Communications in Combinatorics and Optimization
Vol. 6, No. 2, 2021 pp.185-196
DOI: 10.22049/cco.2020.26638.1126

CCO
Commun. Comb. Optim.

Distinct edge geodetic decomposition in graphs

J. John[1] and D. Stalin[2]

[1]Department of Mathematics, Government College of Engineering, Tirunelveli-627 007, S. India
john@gcetly.ac.in

[2]Research and Development Center, Bharathiyar University, Coimbatore-641 046
stalindd@gmail.com

Received: 11 August 2019; Accepted: 19 November 2020
Published Online: 22 November 2020

Abstract: Let $G = (V, E)$ be a simple connected graph of order p and size q. A decomposition of a graph G is a collection π of edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G belongs to exactly one $G_i, (1 \le i \le n)$. The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be a distinct edge geodetic decomposition if $g_1(G_i) \ne g_1(G_j), (1 \le i \ne j \le n)$. The maximum cardinality of π is called the distinct edge geodetic decomposition number of G and is denoted by $\pi_{dg_1}(G)$, where $g_1(G)$ is the edge geodetic number of G. Some general properties satisfied by this concept are studied. Connected graphs of $\pi_{dg_1}(G) \ge 2$ are characterized and connected graphs of order p with $\pi_{dg_1}(G) = p - 2$ are characterized.

Keywords: decomposition, distinct edge geodetic decomposition, distinct edge geodetic decomposition number, edge geodetic number.

AMS Subject classification: 05C51, 05C12, 05C70

1. Terminology and introduction

By a graph $G = (V, E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to Harary [6, 8]. $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ is called the *neighborhood* of the vertex v in G. The *degree* of a vertex $v \in V(G)$ is $|N(v)|$ and is denoted by $deg(v)$. If $e = \{u, v\}$ is an edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then we call e a *pendent edge*, u a leaf and v a support vertex. A vertex of degree $p - 1$ is called a *universal vertex*. A vertex v in a connected graph G is said to be a *semi simplicial vertex* of G if $\Delta(< N(v) >) = |N(v)| - 1$. A vertex v is a *simplicial* vertex of a graph G if $< N(v) >$ is complete. Every simplicial vertex of a graph G is semi simplicial vertex. A graph G is said to be a *semi complete* graph if every vertex of G is a semi simplicial. It is observed that any semi complete graph has at least two vertices of degree $p-2$ or more. For any connected graph G, a vertex $v \in V(G)$ is called a *cut vertex* of G if $V - \{v\}$ is no longer connected. An *independent set* is a set of vertices in a graph, no two of which are adjacent. The number of vertices in a maximum independent vertex set of G is called the *independent vertex number* of G, denoted by β.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the *radius*, *rad* G or $r(G)$ and the maximum eccentricity is its *diameter*, *diamG* of G. A vertex v of G is said to be *peripheral vertex* if $e(v) = \text{diam}(G)$. An u-v path of length $d(u, v)$ is called an *u-v geodesic*. A vertex x is said to lie on a u-v geodesic P if x is a vertex of P including the vertices u and v. For any set S of vertices of G, the *induced subgraph* $< S >$ is the maximal subgraph of G with vertex set S. An *edge geodetic set* of G is a set $S \subseteq V(G)$ such that every edge of G is contained in a geodesic joining some pair of vertices in S. The *edge geodetic number* $g_1(G)$ of G is the minimum order of its edge geodetic sets and any edge geodetic set of order $g_1(G)$ is a *minimum edge geodetic set* of G or g_1-set of G. The edge geodetic number was introduced by Atici [5] and further studied in [1–3, 13, 15, 17–21]. A *decomposition of a graph G* is a collection of edge - disjoint subgraphs $G_1, G_2, \ldots, G_n$ of G such that every edge of G belongs to exactly one G_i, $(1 \leq i \leq n)$. Various types of decompositions of G have been studied in the literature by imposing conditions on the subgraph G_i. In this paper we introduced and studied the concept of *distinct edge geodetic decomposition in graphs*. For references on decomposition parameters in graphs see [4, 7, 9–12, 14, 16]. In number theory and combinatorics, a partition of a positive integer n, also called an *integer partition*, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. We denote by P_p, C_p and $K_{r,s}$, the path on p vertices, the cycle on p vertices and complete bipartite graph in which one partite set has r vertices and the other partite set has s vertices, respectively. Throughout this paper G denotes simple connected graph with at least two vertices. The following theorems are used in sequel.

Theorem A. ([18]) *Each simplicial vertex of G belongs to every edge geodetic dominating set of G.*

Theorem B. ([18]) For any connected graph G, $2 \leq g_1(G) \leq p$.

Theorem C. ([5]) For any non-trivial tree T, the edge geodetic number $g_1(T)$ equals the number of end-vertices in T.

Theorem D. ([5]) For the star $G = K_{1,n}$, $g_1(G) = n = q$.

Theorem E. ([18]) Any connected graph having more than one universal vertex has edge geodetic number p.

Theorem F. ([18]) Any connected graph having exactly one universal vertex has edge geodetic number $p - 1$.

Theorem G. ([18]) For any connected graph G, $g_1(G) = 2$ if and only if there exist two peripheral vertices u and v such that every edges of G lies in a diametral path of u and v.

Theorem H. ([18]) For the cycle $C_p(p \geq 4)$, $g_1(C_p) = 2$ if p is even and $g_1(C_p) = 3$ if p is odd.

Theorem I. ([18]) No cut vertex of G belongs to any minimum edge geodetic set of G.

2. Distinct edge geodetic decomposition in graphs

Definition 1. *The decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of a connected graph G is said to be a* distinct edge geodetic decomposition *if $g_1(G_i) \neq g_1(G_j)(1 \leq i \neq j \leq n)$. The maximum cardinality of π is called the* distinct edge geodetic decomposition number *of G and is denoted by $\pi_{dg_1}(G)$, where $g_1(G)$ is the edge geodetic number of G.*

Example 1. *For the graph G given in Figure 1, G_1 and G_2 [given in Figure 2] is a decomposition of G. Since $g_1(G_1) = 3$ and $g_1(G_2) = 2$, $\pi = \{G_1, G_2\}$ is a distinct edge geodetic decomposition of G. It is easily verified that there is no distinct edge geodetic decomposition of cardinality more than 2. Therefore $\pi_{dg_1}(G) = 2$.*

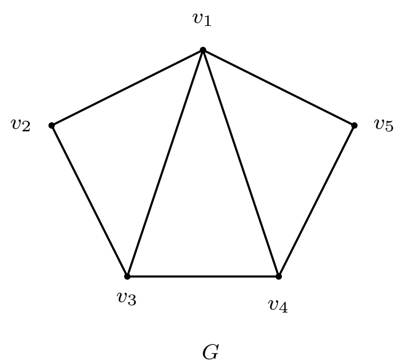

Figure 1. **A graph with $\pi_{dg_1}(G) = 2$.**

Remark 1. *There can be more than one distinct edge geodetic decompositions for a graph. For the graph G given in Figure 1, G_3 and G_4 [given in Figures 2] is a decomposition*

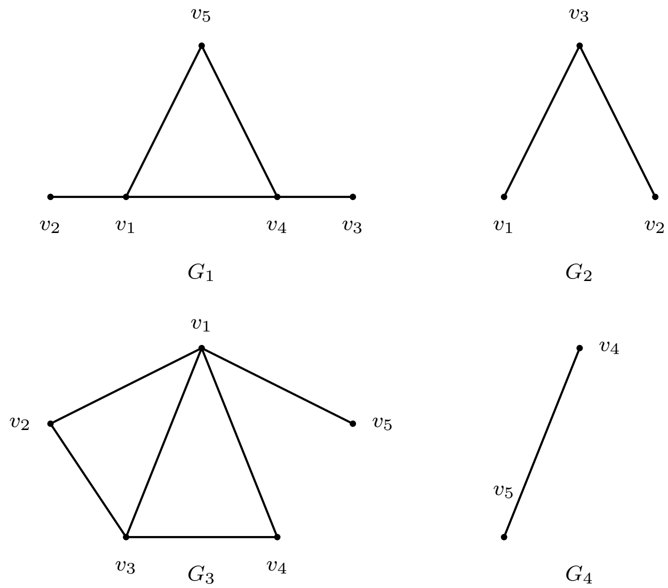

Figure 2.

of G. Since $g_1(G_3) = 4$ and $g_1(G_4) = 2$, $\pi_2 = \{G_3, G_4\}$ is also a distinct edge geodetic decomposition of G.

Observation 1. *For the cycle $G = C_p (p \geq 3)$ and path $G = P_p (p \geq 3)$, $\pi = \{G\}$ so that $\pi_{dg_1}(G) = 1$.*

Observation 2. *Let $G = K_{1,q}$. Then $\pi_{dg_1}(G) \geq 2$ if and only if $q \geq 4$.*

Lemma 1. *For any connected graph G, $g_1(G) = p$ if and only if G is semi complete.*

Proof. Let G be a semi complete graph and S be an edge geodetic set of G. Let v be an vertex of G such that $v \in V \setminus S$. Let u be a vertex of $< N(v) >$ such that $deg_{<N(v)>}(u) = |N(v)| - 1$. Let $u_1, u_2, \ldots, u_k (k \geq 2)$ be the neighbors of $u \in < N(v) >$. Since S is an edge geodetic set of G, the edge uv lies on the x-y geodesic $P : x, x_1, \ldots, u_i, u, v, u_j, \ldots, y$, where $x, y \in S$. Since v is a semi-simplicial vertex of G, u and u_j are adjacent in G and so P is not a x-y geodesic of G, which is a contradiction. This implies that $S = V$ and hence $g_1(G) = p$.
Conversely, suppose $g_1(G) = p$. We claim that G is a semi-simplicial graph. If not, let there exists a vertex v in G such that v is not a semi-simplicial vertex of G. Then for each $w \in N(v)$, there exists $z_w \in [N(v) - \{w\}]$ such that $wz_w \notin E(G)$. Let $S = V(G) - \{v\}$. Consider the edge wv. Since $w, z_w \in S$, the edge wv lies on the

geodesic w, v, z_w. Then S is an edge geodetic set of G with $|S| = p - 1$, which is a contradiction. Therefore, G is a semi-complete graph. □

Theorem 3. *Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a distinct edge geodetic decomposition of G. If any one of the $G_i (1 \leq i \leq k)$ has edge geodetic number p, then none of the $G_i (1 \leq i \leq k)$ has edge geodetic number $p - 1$.*

Proof. Let $G_1 \in \pi$ such that $g_1(G_1) = p$. Then by Lemma 1, G_1 is semi complete. Hence G_1 has at least two vertices of degree $p - 2$ or more. Let $G_2 =< G - G_1 >$. Consider two cases.

Case 1. Let $\Delta(G_1) = p - 2$. Then G_2 has at least two end vertices. Hence G_2 has at least two cut vertices. By Theorem I, $g_1(G_2) \leq p - 2 \neq p - 1$.

Case 2. Let $\Delta(G_1) = p - 1$. Since $g_1(G_1) = p$, G_1 has at least two vertices of degree $p - 1$. Then $|V(G_2)| \leq p - 2$ and so $g_1(G_2) \neq p - 1$. □

Theorem 4. *Let $\pi = \{G_1, G_2, \ldots, G_k\}$ be a distinct edge geodetic decomposition of G. If any one of the $G_i (1 \leq i \leq k)$ has edge geodetic number $p - 1$, then none of the $G_i (1 \leq i \leq k)$ has edge geodetic number p.*

Proof. The proof is similar to the proof of Theorem 3. □

Theorem 5. *For any connected graph G with $p \geq 4$, $1 \leq \pi_{dg_1}(G) \leq p - 2$.*

Proof. From the definition of distinct edge geodetic decomposition of G, $\pi_{dg_1}(G) \geq 1$. Suppose that $\pi_{dg_1}(G) = p - 1$. Let $\pi = \{G_1, G_2, \ldots, G_{p-1}\}$ be a distinct edge geodetic decomposition of G. Since $g_1(G_i) \neq g_1(G_j) (i \neq j)$ and $g_1(G_i) \geq 2 (1 \leq i \leq p - 1)$, there exist $G_l, G_k \in \pi$ such that $g_1(G_1) = p - 1$ and $g_1(G_k) = p$, which is a contradiction to Theorems 3 and 4. Therefore $\pi_{dg_1}(G) \leq p - 2$. Thus $1 \leq \pi_{dg_1}(G) \leq p - 2$. □

Remark 2. The bounds in Theorem 5 are sharp. For the graph $G = K_3$, $\pi_{dg_1}(G) = 1$ and for the complete graph $G = K_p (p \geq 4), \pi_{dg_1}(G) = p - 2$. Also the bounds in Theorem 5 can be strict. For the graph G given in Figure 3, the graphs G_1, G_2 and G_3 illustrated in Figure 3 represents a decomposition of G with $g_1(G_1) = 4$, $g_1(G_2) = 3$ and $g_1(G_3) = 2$. Hence $\pi = \{G_1, G_2, G_3\}$ is a distinct edge geodetic decomposition of G. It is easily verified that there is no distinct edge geodetic decomposition of cardinality more than 3 so that $\pi_{dg_1}(G) = 3$. Therefore $2 < \pi_{dg_1}(G) < p - 2$.

Theorem 6. *For any connected graph G, $\pi_{dg_1}(G) \geq 2$ if and only if $\triangle(G) \geq 3$ and $q \geq 4$.*

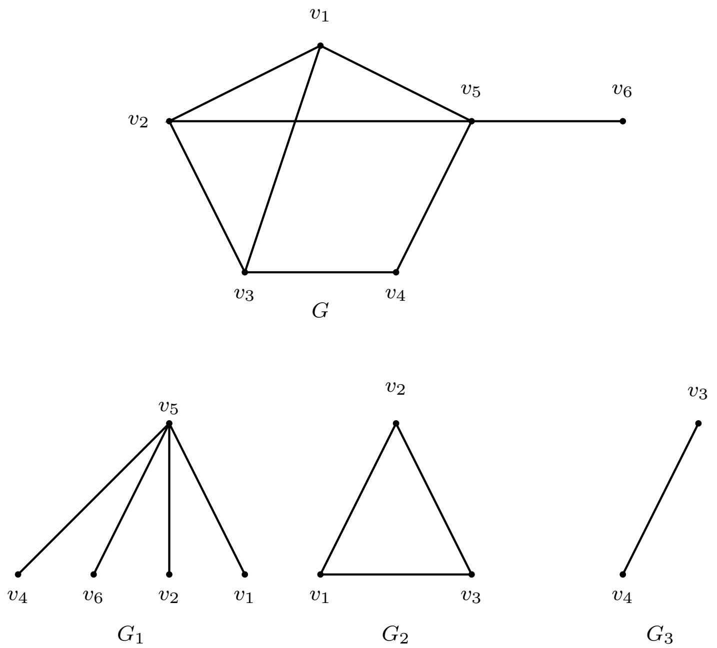

Figure 3. **A graph with $\pi_{dg_1}(G) = 3$.**

Proof. Let $\triangle(G) \geq 3$ and $q \geq 4$. We consider two cases.

Case 1. Suppose G is not a tree.
Then G has at least one cycle. Let $u_1, u_2, u_3, \ldots, u_k, u_1 (k \geq 3)$ be a cycle of G. Let $G_1 = G - \{u_1u_k\}$, $G_2 = K_2 = \{u_1u_k\}$. Also G_1 has at least one edge, which does not lie on the diametral path. Hence by Theorems G, $g_1(G_1) \neq 2$. By Theorem C, $g_1(G_2) = 2$. Therefore $g_1(G_1) \neq g_1(G_2)$ and hence $\pi = \{G_1, G_2\}$ is a distinct geodetic decomposition of G so that $\pi_{dg_1}(G) \geq 2$.

Case 2. Suppose G is a tree.
We distinguish the following situations.

Subcase 2.1. G is a star.
Let $G_1 = \{e\}$ and $G_2 = G - \{e\}$. Then by Theorem C, $g_1(G_1) = 2$. Since $q \geq 4$, by Theorem C, $g_1(G_2) \geq 3$. Then $g_1(G_1) \neq g_1(G_2)$. Hence $\pi = \{G_1, G_2\}$ is a distinct geodetic decomposition of G so that $\pi_{dg}(G) \geq 2$.

Subcase 2.2. G is not a star.
Let u be a vertex of G such that $d(u) \geq 3$. Since $q \geq 4$, there exists at least one pendant vertex x such that x is not adjacent to u. Let $G_1 = K_2 =xy$ and $G_2 = G - \{xy\}$, where y is the adjacent vertex of x. Then by Theorem C, $g_1(G_1) = 2$. Since G_2 has at least three pendant edges, by Theorem A, $g_1(G_1) \geq 3$ such that $g_1(G_1) \neq g_1(G_2)$. Hence $\pi = \{G_1,G_2\}$ is a distinct geodetic decomposition of G so that $\pi_{dg_1}(G) \geq 2$.

Conversely, suppose that $\pi_{dg_1}(G) \geq 2$. If $\triangle(G) \leq 2$, then G is either a path or a cycle. Hence by Observation 1, $\pi_{dg_1}(G) = 1$. Therefore $\triangle(G) \geq 3$. If $q \leq 3$, then G is a star $K_{1,3}$. By Observation 2, $\pi_{dg_1}(G) = 1$, which is a contradiction. Therefore $q \geq 4$. □

Remark 3. *Star is the smallest graph with maximum edge geodetic number. Hence the decomposition of G yields the maximum distinct edge geodetic decomposition if each decomposition G_i is a star.*

Theorem 7. *For any connected graph G, $\pi_{dg_1}(G) = p - 2$ if and only if G has at least $\frac{p^2-p-4}{2}$ edges.*

Proof. Let $\pi_{dg_1}(G) = p - 2$. We have to prove $q \geq \frac{p^2-p-4}{2}$. Let $\pi = \{G_1, G_2, \ldots, G_{p-2}\}$ be a maximum distinct edge geodetic decomposition of G. Let $g_1(G_1) = 2, g_1(G_2) = 3, \ldots, g_1(G_{p-2}) = p - 1$. Since star is the smallest graph with maximum edge geodetic number, let $G_1 = K_2, G_2 = K_{1,3}, G_3 = K_{1,4}, \ldots, G_{p-2} = K_{1,p-1}$. Then $q = 1 + 3 + 4 + \cdots + p - 1 = \frac{p^2-p-4}{2}$. If any one of $G_i(1 \leq i \leq p-2)$, not a star, then $q \geq \frac{p^2-p-4}{2}$.

Conversely, suppose that G has at least $\frac{p^2-p-4}{2}$ edges. Also $\frac{p^2-p-4}{2} = \frac{p(p-1)}{2} - 2$ and addition of two edges to G gives the resulting graph as a complete graph. Hence G has at least $p-3$ universal vertices. As in the first part of the theorem, let $G_1 = K_{1,p-1}$, $G_2 = K_{1,p-2}$, $\ldots$, $G_{p-4} = K_{1,4}$, $G_{p-3} = K_{1,3}$. Then $\pi_{dg_1}(G) \geq p - 3$. Hence by Theorem 5, $\pi_{dg_1}(G) = p - 3$ or $p - 2$. Consider the following cases.

Case 1. Let $q = \frac{p^2-p-4}{2}$. Then $G_{p-2} =< G - G_1 - G_2 - \cdots - G_{p-3} >= K_2$. Hence $g_1(G_{p-2}) = 2 \neq g_1(G_i)(1 \leq i \leq p-3)$ and $\pi = \{G_1, G_2, \ldots, G_{p-2}\}$ is a maximum distinct edge geodetic decomposition of G so that $\pi_{dg_1}(G) = p - 2$.

Case 2. Let $q = \frac{p^2-p-2}{2}$. Then $G_{p-2} =< G - G_1 - G_2 - dots - G_{p-3} >= P_3$. Hence $g_1(G_{p-2}) = 2 \neq g_1(G_i)(1 \leq i \leq p-3)$ and $\pi = \{G_1, G_2, \ldots, G_{p-2}\}$ is a distinct edge geodetic decomposition of G so that $\pi_{dg_1}(G) = p - 2$.

Case 3. Let$q = \frac{p^2-p}{2} = \frac{p(p-1)}{2}$. Then G is complete. As in the Case 1 and Case 2, $G_{p-3} =< G - G_1 - G_2 - \cdots - G_{p-4} >= K_4$. Let $G_{p-2} = P_3$ and $G_{p-3} = K_4 - P_3$. Then by Theorem C, $g_1(G_{p-2}) = 2$. Also it is clear that $g_1(G_{p-3}) = 3$. Therefore $\pi_{dg_1}(G) = p - 2$. □

Corollary 1. *For any connected graph G, $\pi_{dg_1}(G) = p-2$ $(p \geq 4)$ if G has at least $p-2$ universal vertices.*

Proof. Suppose G has at least $p - 2$ universal vertex. Then $q \geq \frac{p^2-p-4}{2}$. Hence by Theorem 7, $\pi_{dg_1}(G) = p - 2$. □

Corollary 2. *For any complete graph $G = K_p(p \geq 4)$, $\pi_{dg_1}(G) = p - 2$.*

Proof. This follows from Corollary 1. □

Theorem 8. *Let G be a connected graph with r universal vertices. Then $r \leq \pi_{dg_1}(G) \leq p-2$.*

Proof. Without loss of generality, assume that $\deg(u_i) = p-1 (1 \leq i \leq r)$. Let $G_1 = K_{1,p-1}$ such that u_1 is the rooted vertex of G_1. Let $V(G_1) = \{u_1, u_2, \ldots, u_p\}$. Then by Theorem D, $g_1(G_1) = p-1$. Let $G_2 = K_{1,p-2}$ such that u_2 is the rooted vertex of G_2. Let $V(G_2) = \{u_2, u_3, \ldots, u_p\}$. Then by Theorem D, $g_1(G_2) = p-2$. Continuing this process, we get $G_{r-1} = K_{1,p-(r-1)}$ such that u_{r-1} is the rooted vertex of G_{r-1} and $g_1(G_{r-1}) = p-r+1$. Let $G_r =< G - G_1 - G_2 - \cdots - G_{r-1} >$. Then the order of G_r is $p-r+1$ and exactly one vertex u_r(say) of G_r has degree $p-r$. Then by Theorem F, $g_1(G_r) = p-r$. Hence $\pi = \{G_1, G_2, \ldots, G_{r-1}, G_r\}$ is a maximum distinct edge geodetic decomposition of G. Hence $\pi_{dg_1}(G) = r$. Suppose $G_r = K_{1,p-r}$ and let $G_{r+1} =< G - G_1 - G_2 - \cdots - G_{r-1} - G_r >$. If $g_1(G_{r+1}) \neq g_1(G_i)(1 \leq i \leq r)$ or any one of $p-i(1 \leq i \leq r)$ can be partitioned into distinct factor, which are differ from $p-1, p-2, \ldots, p-(i-1), p-(i+1), \ldots, p-r$, then by Remark 3, $\pi_{dg_1}(G) \geq r$. If $r \geq p-2$ by Corollary 1, $\pi_{dg_1}(G) = p-2$ and hence $r \leq \pi_{dg_1}(G) \leq p-2$. □

Theorem 9. *Let G be any connected graph having exactly $p-3$ universal vertices. Then $\pi_{dg_1}(G) = p-3$ if and only if $\beta(G) = 3$.*

Proof. Suppose that G has $p-3$ universal vertices and $\pi_{dg_1}(G) = p-3$. We have to prove $\beta(G) = 3$. Suppose that $\beta(G) \neq 3$. Since G has $p-3$ universal vertices, $\beta(G) < 3$. As given in the proof of Theorem 7, let $\pi = \{G_1, G_2, \ldots, G_{p-3}\}$ be a maximum distinct edge geodetic decomposition with $G_{p-2} =< G - G_1 - G_2 - \cdots - G_{p-3} >$. Then $G_{p-2} = K_2$ (otherwise universal vertices will be greater than $p-3$). It is easily verified that $g_1(G_{p-2}) \neq g_1(G_i)(1 \leq i \leq p-3)$. Thus $\pi = \{G_1, G_2, \ldots, G_{p-3}, G_{p-2}\}$ is a maximum distinct edge geodetic decomposition, which is a contradiction.

Conversely, suppose $\beta(G) = 3$. Since G has exactly $p-3$ universal vertices, exactly 3 vertices of G has degree $p-3$. Hence $q = \frac{(p-3)(p-1)+3(p-3)}{2} = \frac{p^2-p-6}{2}$ and $p-1, p-2, p-3, \ldots, 3$ is the maximum partitions of q with distinct parts. Then by Remark 3, $G_1 = K_{1,p-1}, G_2 = K_{1,p-2}, \ldots, G_3 = K_{1,3}$ is a distinct edge geodetic decomposition of G and so $\pi_{dg_1}(G) = p-3$. □

Corollary 3. *Let G be a connected graph G with $\pi_{dg_1}(G) = p-2$. Then $\beta(G) \leq 2$.*

Proof. Let G be a connected graph with $\pi_{dg_1}(G) = p-2$. Then by Theorem 9, G has at least $\frac{p^2-p-4}{2}$ edges and hence G has at least $p-3$ universal vertices so that $\beta(G) \leq 2$. □

Remark 4. The converse of the Corollary 3 need not be true. For the graph G given in Figure 4, $\beta(G) = 2$. The graphs G_1, G_2 in Figure 4 represents a decomposition of G with

$g_1(G_1) = 4$, $g_1(G_2) = 2$. Hence $\pi = \{G_1, G_2\}$ is a distinct edge geodetic decomposition of G. It is easily verified that there is no distinct edge geodetic decomposition of cardinality more than 2 so that $\pi_{dg_1}(G) = 2 \neq p - 2$.

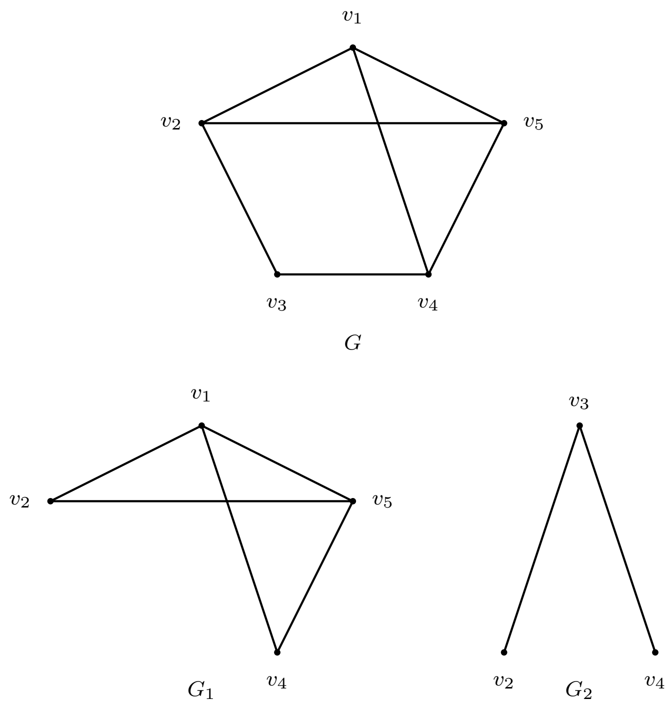

Figure 4. **An example to show that the converse of the Corollary 3 need not be true.**

Lemma 2. *Let G be a connected graph of order p and size q with edge geodetic number $2 \leq g_1(G) \leq p$. Then*

(i) *$g_1(G) = q$ if and only if $G = K_{1,q}$ or K_3.*

(ii) *$g_1(G) = q + 1$ if and only if $G = K_2$.*

Proof. **(*i*):** Let $g_1(G) = q$. Consider the following cases.

Case 1. G is a tree.
If G has at least two cut vertices, then since no cut vertex of G belongs to any minimum edge geodetic set of G, $g_1(G) \leq p - 2 \leq q - 1$, which is a contradiction. Hence G has exactly one cut vertex. This implies that $G = K_{1,q}$.

Case 2. G is not a tree.
Suppose first that G is a cycle C_p. If $p \geq 4$, then by Theorem H, $g_1(G) < q$, which is

a contradiction. Therefore $G = C_3 = K_3$. Suppose now that G is not a cycle. Then by Theorem B, $g_1(G) \leq p < q$, which is a contradiction.
Conversely, let $G = K_{1,q}$ or K_3. Then by Theorems C and A, $g_1(G) = q$.

(ii) Let $g_1(G) = q + 1$ and $G \neq K_2$. Then G has at least two edges. Hence by Theorem B, $g_1(G) \leq p \leq q$, which is a contradiction. Hence $g_1(G) = q+1$ if $G = K_2$. Conversely, suppose $G = K_2$. Then by Theorem C, $g_1(G) = 2 = q + 1$. □

Theorem 10. *For any connected graph G, $\sum g_1(G_i) \leq q + 1$.*

Proof. Let $\pi = \{G_1, G_2, \ldots, G_n\}$ be a maximum distinct edge geodetic decomposition of G and $E(G_i) = q_i(1 \leq i \leq n)$. If $G_i \neq K_2(1 \leq i \leq n)$, then $\sum g_1(G_i) \leq q$. Suppose that $G_i = K_2(1 \leq i \leq n)$. Then $G_i \neq G_j(1 \leq i, j \leq n)$. By Lemma 2, $\sum g_1(G_i) \leq q + 1$. Therefore $\sum g_1(G_i) \leq q + 1$. □

Remark 5. Generally distinct edge geodetic decomposition of a graph G is not a partition of q (size of G). For the graph G given in Figure 1, the maximum distinct edge geodetic decomposition is $\pi = \{G_1, G_2\}$. The edge geodetic numbers of G_1 and G_2 are 3 and 2 respectively. But 2 and 3 are not the partition of $q = 7$.

Definition 2. The distinct edge geodetic decomposition $\pi = \{G_1, G_2, \ldots, G_n\}$ of G is called a distinct edge geodetic star decomposition if each $G_i(1 \leq i \leq n)$ is the star $K_{1,r}(r > 1)$ and $|E(G_i)| \neq |E(G_j)|(1 \leq i \neq j \leq n)$.

Example 2. For the graph G given in Figure 5, G_1 and G_2 [given in Figure 5] is a decomposition of G. Since $|E(G_1)| \neq |E(G_2)|$, $\pi = \{G_1, G_2\}$ is a distinct edge geodetic star decomposition of G and $g_1(G_1) + g_1(G_2) = 6 = q$.

Theorem 11. *A distinct edge geodetic decomposition is the partition of q if the decomposition is a distinct edge geodetic star decomposition.*

Proof. Let $\pi = \{G_1, G_2, \ldots, G_n\}$ be a distinct edge geodetic star decomposition of G. Then $G_1, G_2, \ldots, G_n$ are star graphs and $|E(G_i| \neq |E(G_j)|(1 \leq i \neq j \leq n)$. By Theorem D, $g_1(G_i) = |E(G_i)|, (1 \leq i \leq n)$. This implies $\sum g_1(G_i) = q$ so that the distinct edge geodetic decomposition is the partition of q. □

Theorem 12. *For any partition $n_1 < n_2 < n_3 < \cdots < n_k$ $(2 \leq n_i \leq p - 2)$ of q, there exist a graph G of order p and size q such that G has a distinct edge geodetic decomposition $\pi = \{G_1, G_2, \ldots, G_k\}$, where $g_1(G_i) = n_i(1 \leq i \leq k)$ and $p - q = 1$.*

Proof. Consider the graph $G = K_{1,n_1+n_2+\cdots+n_k}$ and let $G_i = K_{1,n_i}$ for each $i \in \{1, 2, \ldots, k\}$. Then clearly, $q = n_1 + n_2 + \cdots + n_k$ and $p = q + 1$. Moreover, $\pi = \{G_1, G_2, \ldots, G_k\}$ is a distinct edge geodetic decomposition of G. □

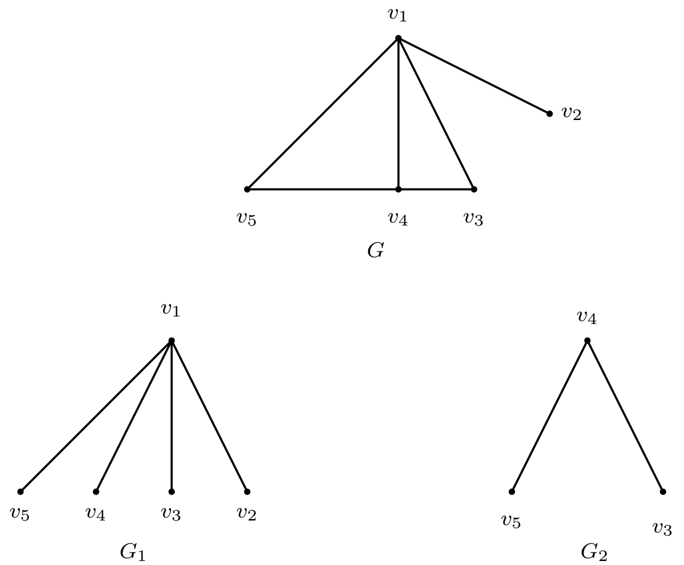

Figure 5. **A graph with a distinct edge geodetic star decomposition**

Conclusion

In this paper, we studied distinct edge geodetic decomposition in graphs. Further, this concept can be extended to monophonic paths related parameters in graphs.

Acknowledgements

The authors are thankful to the referee for the useful suggestions for the revised version of this paper.

References

[1] H. Abdollahzadeh Ahangar, S. Kosari, S.M. Sheikholeslami, and L. Volkmann, *Graphs with large geodetic number*, Filomat **29** (2015), no. 6, 1361–1368.

[2] H. Abdollahzadeh Ahangar and M. Najimi, *Total restrained geodetic number of graphs*, Iran. J. Sci. Technol. Trans. A Sci. **41** (2017), no. 2, 473–480.

[3] H. Abdollahzadeh Ahangar, V. Samodivkin, S.M. Sheikholeslami, and A. Khodkar, *The restrained geodetic number of a graph*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 3, 1143–1155.

[4] J.M. Aroca, A. Lladó, and S. Slamin, *On the ascending subgraph decomposition problem for bipartite graphs*, Electron. Notes Discrete Math. **46** (2014), 19–26.

[5] M. Atici, *On the edge geodetic number of a graph*, Int. J. Comput. Math. **80** (2003), no. 7, 853–861.

[6] F. Buckley and F. Harary, *Distance in Graphs*, Addition-Wesley, Redwood City,

CA, 1990.

[7] S. Dantas, F. Maffray, and A. Silva, *$2K_2$-partition of some classes of graphs*, Discrete Math. **160** (2012), no. 18, 2662–2668.

[8] F. Harary, *Graph Theory*, Narosa Publishing House, 1998.

[9] B. Jamison and S. Olariu, *P-components and the homogeneous decomposition of graphs*, SIAM J. Discrete Math. **8** (1995), no. 3, 448–463.

[10] Z. Jianqin, M. Kejie, and Z. Huishan, *A proof of the Alavi conjecture on integer decomposition*, Acta Math. Sinica **5** (1996), 636–641.

[11] J. John and D. Stalin, *The edge geodetic self decomposition number of a graph*, RAIRO Oper. Res., (in press).

[12] ———, *Edge geodetic self-decomposition in graphs*, Discrete Math. Algorithms Appl. **12** (2020), no. 5, 2050064.

[13] J. John, P.A.P. Sudhahar, and D. Stalin, *On the (M, D)-number of a graph*, Proyecciones J. Math. **38** (2019), no. 2, 255–266.

[14] K. Ma, H. Zhou, and J. Zhou, *On the ascending star subgraph decomposition of star forests*, Combinatorica **14** (1994), no. 3, 307–320.

[15] R.E. Mariano and S.R. Canoy Jr, *Edge geodetic covers in graphs*, Int. Math. Forum **4** (2009), no. 46, 2301–2310.

[16] M. Presler and M. Tarsi, *On the decomposition of graphs into copies of $P_3 \cup tK_2$*, Ars Combin. **35** (1993), 325–333.

[17] V. Samodivkin, *On the edge geodetic and edge geodetic domination numbers of a graph*, Commun. Comb. Optim. **5** (2019), no. 1, 41–54.

[18] A.P. Santhakumaran and J. John, *Edge geodetic number of a graph*, J. Discrete Math. Sci. Cryptogr. **10** (2007), no. 3, 415–432.

[19] ———, *The connected edge geodetic number of a graph*, SCIENTIA Series A: Math. Sci. **17** (2009), 67–82.

[20] ———, *The upper edge geodetic number and the forcing edge geodetic number of a graph*, Opuscula Math. **29** (2009), no. 4, 427–441.

[21] ———, *The upper connected edge geodetic number of a graph*, Filomat **26** (2012), no. 1, 131–141.