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Research Article

Distinct edge geodetic decomposition in graphs

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Abstract: Let $G = (V, E)$ be a simple connected graph of order p and size q. A decomposition of a graph G is a collection π of edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G belongs to exactly one $G_i, (1 \leq i \leq n)$. The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be a distinct edge geodetic decomposition if $g_1(G_i) \neq g_1(G_j), (1 \leq i \neq j \leq n)$. The maximum cardinality of π is called the distinct edge geodetic decomposition number of G and is denoted by $\pi_{dg_1}(G)$, where $g_1(G)$ is the edge geodetic number of G. Some general properties satisfied by this concept are studied. Connected graphs of $\pi_{dg_1}(G) \geq 2$ are characterized and connected graphs of order p with $\pi_{dg_1}(G) = p - 2$ are characterized.

Keywords: decomposition, distinct edge geodetic decomposition, distinct edge geodetic decomposition number, edge geodetic number.

AMS Subject classification: 05C51, 05C12, 05C70

1. Terminology and introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6, 8]. $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$ and is denoted by $deg(v)$. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a *pendent edge*, u a leaf and v a support vertex. A vertex of degree $p - 1$ is called a *universal vertex.* A vertex v in a connected graph G is said to be a *semi simplicial vertex* of G if $\Delta(< N(v) >) = |N(v)| - 1$. A vertex v is a *simplicial* vertex of a graph G if $< N(v) >$ is complete. Every simplicial vertex of a graph G is semi simplicial vertex. A graph G is said to be a *semi complete* graph if every vertex of G is a semi simplicial. It is observed that any semi complete graph has at least two vertices of degree $p-2$ or more. For any connected graph G, a vertex $v \in V(G)$ is called a *cut vertex* of G if $V - \{v\}$ is no longer connected. An *independent set* is a set of vertices in a graph, no two of which are adjacent. The number of vertices in a maximum independent vertex set of G is called the *independent vertex number* of G, denoted by β.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* G or $r(G)$ and the maximum eccentricity is its *diameter*, *diamG* of G. A vertex v of G is said to be *peripheral vertex* if $e(v) = \text{diam}(G)$. An u-v path of length $d(u, v)$ is called an *u-v geodesic*. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. An *edge geodetic set* of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is a *minimum edge geodetic set* of G or g_1-set of G. The edge geodetic number was introduced by Atici [5] and further studied in [1–3, 13, 15, 17–21]. A *decomposition of a graph* G is a collection of edge - disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G belongs to exactly one G_i, $(1 \leq i \leq n)$. Various types of decompositions of G have been studied in the literature by imposing conditions on the subgraph G_i. In this paper we introduced and studied the concept of *distinct edge geodetic decomposition in graphs*. For references on decomposition parameters in graphs see [4, 7, 9–12, 14, 16]. In number theory and combinatorics, a partition of a positive integer n, also called an *integer partition*, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. We denote by P_p, C_p and $K_{r,s}$, the path on p vertices, the cycle on p vertices and complete bipartite graph in which one partite set has r vertices and the other partite set has s vertices, respectively. Throughout this paper G denotes simple connected graph with at least two vertices. The following theorems are used in sequel.

Theorem A. ([18]) *Each simplicial vertex of G belongs to every edge geodetic dominating set of G.*

Theorem B. ([18]) For any connected graph G, $2 \leq g_1(G) \leq p$.

Theorem C. ([5]) For any non-trivial tree T, the edge geodetic number $g_1(T)$ equals the number of end-vertices in T.

Theorem D. ([5]) For the star $G = K_{1,n}$, $g_1(G) = n = q$.

Theorem E. ([18]) Any connected graph having more than one universal vertex has edge geodetic number p.

Theorem F. ([18]) Any connected graph having exactly one universal vertex has edge geodetic number $p-1$.

Theorem G. ([18]) For any connected graph G, $g_1(G) = 2$ if and only if there exist two peripheral vertices u and v such that every edges of G lies in a diametral path of u and v.

Theorem H. ([18]) For the cycle $C_p(p \geq 4)$, $g_1(C_p) = 2$ if p is even and $g_1(C_p) = 3$ if p is odd.

Theorem I. ([18]) No cut vertex of G belongs to any minimum edge geodetic set of G.

2. Distinct edge geodetic decomposition in graphs

Definition 1. *The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be a* distinct edge geodetic decomposition *if $g_1(G_i) \neq g_1(G_j)(1 \leq i \neq j \leq n)$. The maximum cardinality of π is called the* distinct edge geodetic decomposition number *of G and is denoted by $\pi_{dg_1}(G)$, where $g_1(G)$ is the edge geodetic number of G.*

Example 1. *For the graph G given in Figure 1, G_1 and G_2 [given in Figure 2] is a decomposition of G. Since $g_1(G_1) = 3$ and $g_1(G_2) = 2$, $\pi = \{G_1, G_2\}$ is a distinct edge geodetic decomposition of G. It is easily verified that there is no distinct edge geodetic decomposition of cardinality more than 2. Therefore $\pi_{dg_1}(G) = 2$.*

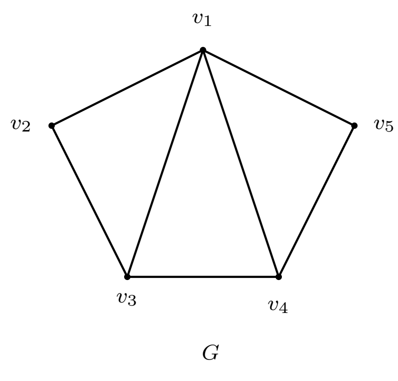

Figure 1. **A graph with $\pi_{dg_1}(G) = 2$.**

Remark 1. *There can be more than one distinct edge geodetic decompositions for a graph. For the graph G given in Figure 1, G_3 and G_4 [given in Figures 2] is a decomposition*

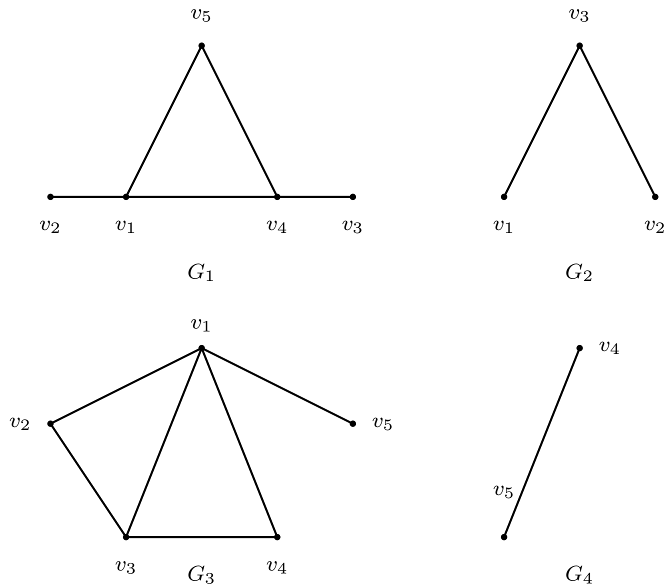

Figure 2.

of G. Since $g_1(G_3) = 4$ and $g_1(G_4) = 2$, $\pi_2 = \{G_3, G_4\}$ is also a distinct edge geodetic decomposition of G.

Observation 1. *For the cycle $G = C_p(p \geq 3)$ and path $G = P_p(p \geq 3)$, $\pi = \{G\}$ so that $\pi_{dg_1}(G) = 1$.*

Observation 2. *Let $G = K_{1,q}$. Then $\pi_{dg_1}(G) \geq 2$ if and only if $q \geq 4$.*

Lemma 1. *For any connected graph G, $g_1(G) = p$ if and only if G is semi complete.*

Proof. Let G be a semi complete graph and S be an edge geodetic set of G. Let v be an vertex of G such that $v \in V \setminus S$. Let u be a vertex of $< N(v) >$ such that $deg_{<N(v)>}(u) = |N(v)| - 1$. Let $u_1, u_2, \ldots, u_k (k \geq 2)$ be the neighbors of $u \in < N(v) >$. Since S is an edge geodetic set of G, the edge uv lies on the x-y geodesic $P : x, x_1, \ldots, u_i, u, v, u_j, \ldots, y$, where $x, y \in S$. Since v is a semi-simplicial vertex of G, u and u_j are adjacent in G and so P is not a x-y geodesic of G, which is a contradiction. This implies that $S = V$ and hence $g_1(G) = p$.
Conversely, suppose $g_1(G) = p$. We claim that G is a semi-simplicial graph. If not, let there exists a vertex v in G such that v is not a semi-simplicial vertex of G. Then for each $w \in N(v)$, there exists $z_w \in [N(v) - \{w\}]$ such that $wz_w \notin E(G)$. Let $S = V(G) - \{v\}$. Consider the edge wv. Since $w, z_w \in S$, the edge wv lies on the

geodesic w, v, z_w. Then S is an edge geodetic set of G with $|S| = p - 1$, which is a contradiction. Therefore, G is a semi-complete graph. □

Theorem 3. *Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a distinct edge geodetic decomposition of G. If any one of the $G_i (1 \leq i \leq k)$ has edge geodetic number p, then none of the $G_i (1 \leq i \leq k)$ has edge geodetic number $p - 1$.*

Proof. Let $G_1 \in \pi$ such that $g_1(G_1) = p$. Then by Lemma 1, G_1 is semi complete. Hence G_1 has at least two vertices of degree $p - 2$ or more. Let $G_2 =< G - G_1 >$. Consider two cases.

Case 1. Let $\Delta(G_1) = p - 2$. Then G_2 has at least two end vertices. Hence G_2 has at least two cut vertices. By Theorem I, $g_1(G_2) \leq p - 2 \neq p - 1$.

Case 2. Let $\Delta(G_1) = p - 1$. Since $g_1(G_1) = p$, G_1 has at least two vertices of degree $p - 1$. Then $|V(G_2)| \leq p - 2$ and so $g_1(G_2) \neq p - 1$. □

Theorem 4. *Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a distinct edge geodetic decomposition of G. If any one of the $G_i (1 \leq i \leq k)$ has edge geodetic number $p - 1$, then none of the $G_i (1 \leq i \leq k)$ has edge geodetic number p.*

Proof. The proof is similar to the proof of Theorem 3. □

Theorem 5. *For any connected graph G with $p \geq 4$, $1 \leq \pi_{dg_1}(G) \leq p - 2$.*

Proof. From the definition of distinct edge geodetic decomposition of G, $\pi_{dg_1}(G) \geq 1$. Suppose that $\pi_{dg_1}(G) = p - 1$. Let $\pi = \{G_1, G_2, \ldots, G_{p-1}\}$ be a distinct edge geodetic decomposition of G. Since $g_1(G_i) \neq g_1(G_j)(i \neq j)$ and $g_1(G_i) \geq 2(1 \leq i \leq p - 1)$, there exist $G_l, G_k \in \pi$ such that $g_1(G_1) = p - 1$ and $g_1(G_k) = p$, which is a contradiction to Theorems 3 and 4. Therefore $\pi_{dg_1}(G) \leq p - 2$. Thus $1 \leq \pi_{dg_1}(G) \leq p - 2$. □

Remark 2. The bounds in Theorem 5 are sharp. For the graph $G = K_3$, $\pi_{dg_1}(G) = 1$ and for the complete graph $G = K_p(p \geq 4), \pi_{dg_1}(G) = p - 2$. Also the bounds in Theorem 5 can be strict. For the graph G given in Figure 3, the graphs G_1, G_2 and G_3 illustrated in Figure 3 represents a decomposition of G with $g_1(G_1) = 4$, $g_1(G_2) = 3$ and $g_1(G_3) = 2$. Hence $\pi = \{G_1, G_2, G_3\}$ is a distinct edge geodetic decomposition of G. It is easily verified that there is no distinct edge geodetic decomposition of cardinality more than 3 so that $\pi_{dg_1}(G) = 3$. Therefore $2 < \pi_{dg_1}(G) < p - 2$.

Theorem 6. *For any connected graph G, $\pi_{dg_1}(G) \geq 2$ if and only if $\triangle(G) \geq 3$ and $q \geq 4$.*

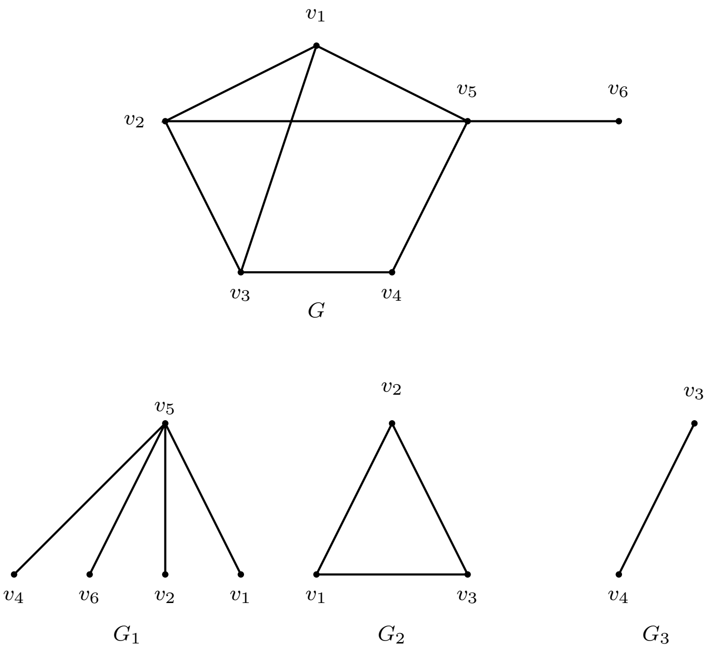

Figure 3. **A graph with $\pi_{dg_1}(G) = 3$.**

Proof. Let $\triangle(G) \geq 3$ and $q \geq 4$. We consider two cases.

Case 1. Suppose G is not a tree.
Then G has at least one cycle. Let $u_1, u_2, u_3, \ldots, u_k, u_1 (k \geq 3)$ be a cycle of G. Let $G_1 = G - \{u_1u_k\}$, $G_2 = K_2 = \{u_1u_k\}$. Also G_1 has at least one edge, which does not lie on the diametral path. Hence by Theorems G, $g_1(G_1) \neq 2$. By Theorem C, $g_1(G_2) = 2$. Therefore $g_1(G_1) \neq g_1(G_2)$ and hence $\pi = \{G_1, G_2\}$ is a distinct geodetic decomposition of G so that $\pi_{dg_1}(G) \geq 2$.

Case 2. Suppose G is a tree.
We distinguish the following situations.

Subcase 2.1. G is a star.
Let $G_1 = \{e\}$ and $G_2 = G - \{e\}$. Then by Theorem C, $g_1(G_1) = 2$. Since $q \geq 4$, by Theorem C, $g_1(G_2) \geq 3$. Then $g_1(G_1) \neq g_1(G_2)$. Hence $\pi = \{G_1, G_2\}$ is a distinct geodetic decomposition of G so that $\pi_{dg}(G) \geq 2$.

Subcase 2.2. G is not a star.
Let u be a vertex of G such that $d(u) \geq 3$. Since $q \geq 4$, there exists at least one pendant vertex x such that x is not adjacent to u. Let $G_1 = K_2 = xy$ and $G_2 = G - \{xy\}$, where y is the adjacent vertex of x. Then by Theorem C, $g_1(G_1) = 2$. Since G_2 has at least three pendant edges, by Theorem A, $g_1(G_1) \geq 3$ such that $g_1(G_1) \neq g_1(G_2)$. Hence $\pi = \{G_1, G_2\}$ is a distinct geodetic decomposition of G so that $\pi_{dg_1}(G) \geq 2$.

Conversely, suppose that $\pi_{dg_1}(G) \geq 2$. If $\triangle(G) \leq 2$, then G is either a path or a cycle. Hence by Observation 1, $\pi_{dg_1}(G) = 1$. Therefore $\triangle(G) \geq 3$. If $q \leq 3$, then G is a star $K_{1,3}$. By Observation 2, $\pi_{dg_1}(G) = 1$, which is a contradiction. Therefore $q \geq 4$. □

Remark 3. *Star is the smallest graph with maximum edge geodetic number. Hence the decomposition of G yields the maximum distinct edge geodetic decomposition if each decomposition G_i is a star.*

Theorem 7. *For any connected graph G, $\pi_{dg_1}(G) = p-2$ if and only if G has at least $\frac{p^2-p-4}{2}$ edges.*

Proof. Let $\pi_{dg_1}(G) = p-2$. We have to prove $q \geq \frac{p^2-p-4}{2}$. Let $\pi = \{G_1, G_2, \ldots, G_{p-2}\}$ be a maximum distinct edge geodetic decomposition of G. Let $g_1(G_1) = 2, g_1(G_2) = 3, \ldots, g_1(G_{p-2}) = p-1$. Since star is the smallest graph with maximum edge geodetic number, let $G_1 = K_2, G_2 = K_{1,3}, G_3 = K_{1,4}, \ldots, G_{p-2} = K_{1,p-1}$. Then $q = 1+3+4+\cdots+p-1 = \frac{p^2-p-4}{2}$. If any one of $G_i (1 \leq i \leq p-2)$, not a star, then $q \geq \frac{p^2-p-4}{2}$.

Conversely, suppose that G has at least $\frac{p^2-p-4}{2}$ edges. Also $\frac{p^2-p-4}{2} = \frac{p(p-1)}{2} - 2$ and addition of two edges to G gives the resulting graph as a complete graph. Hence G has at least $p-3$ universal vertices. As in the first part of the theorem, let $G_1 = K_{1,p-1}$, $G_2 = K_{1,p-2}$, $\ldots$, $G_{p-4} = K_{1,4}$, $G_{p-3} = K_{1,3}$. Then $\pi_{dg_1}(G) \geq p-3$. Hence by Theorem 5, $\pi_{dg_1}(G) = p-3$ or $p-2$. Consider the following cases.

Case 1. Let $q = \frac{p^2-p-4}{2}$. Then $G_{p-2} =< G - G_1 - G_2 - \cdots - G_{p-3} >= K_2$. Hence $g_1(G_{p-2}) = 2 \neq g_1(G_i)(1 \leq i \leq p-3)$ and $\pi = \{G_1, G_2, \ldots, G_{p-2}\}$ is a maximum distinct edge geodetic decomposition of G so that $\pi_{dg_1}(G) = p-2$.

Case 2. Let $q = \frac{p^2-p-2}{2}$. Then $G_{p-2} =< G - G_1 - G_2 - dots - G_{p-3} >= P_3$. Hence $g_1(G_{p-2}) = 2 \neq g_1(G_i)(1 \leq i \leq p-3)$ and $\pi = \{G_1, G_2, \ldots, G_{p-2}\}$ is a distinct edge geodetic decomposition of G so that $\pi_{dg_1}(G) = p-2$.

Case 3. Let$q = \frac{p^2-p}{2} = \frac{p(p-1)}{2}$. Then G is complete. As in the Case 1 and Case 2, $G_{p-3} =< G - G_1 - G_2 - \cdots - G_{p-4} >= K_4$. Let $G_{p-2} = P_3$ and $G_{p-3} = K_4 - P_3$. Then by Theorem C, $g_1(G_{p-2}) = 2$. Also it is clear that $g_1(G_{p-3}) = 3$. Therefore $\pi_{dg_1}(G) = p-2$. □

Corollary 1. *For any connected graph G, $\pi_{dg_1}(G) = p-2$ $(p \geq 4)$ if G has at least $p-2$ universal vertices.*

Proof. Suppose G has at least $p-2$ universal vertex. Then $q \geq \frac{p^2-p-4}{2}$. Hence by Theorem 7, $\pi_{dg_1}(G) = p-2$. □

Corollary 2. *For any complete graph $G = K_p (p \geq 4)$, $\pi_{dg_1}(G) = p-2$.*

Proof. This follows from Corollary 1. □

Theorem 8. *Let G be a connected graph with r universal vertices. Then $r \leq \pi_{dg_1}(G) \leq p-2$.*

Proof. Without loss of generality, assume that $\deg(u_i) = p-1 (1 \leq i \leq r)$. Let $G_1 = K_{1,p-1}$ such that u_1 is the rooted vertex of G_1. Let $V(G_1) = \{u_1, u_2, \ldots, u_p\}$. Then by Theorem D, $g_1(G_1) = p-1$. Let $G_2 = K_{1,p-2}$ such that u_2 is the rooted vertex of G_2. Let $V(G_2) = \{u_2, u_3, \ldots, u_p\}$. Then by Theorem D, $g_1(G_2) = p-2$. Continuing this process, we get $G_{r-1} = K_{1,p-(r-1)}$ such that u_{r-1} is the rooted vertex of G_{r-1} and $g_1(G_{r-1}) = p-r+1$. Let $G_r =< G - G_1 - G_2 - \cdots - G_{r-1} >$. Then the order of G_r is $p-r+1$ and exactly one vertex u_r(say) of G_r has degree $p-r$. Then by Theorem F, $g_1(G_r) = p-r$. Hence $\pi = \{G_1, G_2, \ldots, G_{r-1}, G_r\}$ is a maximum distinct edge geodetic decomposition of G. Hence $\pi_{dg_1}(G) = r$. Suppose $G_r = K_{1,p-r}$ and let $G_{r+1} =< G - G_1 - G_2 - \cdots - G_{r-1} - G_r >$. If $g_1(G_{r+1}) \neq g_1(G_i)(1 \leq i \leq r)$ or any one of $p - i(1 \leq i \leq r)$ can be partitioned into distinct factor, which are differ from $p-1, p-2, \ldots, p-(i-1), p-(i+1), \ldots, p-r$, then by Remark 3, $\pi_{dg_1}(G) \geq r$. If $r \geq p-2$ by Corollary 1, $\pi_{dg_1}(G) = p-2$ and hence $r \leq \pi_{dg_1}(G) \leq p-2$. □

Theorem 9. *Let G be any connected graph having exactly $p-3$ universal vertices. Then $\pi_{dg_1}(G) = p-3$ if and only if $\beta(G) = 3$.*

Proof. Suppose that G has $p-3$ universal vertices and $\pi_{dg_1}(G) = p-3$. We have to prove $\beta(G) = 3$. Suppose that $\beta(G) \neq 3$. Since G has $p-3$ universal vertices, $\beta(G) < 3$. As given in the proof of Theorem 7, let $\pi = \{G_1, G_2, \ldots, G_{p-3}\}$ be a maximum distinct edge geodetic decomposition with $G_{p-2} =< G - G_1 - G_2 - \cdots - G_{p-3} >$. Then $G_{p-2} = K_2$ (otherwise universal vertices will be greater than $p-3$). It is easily verified that $g_1(G_{p-2}) \neq g_1(G_i)(1 \leq i \leq p-3)$. Thus $\pi = \{G_1, G_2, \ldots, G_{p-3}, G_{p-2}\}$ is a maximum distinct edge geodetic decomposition, which is a contradiction.

Conversely, suppose $\beta(G) = 3$. Since G has exactly $p-3$ universal vertices, exactly 3 vertices of G has degree $p-3$. Hence $q = \frac{(p-3)(p-1)+3(p-3)}{2} = \frac{p^2-p-6}{2}$ and $p-1, p-2, p-3, \ldots, 3$ is the maximum partitions of q with distinct parts. Then by Remark 3, $G_1 = K_{1,p-1}, G_2 = K_{1,p-2}, \ldots, G_3 = K_{1,3}$ is a distinct edge geodetic decomposition of G and so $\pi_{dg_1}(G) = p-3$. □

Corollary 3. *Let G be a connected graph G with $\pi_{dg_1}(G) = p-2$. Then $\beta(G) \leq 2$.*

Proof. Let G be a connected graph with $\pi_{dg_1}(G) = p-2$. Then by Theorem 9, G has at least $\frac{p^2-p-4}{2}$ edges and hence G has at least $p-3$ universal vertices so that $\beta(G) \leq 2$. □

Remark 4. The converse of the Corollary 3 need not be true. For the graph G given in Figure 4, $\beta(G) = 2$. The graphs G_1, G_2 in Figure 4 represents a decomposition of G with

$g_1(G_1) = 4$, $g_1(G_2) = 2$. Hence $\pi = \{G_1, G_2\}$ is a distinct edge geodetic decomposition of G. It is easily verified that there is no distinct edge geodetic decomposition of cardinality more than 2 so that $\pi_{dg_1}(G) = 2 \neq p - 2$.

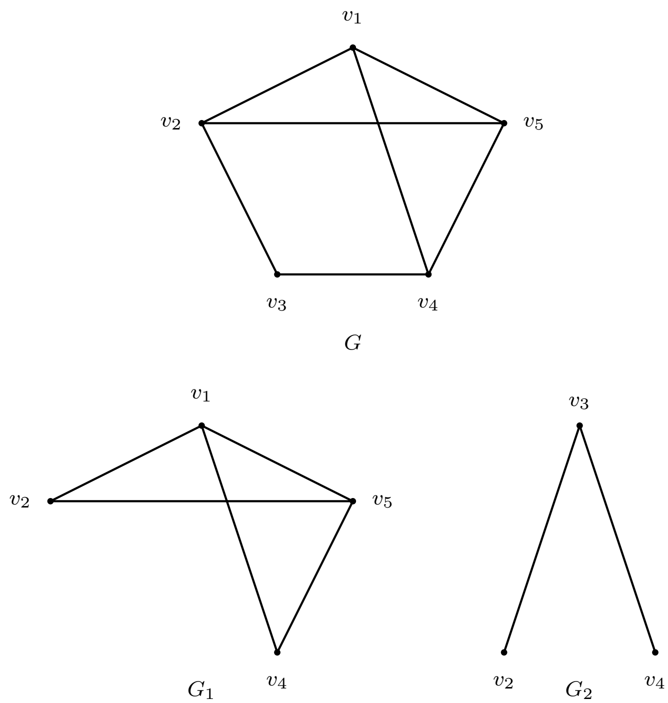

Figure 4. **An example to show that the converse of the Corollary 3 need not be true.**

Lemma 2. *Let G be a connected graph of order p and size q with edge geodetic number $2 \leq g_1(G) \leq p$. Then*

(i) *$g_1(G) = q$ if and only if $G = K_{1,q}$ or K_3.*

(ii) *$g_1(G) = q + 1$ if and only if $G = K_2$.*

Proof. ***(i):*** Let $g_1(G) = q$. Consider the following cases.

Case 1. G is a tree.
If G has at least two cut vertices, then since no cut vertex of G belongs to any minimum edge geodetic set of G, $g_1(G) \leq p - 2 \leq q - 1$, which is a contradiction. Hence G has exactly one cut vertex. This implies that $G = K_{1,q}$.

Case 2. G is not a tree.
Suppose first that G is a cycle C_p. If $p \geq 4$, then by Theorem H, $g_1(G) < q$, which is

a contradiction. Therefore $G = C_3 = K_3$. Suppose now that G is not a cycle. Then by Theorem B, $g_1(G) \leq p < q$, which is a contradiction.
Conversely, let $G = K_{1,q}$ or K_3. Then by Theorems C and A, $g_1(G) = q$.

(ii) Let $g_1(G) = q+1$ and $G \neq K_2$. Then G has at least two edges. Hence by Theorem B, $g_1(G) \leq p \leq q$, which is a contradiction. Hence $g_1(G) = q+1$ if $G = K_2$. Conversely, suppose $G = K_2$. Then by Theorem C, $g_1(G) = 2 = q+1$. □

Theorem 10. *For any connected graph G, $\sum g_1(G_i) \leq q+1$.*

Proof. Let $\pi = \{G_1, G_2, \dots, G_n\}$ be a maximum distinct edge geodetic decomposition of G and $E(G_i) = q_i (1 \leq i \leq n)$. If $G_i \neq K_2 (1 \leq i \leq n)$, then $\sum g_1(G_i) \leq q$. Suppose that $G_i = K_2 (1 \leq i \leq n)$. Then $G_i \neq G_j (1 \leq i, j \leq n)$. By Lemma 2, $\sum g_1(G_i) \leq q+1$. Therefore $\sum g_1(G_i) \leq q+1$. □

Remark 5. Generally distinct edge geodetic decomposition of a graph G is not a partition of q (size of G). For the graph G given in Figure 1, the maximum distinct edge geodetic decomposition is $\pi = \{G_1, G_2\}$. The edge geodetic numbers of G_1 and G_2 are 3 and 2 respectively. But 2 and 3 are not the partition of $q = 7$.

Definition 2. The distinct edge geodetic decomposition $\pi = \{G_1, G_2, \dots, G_n\}$ of G is called a distinct edge geodetic star decomposition if each $G_i (1 \leq i \leq n)$ is the star $K_{1,r} (r > 1)$ and $|E(G_i)| \neq |E(G_j)| (1 \leq i \neq j \leq n)$.

Example 2. For the graph G given in Figure 5, G_1 and G_2 [given in Figure 5] is a decomposition of G. Since $|E(G_1)| \neq |E(G_2)|$, $\pi = \{G_1, G_2\}$ is a distinct edge geodetic star decomposition of G and $g_1(G_1) + g_1(G_2) = 6 = q$.

Theorem 11. *A distinct edge geodetic decomposition is the partition of q if the decomposition is a distinct edge geodetic star decomposition.*

Proof. Let $\pi = \{G_1, G_2, \dots, G_n\}$ be a distinct edge geodetic star decomposition of G. Then $G_1, G_2, \dots, G_n$ are star graphs and $|E(G_i| \neq |E(G_j)| (1 \leq i \neq j \leq n)$. By Theorem D, $g_1(G_i) = |E(G_i)|, (1 \leq i \leq n)$. This implies $\sum g_1(G_i) = q$ so that the distinct edge geodetic decomposition is the partition of q. □

Theorem 12. *For any partition $n_1 < n_2 < n_3 < \cdots < n_k$ $(2 \leq n_i \leq p-2)$ of q, there exist a graph G of order p and size q such that G has a distinct edge geodetic decomposition $\pi = \{G_1, G_2, \dots, G_k\}$, where $g_1(G_i) = n_i (1 \leq i \leq k)$ and $p - q = 1$.*

Proof. Consider the graph $G = K_{1,n_1+n_2+\cdots+n_k}$ and let $G_i = K_{1,n_i}$ for each $i \in \{1, 2, \dots, k\}$. Then clearly, $q = n_1 + n_2 + \cdots + n_k$ and $p = q+1$. Moreover, $\pi = \{G_1, G_2, \dots, G_k\}$ is a distinct edge geodetic decomposition of G. □

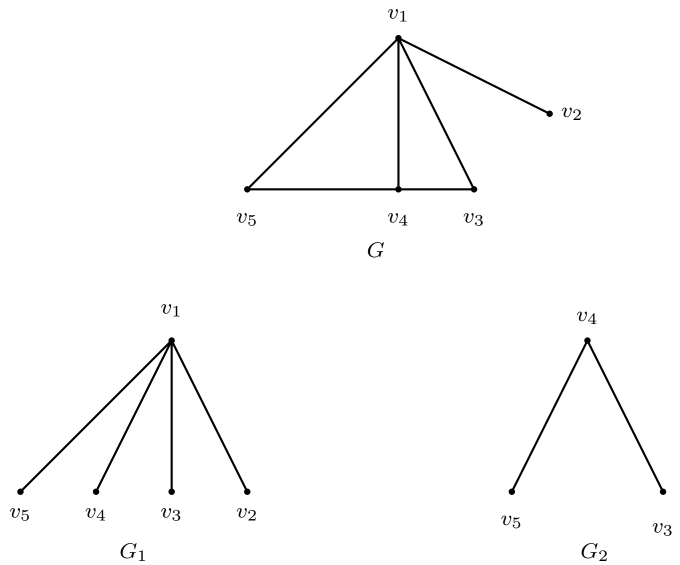

Figure 5. **A graph with a distinct edge geodetic star decomposition**

Conclusion

In this paper, we studied distinct edge geodetic decomposition in graphs. Further, this concept can be extended to monophonic paths related parameters in graphs.

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