Then:*
(a) Entries $a_{i,i} = 1$, $1 \le i \le t$.
(b) Entries $a_{i,i-1} = a_{i-1,i-2} + (i-1)$, $3 \le i \le t$.
(c) Through immediate induction (b) holds for all $t \ge 3$, $t \in \mathbb{N}$.

Proof. (a) The result follows directly from Theorem 1(a).
(b) It is known that $L(2,1) = 1$ and $L(3,2) = 3$. Since, $a_{3,2} = a_{2,1} + (3-1) = 1+2 = 3$, the result holds for $i = 3$. Assume the result holds for $n = q$, $4 \le q \le t-1$. Therefore, $a_{q,q-1} = a_{q-1,q-2} + (q-1)$.
Let $i = q+1$. Clearly, because of the Lucky partition form we only have to determine, $\binom{q+1}{2} - \binom{q}{2}$ to close the result. It follows easily that,

$$\frac{(q+1)!}{2!(q-1)!} - \frac{q!}{2!(q-2)!} = q = (q+1) - 1.$$

Thus the result is true for all $3 \le i \le t$.
(c) Trivial. □

4. Conclusion

Clearly, Lucky partition of a finite set is a specialization of Stirling partition of the fourth kind. The latter is a specialization of Stirling partitions of the second kind. It is also clear that integer partitions play a key role in this avenue of research.
We now present some research problems.
Problem 1. Find a recurrence relation, if such exists, to determine $S(P_i(n,k))$, $i = 1, 2, 3, \ldots, p_k(n)$.
Problem 2. Find an explicit formula, if such exists, to determine $S(P_i(n,k))$,

$i = 1, 2, 3, \ldots, p_k(n)$.

Problem 3. For a $k \in \mathbb{N}$, $n = 1, 2, 3, \ldots,$ determine the growth rate of Lucky partition numbers.

Problem 4. From the table depicting the Lucky partition numbers read together with Theorem 1(b),(c) it follows that for $k = 2$, $n = 2, 3, 4, \ldots$ the Lucky partition numbers is a sequence, $1, 3, 3, 10, 10, \ldots, a_{i,1}, a_{i,2}, a_{(i+1),1}, a_{(i+1),2}, \cdots$ with $a_{j,1} = a_{j,2}$. Note that $a_{j,1}$ corresponds to odd j and $a_{j,2}$ corresponds to even $j+1$.

Furthermore, for $1, 3, 3, 10, 10, \ldots, a_{i,1}, a_{i,2}, a_{(i+1),1}, a_{(i+1),2}, \cdots$ the sequence of non-zero first differences is, $2, 7, \ldots, (a_{(i+1),1} - a_{i,2}), \ldots$.

Conjecture. Consider the sequence, $s = 1, 3, 3, a_{5,1}, a_{5,2}, a_{7,1}, a_{7,2}, \cdots$ Let the non-zero first difference be defined as,

$3 - 1 = 2 = t_3,$

$a_{5,1} - 3 = t_5,$

$a_{7,1} - a_{5,2} = t_7,$

.

.

.

Then, for $n \geq 5$ and n is odd, $L(n, 2) = 2(a_{n-1} + t_{n-2})$.

Problem 5. In order to apply Lucky partitions to graph colouring, chromatic completion and finding Lucky χ- and Lucky k-polynomials the partitions *per se* are required because testing for adjacency is required [5–8].

Write a computer code (algorithm) to generate the partitions for the respective Stirling of the fourth kind partition forms. Also prove, uniqueness of solution and convergence of algorithm.

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