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# A characterization relating domination, semitotal domination and total Roman domination in trees

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**Abstract:** A total Roman dominating function on a graph $G$ is a function $f : V(G) \to \{0,1,2\}$ such that for every vertex $v \in V(G)$ with $f(v) = 0$ there exists a vertex $u \in V(G)$ adjacent to $v$ with $f(u) = 2$, and the subgraph induced by the set $\{x \in V(G) : f(x) \geq 1\}$ has no isolated vertices. The total Roman domination number of $G$, denoted $\gamma_{tR}(G)$, is the minimum weight $\omega(f) = \sum_{v\in V(G)} f(v)$ among all total Roman dominating functions $f$ on $G$. It is known that $\gamma_{tR}(G) \geq \gamma_{t2}(G) + \gamma(G)$ for any graph $G$ with neither isolated vertex nor components isomorphic to $K_2$, where $\gamma_{t2}(G)$ and $\gamma(G)$ represent the semitotal domination number and the classical domination number, respectively. In this paper we give a constructive characterization of the trees that satisfy the equality above.

**Keywords:** Total Roman domination; semitotal domination; domination; trees.

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## 1. Introduction

Throughout this paper we consider $G = (V(G), E(G))$ as a simple graph of order $n = |V(G)|$. Given a vertex $v$ of $G$, $N(v)$ and $N[v]$ represent the *open neighbourhood* and the *closed neighbourhood* of $v$, respectively.

A *dominating set* of a graph $G$ is a set $D \subseteq V(G)$ such that every vertex not in $D$ is adjacent to at least one vertex in $D$. The minimum cardinality among all dominating sets is called the *domination number* of $G$ and is denoted by $\gamma(G)$. We refer to [9, 10] for numerous results on this parameter. Now, we consider a recent variant of the concept of domination. A *semitotal dominating set* (STDS) of a graph $G$ without isolated vertices, is a dominating set $D$ of $G$ such that every vertex in $D$ is within distance two of another vertex of $D$. The *semitotal domination number*, denoted by $\gamma_{t2}(G)$, is the minimum cardinality among all STDSs of $G$. This parameter was introduced by Goddard et al. in [8], and was also further studied in [11, 12].

Functions defined on graphs is another variant very studied in domination theory. Let $f : V(G) \to \{0, 1, 2\}$ be a function on a graph $G$. The function $f$ generates three sets $V_0$, $V_1$ and $V_2$, where $V_i = \{v \in V(G) : f(v) = i\}$ for $i \in \{0, 1, 2\}$. We will write $f(V_0, V_1, V_2)$ so as to refer to the function $f$. For a set $S \subseteq V(G)$, $f(S) = \sum_{v\in S} f(v)$ and we define the *weight* of $f$ as $\omega(f) = f(V(G)) = |V_1| + 2|V_2|$. In this sense, by an $f(V(G))$-*function*, we mean a function of weight $f(V(G))$. We shall also use the following notations: $V_{1,2} = \{v \in V_1 : N(v) \cap V_2 \neq \emptyset\}$ and $V_{1,1} = V_1 \setminus V_{1,2}$.

One of the most remarkable dominating functions defined on graphs are the total Roman dominating functions, which were introduced by Liu and Chang [13]. A *total Roman dominating function* (TRDF) on a graph $G$ without isolated vertices, is a function $f(V_0, V_1, V_2)$ such that for every vertex $v \in V_0$ there exists a vertex $u \in N(v) \cap V_2$, and the subgraph induced by $V_1 \cup V_2$ has no isolated vertices. The *total Roman domination number* of $G$, denoted by $\gamma_{tR}(G)$, is the minimum weight among all TRDFs on $G$.

The total Roman domination number was first presented and deeply studied in [1]. Further results on total Roman domination can be found for example, in [2–7].

In [4], Cabrera et al. established the following lower bound for the total Roman domination number of a graph.

**Theorem 1.** [4] *For any graph $G$ with neither isolated vertex nor components isomorphic to $K_2$,*

$$\gamma_{tR}(G) \geq \gamma_{t2}(G) + \gamma(G).$$

Also, the authors [4] posed the following open problem.

**Problem 1.** Characterize the graphs $G$ satisfying $\gamma_{tR}(G) = \gamma_{t2}(G) + \gamma(G)$.

In this article, we address this open problem by giving a constructive characterization of trees satisfying the equality above.

### 1.1. Some Additional Concepts and Notation

For a set $D \subseteq V(G)$, as usual, its *open neighbourhood* and *closed neighbourhood* are $N(D) = \cup_{v\in D} N(v)$ and $N[D] = N(D) \cup D$, respectively. The boundary of the set $D$ is defined as $\partial(D) = N(D) \setminus D$. The *private neighbourhood* $pn(v, D)$ of $v \in D \subseteq V(G)$ is defined by $pn(v, D) = \{u \in V(G)\colon N(u) \cap D = \{v\}\}$. Each vertex in $pn(v, D)$ is called a private neighbour of $v$ with respect to $D$. The *external private neighbourhood* $epn(v, D)$ of $v$ consists of the private neighbours of $v$ in $V(G) \setminus D$. Thus, $epn(v, D) = pn(v, D) \cap (V(G) \setminus D)$. Also, by $G - D$ we denote the graph obtained from $G$ when removing all the vertices in $D$, and all the edges incident with vertices in $D$. The subgraph induced by $D \subseteq V(G)$ is denoted by $G[D]$.
For any two vertices $u$ and $v$, the *distance* $d(u, v)$ between $u$ and $v$ is the minimum length of a $u - v$ path. The *diameter* of $G$, denoted by $diam(G)$, is the maximum distance among pairs of vertices of $G$.
A *leaf vertex* of a graph $G$ is a vertex of degree one, and a *support vertex* of $G$ is a vertex adjacent to a leaf. The set of leaves and support vertices are denoted by $L(G)$ and $S(G)$, respectively. Let $SS(G) = N(S(G)) \setminus (L(G) \cup S(G))$, $S_{adj}(G) = S(G) \cap N(S(G))$ and $S_s(G) = \{v \in S(G) : |N(v) \cap L(G)| \geq 2\}$. Also, given a set $D \subseteq V(G)$ we denote $I(D)$ as an independent set of maximum cardinality in $G[D]$ such that $|I(D) \cap S(G)|$ is maximum.
A *tree* is a connected and acyclic graph. A *star* $K_{1,n-1}$ is a tree of order $n \geq 3$ with a central vertex of degree $n - 1$ and the remaining vertices are leaves. A *double star* is a tree with exactly two vertices that are not leaves. A *rooted tree* $T$ is a tree with a distinguished special vertex $r$, called the root. For each vertex $v \neq r$ of $T$, the *parent* of $v$ is the neighbour of $v$ on the unique $r - v$ path, while a *child* of $v$ is any other neighbour of $v$. A *descendant* of $v$ is a vertex $u \neq v$ such that the unique $r - u$ path contains $v$. Thus, every child of $v$ is a descendant of $v$. The set of descendants of $v$ is denoted by $D(v)$, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

## 2. Trees $T$ with $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$

We begin with a useful result which provides some properties that satisfies a specific TRDF for the trees $T$ with $\gamma_{tR}(T) = \gamma_{t2}(T)+\gamma(T)$. Before, we shall need the following lemma.

**Lemma 1.** [1] *If $G$ is a graph with no isolated vertex, then there exists a $\gamma_{tR}(G)$-function $f(V_0, V_1, V_2)$ such that either $V_2$ is a dominating set in $G$, or the set $S$ of vertices not dominated by $V_2$ satisfies $G[S] = kK_2$ for some $k \geq 1$, where $S \subseteq V_1$ and $\partial(S) \subseteq V_0$.*

**Theorem 2.** *Let $T$ be a tree with $diam(T) \geq 3$ such that $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. Then there exists a $\gamma_{tR}(T)$-function $f(V_0, V_1, V_2)$ satisfying the following conditions.*

(i) *Either $V_2$ is a dominating set of $T$, or the set $V_{1,1}$ satisfies $T[V_{1,1}] = kK_2$ for some $k \geq 1$, where $\partial(V_{1,1}) \subseteq V_0$.*

(ii) *$V_2 \cup I(V_{1,1})$ is a $\gamma(T)$-set and $V_2 \cup V_{1,2} \cup I(V_{1,1})$ is a $\gamma_{t2}(T)$-set.*

(iii) *$V_{1,2} = \emptyset$ or $T[V_{1,2}]$ is isomorphic to an edgeless graph. Furthermore, if $v \in V_{1,2}$, then $|N(v) \cap V_2| = 1$.*

(iv) *If $v \in V_2$, then $V_0 \cap epn(v, V_2) \neq \emptyset$.*

(v) *If $v \in V_2 \cap N(V_{1,2})$, then $N(v) \cap V_2 = \emptyset$.*

(vi) *If $v \in L(T)$, then $v \in V_0 \cup V_{1,1}$.*

(vii) *If $v \in S_s(T)$, then $v \in V_2$ and $N(v) \cap L(T) \subseteq V_0$.*

*Proof.* Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}(T)$-function which satisfies Lemma 1 such that $f(L(T))$ is minimum. Observe that the set $S \subseteq V_1$ of vertices not dominated by $V_2$ is $V_{1,1}$, i.e., $S = V(T) \setminus N[V_2] = V_{1,1}$. Hence, condition (i) holds.

Now, we proceed to prove (ii). First, we notice that $A = V_2 \cup I(V_{1,1})$ and $B = V_2 \cup V_{1,2} \cup I(V_{1,1})$ are a dominating set and a semitotal dominating set, respectively. Hence, $\gamma(T) \leq |A|$ and $\gamma_{t2}(T) \leq |B|$. Since $|A| + |B| = \gamma_{tR}(T)$ and $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$, we obtain that $|B| + |A| = \gamma_{t2}(T) + \gamma(T)$. If $|A| > \gamma(T)$, then $|B| < \gamma_{t2}(T)$, which is a contradiction. Therefore, $|A| = \gamma(T)$ and so, $|B| = \gamma_{t2}(T)$, which completes the proof of (ii).

Next, we proceed to prove (iii). Let $v \in V_{1,2}$. Clearly, $N(v) \cap V_2 \neq \emptyset$. If $N(v) \cap V_{1,2} \neq \emptyset$ or $|N(v) \cap V_2| > 1$, then $(V_2 \cup V_{1,2} \cup I(V_{1,1})) \setminus \{v\}$ is a semitotal dominating set of $T$, which is a contradiction with the fact that $V_2 \cup V_{1,2} \cup I(V_{1,1})$ is a $\gamma_{t2}(T)$-set by (ii). Therefore, $N(v) \cap V_{1,2} = \emptyset$ and $|N(v) \cap V_2| = 1$, which implies that $T[V_{1,2}]$ is isomorphic to an edgeless graph, and that $|N(v) \cap V_2| = 1$, which completes the proof of (iii).

In order to prove (iv), let $v \in V_2$ and suppose that $V_0 \cap epn(v, V_2) = \emptyset$. Let $f'(V_0', V_1', V_2')$ be a function defined on $T$ as follows: $V_2' = V_2 \setminus \{v\}$, $V_1' = V_1 \cup \{v\}$ and $V_0' = V_0$. We claim that $f'$ is a TRDF on $T$. Since $f$ is a TRDF on $T$, then by the definition of $f'$, we only need to prove that every vertex $x \in N(v) \cap V_0'$ has a neighbour in $V_2'$. Let $x \in N(v) \cap V_0'$. As $V_0 \cap epn(v, V_2) = \emptyset$ and $V_0' = V_0$, then there exists a vertex $y \in N(x) \cap (V_2 \setminus \{v\}) \subseteq V_2'$, as desired. Hence, $f'$ is a TRDF on $T$ and satisfies that $\omega(f') < \omega(f) = \gamma_{tR}(T)$, which is a contradiction. Therefore, $V_0 \cap epn(v, V_2) \neq \emptyset$, as desired.

Now, we proceed to prove (v). Let $v \in V_2 \cap N(V_{1,2})$ and $u \in V_{1,2} \cap N(v)$. By (iii) we have that $N(u) \cap V_2 = \{v\}$. If $N(v) \cap V_2 \neq \emptyset$, then as $f$ is a TRDF on $T$ and $N(u) \cap V_2 = \{v\}$, we deduce that the function $f''$ defined by $f''(u) = 0$ and $f''(x) = f(x)$ whenever $x \in V(T) \setminus \{u\}$, is a TRDF on $T$ and satisfies that $\omega(f'') < \omega(f) = \gamma_{tR}(T)$, which is a contradiction. Therefore, $N(v) \cap V_2 = \emptyset$, as desired.

Next, we proceed to prove (vi). Let $v \in L(T)$. If $v \in V_2$, then as $T[V_1 \cup V_2]$ has no isolated vertex, we obtain that $V_0 \cap epn(v, V_2) = \emptyset$, which is a contradiction with condition (iv). Now, if $v \in V_{1,2}$, then the support associated to $v$, namely $u$, satisfies that $u \in V_2$. Let $z \in N(u) \cap V_0$. We consider the function $g$, defined by

$g(v) = 0$, $g(z) = 1$ and $g(x) = f(x)$ whenever $x \in V(T) \setminus \{v, z\}$. Notice that $g$ is a $\gamma_{tR}(T)$-function as well and satisfies that $g(L(T)) < f(L(T))$, which is a contradiction. Therefore, $v \notin V_{1,2} \cup V_2$, i.e. $v \in V_0 \cup V_{1,1}$ as desired.
Finally, we proceed to prove (vii). Let $v \in S_s(T)$. Hence, $|N(v) \cap L(T)| \geq 2$. Notice that every support vertex has positive weight under $f$. Hence $v \in V_1 \cup V_2$. If $v \in V_1$, then for every $h \in N(v) \cap L(T)$ we have that $h \in V_1$. Let $w \in N(v) \setminus L(T)$ (notice that $w$ exists because $diam(T) \geq 3$). Now, the function $g$, defined by $g(v) = 2$, $g(h) = 0$ if $h \in N(v) \cap L(T)$, $g(w) = \max\{1, f(w)\}$ and $g(x) = f(x)$ otherwise, is a $\gamma_{tR}(T)$-function as well and satisfies that $g(L(T)) < f(L(T))$, which is a contradiction. Therefore, $v \in V_2$ and $N(v) \cap L(T) \subseteq V_0$, which completes the proof. □

**Remark 1.** If $T'$ is a subtree of a tree $T$, then $\gamma(T') \leq \gamma(T)$ and $\gamma_{t2}(T') \leq \gamma_{t2}(T)$.

We next provide a constructive characterization of the trees $T$ satisfying $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. To this end, we need to introduce some additional terminology.

An *almost semitotal dominating set of a tree $T$ relative to a vertex $v$* (ASTDS of $T$ relative to $v$) is a dominating set $S$ of $T$ (with $|S| \geq 2$) such that every vertex in $S \setminus N[v]$ is within distance 2 of another vertex of $S$. The *almost semitotal domination number of $T$ relative to $v$*, denoted $\gamma_{t2}(T; v)$, is the minimum cardinality among all ASTDSs of $T$ relative to $v$. An ASTDS of $T$ relative to $v$ of cardinality $\gamma_{t2}(T; v)$ we call a $\gamma_{t2}(T; v)$-set. Notice that for any vertex $v$ of a tree $T$, we have that any STDS is an ASTDS relative to $v$. Hence, $\gamma_{t2}(T; v) \leq \gamma_{t2}(T)$ for any vertex $v$ of $T$.
We define a vertex $v \in V(T)$ to be a *stable vertex* if $\gamma(T - v) \geq \gamma(T)$ and $\gamma_{t2}(T; v) = \gamma_{t2}(T)$. Also, we define a vertex $v \in V(T)$ to be a *semi-stable vertex* if $\gamma_{t2}(T; v) = \gamma_{t2}(T)$. Finally, we consider the following sets.

$$W^1_{t2}(T) = \{v \in V(T) : v \text{ belongs to some } \gamma_{t2}(T)\text{-set}\}.$$

$$W^a_{t2}(T) = \{v \in V(T) : v \text{ belongs to some } \gamma_{t2}(T; v)\text{-set}\}.$$

$$S_2(T) = \{v \in S(T) : f(v) = 2 \text{ for some } \gamma_{tR}(T)\text{-function } f\}.$$

Next, we show an example of some definitions above. For the tree $T$ given in the Figure 1 we have the following.

- The set $\{s_1, s_2, s_3\}$ is the only $\gamma_{t2}(T)$-set and the only $\gamma_{t2}(T; v)$-set.
- The sets $\{h, s_2, s_3\}$ and $\{s_1, s_2, s_3\}$ are the only $\gamma_{t2}(T; h)$-sets.
- $W^1_{t2}(T) = \{s_1, s_2, s_3\}$ and $W^a_{t2}(T) = \{h, s_1, s_2, s_3\}$.

For integers $a, b, c$ with $a \geq 1$, $b \in \{0, 1\}$ and $c \geq 0$, the graph $T_{a,b,c}$ is defined as the graph obtained from $P_4$, $P_3$, $P_2$ and $N_1$ by taking one copy of $N_1$, $a$ copies of $P_4$, $b$ copies of $P_3$ and $c$ copies of $P_2$ and joining by an edge one support vertex of each copy of $P_4$ and one leaf vertex of each copy of $P_3$ and $P_2$ with the vertex of $N_1$. The vertex associated to the copy of $N_1$ is called the *special vertex* of $T_{a,b,c}$. In Figure 2 we show the tree $T_{1,1,1}$.

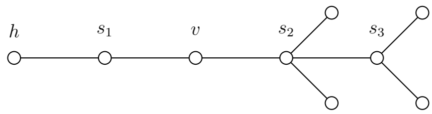


**Figure 1.** **A tree $T$ where $h \in W^a_{t2}(T) \setminus W^1_{t2}(T)$ and $v \in V(T) \setminus (W^1_{t2}(T) \cup W^a_{t2}(T))$.**

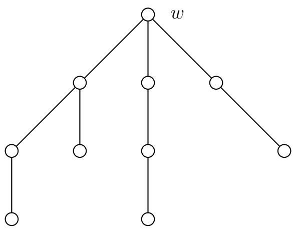


**Figure 2.** **The structure of the tree $T_{1,1,1}$, where $w$ is the special vertex.**

**Remark 2.** For any integers $a, b, c$ with $a \geq 1$, $b \in \{0, 1\}$ and $c \geq 0$,

$$\gamma_{tR}(T_{a,b,c}) = \gamma_{t2}(T_{a,b,c}) + \gamma(T_{a,b,c}).$$

*Proof.* Let $T_{a,b,c}$ be the tree defined above with special vertex $w \in V(T_{a,b,c})$. Let $S_b(T_{a,b,c}) = \{x \in S(T_{a,b,c}) \setminus S_{adj}(T_{a,b,c}) : d(x, w) = 2\}$. Now, we define a function $f(V_0, V_1, V_2)$ on $T_{a,b,c}$ as follows: $V_2 = S_{adj}(T_{a,b,c}) \cup S_b(T_{a,b,c})$, $V_0 = \partial(S_{adj}(T_{a,b,c})) \cup (N(S_b(T_{a,b,c})) \cap L(T_{a,b,c}))$ and $V_1 = V(T_{a,b,c}) \setminus (V_2 \cup V_0)$. It is easy to see that $f$ is a TRDF on $T_{a,b,c}$, which implies that $\gamma_{tR}(T_{a,b,c}) \leq \omega(f) = 2|V_2| + |V_1| = \gamma_{t2}(T_{a,b,c}) + \gamma(T_{a,b,c})$. The result follows by Theorem 1. □

**Lemma 2.** *Let $T_2$ be a tree obtained from a tree $T$ by attaching a path $P_2$ to a stable vertex $v$ of $T$. Then $\gamma(T_2) = \gamma(T) + 1$ and $\gamma_{t2}(T_2) = \gamma_{t2}(T) + 1$.*

*Proof.* Assume $T_2$ is obtained from $T$ by adding the path $uu_1$ and the edge $uv$, where $v$ is a stable vertex of $T$. Notice that any dominating set of $T$ can be extended to a dominating set of $T_2$ by adding the vertex $u$. Hence, $\gamma(T_2) \leq \gamma(T) + 1$. Let $D$ be a $\gamma(T_2)$-set containing $u$. If $D \setminus \{u\}$ is a dominating set of $T$, then $\gamma(T) \leq \gamma(T_2) - 1$. Conversely, if $D \setminus \{u\}$ is not a dominating set of $T$, then $D \setminus \{u\}$ is a dominating set of $T - v$, and as $v$ is a stable vertex of $T$, we deduce that $\gamma(T) \leq \gamma(T - v) \leq \gamma(T_2) - 1$. Therefore, in both cases, we obtain that $\gamma(T_2) = \gamma(T) + 1$.
Moreover, observe that any semitotal dominating set of $T$ can be extended to a semitotal dominating set of $T_2$ by adding the vertex $u$. Hence, $\gamma_{t2}(T_2) \leq \gamma_{t2}(T) + 1$. Let $S$ be a $\gamma_{t2}(T_2)$-set containing $u$. Notice that $D \setminus \{u\}$ is an ASTDS of $T$ relative to $v$. Since $v$ is a stable vertex of $T$, we have that $\gamma_{t2}(T) = \gamma_{t2}(T; v) \leq \gamma_{t2}(T_2) - 1$, which implies that $\gamma_{t2}(T_2) = \gamma_{t2}(T) + 1$. □

Let $\mathcal{H}$ be the family of trees $T$ that can be obtained from a sequence of trees $T_0, \ldots, T_k = T$, with $k \geq 0$ in the following way. First, we consider $T_0 = P_4$. Then, for any $i \in \{1, \ldots, k\}$, the tree $T_i$ can be obtained from the tree $T' = T_{i-1}$ by one of the following operations.

**Operation $O_1$:** Add a new vertex $u$ to $T'$ and join $u$ to a vertex $v \in S_2(T')$.

**Operation $O_2$:** Add a path $P_2$, and join a leaf of the path to a stable vertex $v$ of $T'$.

**Operation $O_3$:** Add a path $P_3$, and join a leaf of the path to a semi-stable vertex $v \in V(T') \setminus (W^1_{t2}(T') \cup W^a_{t2}(T'))$.

**Operation $O_4$:** Add a tree $T_{a,b,c}$ with special vertex $u$, and join $u$ to an arbitrary vertex $v$ of $T'$.

For instance, notice the tree $T$ given in Figure 1 belongs to the family $\mathcal{H}$. First, we consider the subtree $T' = T[N[\{s_2, s_3\}]]$. Observe that $T' \in \mathcal{H}$ because can be obtained from $P_4$ by repeatedly applying Operation $O_1$ three times. Now, note that $v \in L(T')$ is a stable vertex of $T'$. Therefore, $T$ can be obtained from $T'$ by Operation $O_2$, which implies that $T \in \mathcal{H}$, as required.
We will now show that every tree $T$ of the family $\mathcal{H}$ satisfies that $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$.

**Theorem 3.** *If $T \in \mathcal{H}$, then $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$.*

*Proof.* We proceed by induction on the number $r(T)$ of operations required to construct the tree $T$. If $r(T) = 0$, then $T = P_4$ and satisfies that $\gamma_{tR}(T) = 4 = \gamma_{t2}(T) + \gamma(T)$. This establishes the base case. Hence, we now assume that $k \geq 1$ is an integer and that each tree $T' \in \mathcal{H}$ with $r(T') < k$ satisfies that $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$.
Let $T \in \mathcal{H}$ be a tree with $r(T) = k$, which is obtained from a tree $T' \in \mathcal{H}$ with $r(T') = k - 1$ by one of the operations defined above. We shall prove that $T$ satisfies that $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. For this, we consider the following cases, depending on which operation is used to construct the tree $T$ from $T'$.

**Case 1.** $T$ is obtained from $T'$ by Operation $O_1$. Assume $T$ is obtained from $T'$ by adding a new vertex $u$ and the edge $uv$, where $v \in S_2(T')$. Thus, there exists a $\gamma_{tR}(T')$-function $f'$ satisfying that $f'(v) = 2$. Observe that the function $f$, defined by $f(u) = 0$ and $f(x) = f'(x)$ whenever $x \in V(T')$, is a TRDF on $T$. Hence, $\gamma_{tR}(T) \leq \omega(f) = \omega(f') = \gamma_{tR}(T')$. Thus, by Theorem 1, the inequality above, the inductive hypothesis and Remark 1, it follows that $\gamma_{t2}(T) + \gamma(T) \leq \gamma_{tR}(T) \leq \gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T') \leq \gamma_{t2}(T) + \gamma(T)$. Therefore, we must have equalities throughout the inequality chain above. In particular, $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$.

**Case 2.** $T$ is obtained from $T'$ by Operation $O_2$. Assume $T$ is obtained from $T'$ by adding a path $uu_1$ and the edge $uv$ where $v$ is a stable vertex of $T'$. Again, notice that any $\gamma_{tR}(T')$-function can be extended to a TRDF on $T$ by assigning the weight 1 to $u$ and $u_1$. Hence, $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 2$. Thus, by Theorem 1, the inequality above, the inductive hypothesis and Lemma 2, we obtain $\gamma_{t2}(T) + \gamma(T) \leq \gamma_{tR}(T) \leq \gamma_{tR}(T') + 2 = \gamma_{t2}(T') + \gamma(T') + 2 = \gamma_{t2}(T) + \gamma(T)$. Thus, we must have equalities throughout the above inequality chain. In particular, $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$.

**Case 3.** $T$ is obtained from $T'$ by Operation $O_3$. Assume $T$ is obtained from $T'$ by adding a path $uu_1u_2$ and the edge $uv$ where $v$ is a semi-stable vertex and belongs to $V(T') \setminus (W^1_{t2}(T') \cup W^a_{t2}(T'))$. Observe that any $\gamma_{tR}(T')$-function can be extended to a TRDF on $T$ by assigning the weight 1 to $u$, $u_1$ and $u_2$. Hence, $\gamma_{tR}(T) \leq \gamma_{tR}(T') + 3$. Moreover, notice that $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 1$. Now, suppose that $\gamma_{t2}(T') = \gamma_{t2}(T) - 1$ and let $S$ be a $\gamma_{t2}(T)$-set such that $v, u_1 \in S$. If $S' = S \setminus \{u_1\}$ is a STDS of $T'$, then $S'$ is a $\gamma_{t2}(T')$-set containing $v$, which is a contradiction with the fact that $v \notin W^1_{t2}(T')$. Hence, $S'$ is not a STDS of $T'$, however $S'$ is an ASTDS of $T'$ relative to $v$ and as $v$ is a semi-stable vertex of $T'$, it follows that $\gamma_{t2}(T') = \gamma_{t2}(T'; v) \leq |S'| = |S \setminus \{u_1\}| = \gamma_{t2}(T) - 1 = \gamma_{t2}(T')$. Thus, $S'$ is a $\gamma_{t2}(T'; v)$-set containing $v$, which is a contradiction with the fact that $v \notin W^a_{t2}(T')$. So, $\gamma_{t2}(T') \leq \gamma_{t2}(T) - 2$. Also, it is clear that $\gamma(T) = \gamma(T') + 1$. Hence, by Theorem 1, the inequalities above and the inductive hypothesis we obtain $\gamma_{t2}(T) + \gamma(T) \leq \gamma_{tR}(T) \leq \gamma_{tR}(T') + 3 = \gamma_{t2}(T') + \gamma(T') + 3 \leq \gamma_{t2}(T) + \gamma(T)$. Thus, we must have equalities throughout the inequality chain above. In particular, $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$.

**Case 4.** $T$ is obtained from $T'$ by Operation $O_4$. Assume $T$ is obtained from $T'$ by adding the tree $T_{a,b,c}$ with special vertex $u$, and join $u$ to an arbitrary vertex $v$ of $T'$. Let $S_b(T_{a,b,c}) = \{x \in S(T_{a,b,c}) \setminus S_{adj}(T_{a,b,c}) : d(x, u) = 2\}$. Notice that any $\gamma_{tR}(T')$-function $f'(V'_0, V'_1, V'_2)$ can be extended to a TRDF $f(V_0, V_1, V_2)$ on $T$ as follows: $V_2 = V'_2 \cup S_{adj}(T_{a,b,c}) \cup S_b(T_{a,b,c})$, $V_0 = V'_0 \cup \partial(S_{adj}(T_{a,b,c})) \cup (N(S_b(T_{a,b,c})) \cap L(T_{a,b,c}))$ and $V_1 = V(T) \setminus (V_2 \cup V_0)$. Hence, $\gamma_{tR}(T) \leq \omega(f) = \gamma_{tR}(T') + (4a + 3b + 2c)$. Notice that the definition of $f$ is similar to the one given in the proof of Remark 2, where we deduce that $\gamma_{tR}(T_{a,b,c}) = \gamma_{t2}(T_{a,b,c}) + \gamma(T_{a,b,c})$. In that a sense, it is easy to see that $\gamma_{t2}(T) = \gamma_{t2}(T') + (2a + 2b + c)$ and $\gamma(T) = \gamma(T') + (2a + b + c)$. Thus, by Theorem 1, the inequalities above and the inductive hypothesis we obtain that $\gamma_{t2}(T) + \gamma(T) \leq \gamma_{tR}(T) \leq \gamma_{tR}(T') + (4a + 3b + 2c) = \gamma_{t2}(T') + \gamma(T') + (4a + 3b + 2c) = \gamma_{t2}(T) + \gamma(T)$. Thus, we must have equalities throughout this inequality chain. In particular, $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. □

Now, we prove that any tree $T$ with $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$ belongs to the family $\mathcal{H}$.

**Theorem 4.** *Let $T$ be a tree with $diam(T) \geq 3$. If $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$, then $T \in \mathcal{H}$.*

*Proof.* We proceed by induction on the order $n \geq 4$ of the trees $T$ with $diam(T) \geq 3$ that satisfy $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. We observe that if $diam(T) = 3$, then $T$ is a double star and so, $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. Thus, either $T$ is a path $P_4$ or $T$ can be obtained from $P_4$ by repeatedly applying Operation $O_1$. Therefore, $T \in \mathcal{H}$. This establishes the base case. We assume now that $diam(T) \geq 4$. Also, we consider an integer $n > 4$ such that each tree $T'$ with $|V(T')| < n$ and $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$ satisfies that $T' \in \mathcal{H}$. Let $T$ be a tree with $|V(T)| = n$ and $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$. We shall prove that $T \in \mathcal{H}$. For this, we root the tree $T$ at a leaf vertex $r$ belonging to a longest path in $T$. Let $h$ be a vertex at maximum distance from $r$. Clearly, $h$ is a leaf. Let $s$ be the parent of $h$, let $v$ be the parent of $s$, let $w$ be the parent of $v$, and let $z$ be the parent of $w$ (note that it could happen $z = r$). We now proceed with the following claims.

**Claim I.** If $\delta_T(s) \geq 3$, then $T \in \mathcal{H}$.

**Proof.** Let $f$ be a $\gamma_{tR}(T)$-function which satisfies Theorem 2. Since $\delta_T(s) \geq 3$, it follows that $s \in S_s(T)$, and so, by Theorem 2 (vii), $f(s) = 2$ and $f(h) = 0$. Let $T' = T - h$. Hence, $f$ restricted to $V(T')$ is a TRDF on $T'$, which implies that $\gamma_{tR}(T') \leq \gamma_{tR}(T)$. Also, since $s \in S(T')$, we have that $\gamma_{t2}(T) = \gamma_{t2}(T')$ and $\gamma(T) = \gamma(T')$ by Remark 1. Thus, by Theorem 1, inequalities above and the hypothesis of the theorem we obtain that $\gamma_{t2}(T')+\gamma(T') \leq \gamma_{tR}(T') \leq \gamma_{tR}(T) = \gamma_{t2}(T)+\gamma(T) = \gamma_{t2}(T')+\gamma(T')$. Consequently, we must have equality throughout this inequality chain. In particular $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$. Applying the hypothesis inductive to $T'$, it follows that $T' \in \mathcal{H}$.
Another consequence of the equalities given in the inequality chain above is that $\gamma_{tR}(T') = \gamma_{tR}(T)$. This implies that $f$ restricted to $V(T')$ is a $\gamma_{tR}(T')$-function. Hence, $s \in S_2(T')$. Therefore, $T$ can be obtained from $T'$ by Operation $O_1$, and consequently, $T \in \mathcal{H}$. $(\diamond)$

Notice that the position of vertex $s$ in $T$ with respect to $r$ is not important in the presented proof of **Claim I**. Hence, we may henceforth assume that $S_s(T) = \emptyset$.

**Claim II.** If $\delta_T(s) = 2$ and $\delta_T(v) \geq 3$, then $T \in \mathcal{H}$.

**Proof.** In this case, we have that $v \in SS(T) \cup S(T)$. Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}(T)$-function which satisfies Theorem 2. Now, we analyse the following two cases.

Case 1. $v \in SS(T)$. In this case we have that $v \notin S(T)$, and as $|N(v) \cap S(T)| \geq 2$, Theorem 2 (iii) leads to $f(v) \neq 1$. Hence, if $f(v) = 0$, then $f(s) = f(h) = 1$. Otherwise, if $f(v) = 2$, then $f(s) = 2$ and $f(h) = 0$. Let $T' = T - \{s, h\}$. Notice that, in both cases, $f$ restricted to $V(T')$ is a TRDF on $T'$, and so $\gamma_{tR}(T') \leq \gamma_{tR}(T) - 2$. Also, since $v \in SS(T')$, it is easy to check that $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1$ and $\gamma(T) \leq \gamma(T') + 1$. Thus, by Theorem 1, the inequalities above and the hypothesis of the theorem, we obtain that

$$\gamma_{t2}(T') + \gamma(T') \leq \gamma_{tR}(T') \leq \gamma_{tR}(T) - 2 = \gamma_{t2}(T) + \gamma(T) - 2 \leq \gamma_{t2}(T') + \gamma(T'). \quad (1)$$

As a consequence, we must have equalities throughout the inequality chain (1). In particular, $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$ and by the inductive hypothesis, $T' \in \mathcal{H}$. Also, we obtain that $\gamma_{t2}(T) = \gamma_{t2}(T')+1$ and $\gamma(T) = \gamma(T')+1$. Notice that any dominating set of $T'-v$ (respectively, ASTDS of $T'$ relative to $v$ ) can be extended to a dominating set (respectively, STDS) of $T$ by adding the vertex $s$. Hence $\gamma(T) \le \gamma(T'-v)+1$ and $\gamma_{t2}(T) \le \gamma_{t2}(T';v)+1$. If $v$ is not a stable vertex of $T'$, then we obtain a contradiction with at least one of the equalities $\gamma_{t2}(T) = \gamma_{t2}(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. So, $v$ is a stable vertex of $T'$. Therefore, $T$ can be obtained from $T'$ by Operation $O_2$, and consequently, $T \in \mathcal{H}$.

Case 2. $v \in S(T)$. Let $N(v)\cap L(T) = \{h'\}$ (recall that $S_s(T) = \emptyset$). Since $v, s \in S(T)$, we have that $f(v) = f(s) = 2$ and $f(h) = f(h') = 0$. We analyse the following two subcases.

Subcase 2.1. $f(w) > 0$. In this subcase, $f$ restricted to $V(T) \setminus \{s, h\}$ is a TRDF on $T' = T-\{s, h\}$. Hence $\gamma_{tR}(T') \le \gamma_{tR}(T)-2$. Also, since $v \in S(T')$, it is easy to check to $\gamma_{t2}(T) \le \gamma_{t2}(T') + 1$ and $\gamma(T) = \gamma(T') + 1$. Thus, by Theorem 1, the inequalities above and the hypothesis of the theorem, we obtain Inequality chain (1). So, we must have equalities throughout this inequality chain. In particular, $\gamma_{t2}(T) = \gamma_{t2}(T') + 1$ and also, $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$, which implies that $T' \in \mathcal{H}$ by the inductive hypothesis,. By proceeding analogously to the Case 1 of Claim II, we deduce that $v$ is a stable vertex of $T'$. Hence, $T$ can be obtained from $T'$ by Operation $O_2$. Therefore, $T \in \mathcal{H}$.

Subcase 2.2. $f(w) = 0$. First, we observe that $w \notin S(T)$. Now, we notice that $w$ does not have two children $v_1$ and $v_2$ such that $T_{v_1} = T_{v_2} = P_3$ (otherwise, the set $(V_2 \cup V_{1,2} \cup I(V_{1,1}) \cup \{w\}) \setminus \{v_1, v_2\}$ is a STDS of $T$, which is a contradiction with Theorem 2 (ii)). Hence, by Theorem 2 we have that $T_w \cong T_{a,b,c}$. Let $T' = T-V(T_w)$. Since $f(w) = 0$, we have that $f$ restricted to $V(T')$ is a TRDF on $T'$. So, $\gamma_{tR}(T') \le \omega(f)-f(V(T_w)) = \gamma_{tR}(T)-(4a+3b+2c)$. Also, by Theorem 2, it is easy to check to $\gamma_{t2}(T) = \gamma_{t2}(T')+(2a+2b+c)$ and $\gamma(T) = \gamma(T')+(2a+b+c)$. Thus, by Theorem 1, the inequalities above and the hypothesis of the theorem, we obtain $\gamma_{t2}(T')+\gamma(T') \le \gamma_{tR}(T') \le \gamma_{tR}(T)-(4a+3b+2c) = \gamma_{t2}(T)+\gamma(T)-(4a+3b+2c) = \gamma_{t2}(T')+\gamma(T')$. So, we must have equalities throughout the inequality chain above. In particular, $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$.
If $diam(T') \ge 3$, then by the inductive hypothesis, $T' \in \mathcal{H}$. Therefore, $T$ can be obtained from $T'$ by Operation $O_4$, and consequently, $T \in \mathcal{H}$.
Now, we suppose that $diam(T') \in \{1, 2\}$. This implies that $T' \in \{P_2, P_3\}$. Since $T_w \cong T_{a,b,c}$ and $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$, we deduce that $T \cong T_{a,b',c'}$, and so, by the structure of $T$, it is easy to check that $T \in \mathcal{H}$. $(\diamond)$

**Claim III.** If $\delta_T(s) = \delta_T(v) = 2$ and $\delta_T(w) \ge 3$, then $T \in \mathcal{H}$.

**Proof.** Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}(T)$-function which satisfies Theorem 2. Let $v' \in N(w) \setminus \{z, v\}$. Notice that $v' \in L(T) \cup S(T) \cup SS(T)$. Suppose that $f(v) = 2$. If $v' \in L(T) \cup S(T)$, then the set $(V_2 \cup I(V_{1,1})) \setminus \{v\}$ is a dominating set of $T$, which is a contradiction with Theorem 2 (ii). So, $v' \in SS(T)$, which implies that $f(v') = 1$ as consequence of Theorem 2. Notice that the set $(V_2 \cup V_{1,2} \cup I(V_{1,1}) \cup \{w\}) \setminus \{v', v\}$ is a STDS of $T$, which is again a contradiction with Theorem 2 (ii). Thus, $f(v) \leq 1$. Now, we analyse the following two cases.

Case 1. $f(v) = 0$. Notice that $f(w) = 2$ and $f(s) = f(h) = 1$. Let $T' = T - \{h, s\}$. By proceeding analogously to the Case 1 of Claim II, we deduce that $T' \in \mathcal{H}$ and that $v$ is a stable vertex of $T'$. Hence, $T$ can be obtained from $T'$ by Operation $O_2$, and consequently, $T \in \mathcal{H}$.

Case 2. $f(v) = 1$. In this case, Theorem 2 leads to $f(s) = 2$ and $f(h) = f(w) = 0$. If $w \in S(T)$ then $f(w) = 2$, which is a contradiction with Theorem 2 (ii) because the set $(V_2 \cup I(V_{1,1})) \setminus \{v\}$ is a dominating set of $T$. Hence, $w \notin S(T)$. If there exists a child of $w$ different from $v$, belonging to $SS(T)$, then, as proceeding as before, we can obtain a STDS of $T$ of cardinality less than $\gamma_{t2}(T)$, which is a contradiction. So, $N(w) \setminus \{v, z\} \subseteq S(T)$. Next, we consider the following two subcases.

Subcase 2.1. There exists $x \in N(w) \setminus \{v, z\}$ such that $f(x) = 2$. In this subcase, it is easy to see that $T_w \cong T_{a,1,c}$. Hence, by proceeding analogously to the Subcase 2.2 (Case 2) of Claim II, we deduce that $T \in \mathcal{H}$.

Subcase 2.2. $N(w)\setminus\{v, z\} \subseteq V_{1,1}$. This implies that $f(z) = 2$. Let $N(v') \cap L(T) = \{s'\}$ and $T'' = T - \{v', s'\}$. Again, by proceeding analogously to the Case 1 of Claim II, we deduce that $T'' \in \mathcal{H}$ and that $w$ is a stable vertex of $T''$. Hence, $T$ can be obtained from $T''$ by Operation $O_2$, and consequently, $T \in \mathcal{H}$. $(\diamond)$

**Claim IV.** If $\delta_T(s) = \delta_T(v) = \delta_T(w) = 2$, then $T \in \mathcal{H}$.

**Proof.** Let $f(V_0, V_1, V_2)$ be a $\gamma_{tR}(T)$-function which satisfies Theorem 2. If $f(v) = 0$, then $f(s) = f(h) = 1$, $f(w) = 2$ and $f(z) > 0$. If $f(z) = 2$, then $(V_2 \cup I(V_{1,1})) \setminus \{w\}$ is a dominating set of $T$ , which is a contradiction with Theorem 2 (ii). Otherwise, if $f(z) = 1$, then $(V_2 \cup V_{1,2} \cup I(V_{1,1})) \setminus \{z\}$ is a STDS of $T$, which is again a contradiction with Theorem 2 (ii). Thus $f(v) \in \{1, 2\}$ and we analyse the following two cases.

Case 1. $f(v) = 1$. In this case, we obtain that $f(s) = f(z) = 2$ and $f(h) = f(w) = 0$. Let $T' = T - \{h, s, v\}$. Since $f(z) = 2$ and by Theorem 2 (vi), we have that $diam(T) \geq 6$. Notice that $f$ restricted to $V(T')$ is a TRDF on $T'$. Hence, $\gamma_{tR}(T') \leq \gamma_{tR}(T) - 3$. Moreover, it is clear that $\gamma(T) = \gamma(T') + 1$. Also, every STDS of $T'$ can be extended to a STDS of $T$ by adding the vertices $v$ and $s$. So, $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 2$. Therefore, by Theorem 1, the inequalities above and the hypothesis of the theorem, we obtain $\gamma_{t2}(T') + \gamma(T') \leq \gamma_{tR}(T') \leq \gamma_{tR}(T) - 3 = \gamma_{t2}(T) + \gamma(T) - 3 \leq \gamma_{t2}(T') + \gamma(T')$. Hence, we must have equalities throughout this inequality chain. In particular, $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$ and by the inductive hypothesis, $T' \in \mathcal{H}$.
Another consequence of equalities above is that $\gamma_{t2}(T) = \gamma_{t2}(T') + 2$. Notice that any ASTDS of $T'$ relative to $w$ can be extended to a STDS of $T$ by adding the vertices $v$ and $s$. Hence $\gamma_{t2}(T) \leq \gamma_{t2}(T'; w) + 2$. If $w$ is not a semi-stable vertex of $T'$, then we obtain a contradiction with the fact that $\gamma_{t2}(T) = \gamma_{t2}(T') + 2$. So, $w$ is a semi-stable vertex of $T'$.

Moreover, if $w \in W^1_{t2}(T') \cup W^a_{t2}(T')$, then there exists either a $\gamma_{t2}(T'; w)$-set $S$ or a $\gamma_{t2}(T')$-set $D$ containing to $w$. Observe that the sets $S \cup \{s\}$ and $D \cup \{s\}$ are STDSs of $T$. This implies, in both cases, that $\gamma_{t2}(T) \leq \gamma_{t2}(T') + 1$, which is a contradiction. So, $w \in V(T') \setminus (W^1_{t2}(T') \cup W^a_{t2}(T'))$.
Thus, as $diam(T') \geq 3$, we have that $T$ can be obtained from $T'$ by Operation $O_3$. Therefore, $T \in \mathcal{H}$.

Case 2. $f(v) = 2$. In this case, we obtain that $f(s) = 2$ and $f(h) = f(w) = 0$. Theorem 2 (iv) leads to $f(z) \neq 2$. Let $T' = T - \{h, s\}$. Notice that the function $f'$, defined by $f'(x) = f(x)$ if $x \in V(T') \setminus \{v, w\}$ and $f'(v) = f'(w) = 1$, is a TRDF on $T'$. Hence, $\gamma_{tR}(T') \leq \omega(f') = \gamma_{tR}(T) - 2$. Also, every dominating set (respectively, STDS) of $T'$ can be extended to a dominating set (respectively, STDS) of $T$ by adding the vertex $s$. So, $\gamma(T) \leq \gamma(T')+1$ and $\gamma_{t2}(T) \leq \gamma_{t2}(T')+1$. Therefore, by Theorem 1, the inequalities above and the hypothesis of the theorem, we obtain $\gamma_{t2}(T')+\gamma(T') \leq \gamma_{tR}(T') \leq \gamma_{tR}(T) - 2 = \gamma_{t2}(T) + \gamma(T) - 2 \leq \gamma_{t2}(T') + \gamma(T')$. Hence, we must have equalities throughout this inequality chain. In particular, $\gamma_{tR}(T') = \gamma_{t2}(T') + \gamma(T')$ and by the inductive hypothesis, $T' \in \mathcal{H}$.
Another consequence of equalities above is that $\gamma(T) = \gamma(T') + 1$ and $\gamma_{t2}(T) = \gamma_{t2}(T') + 1$. By proceeding analogously to the Case 1 of Claim II, it is easy to see that $v$ is a stable vertex of $T'$. Thus, $T$ can be obtained from $T'$ by Operation $O_2$. Therefore, $T \in \mathcal{H}$, which completes the proof. □

As an immediate consequence of Theorems 3 and 4, we have the desired characterization.

**Theorem 5.** *A tree $T$ of order $n \geq 3$ satisfies that $\gamma_{tR}(T) = \gamma_{t2}(T) + \gamma(T)$ if and only if $T \cong K_{1,n-1}$ or $T \in \mathcal{H}$.*

Finally, the examples given in the Figure 3 show that the operations $O_1$, $O_2$, $O_3$ and $O_4$ are required in the previous characterization.

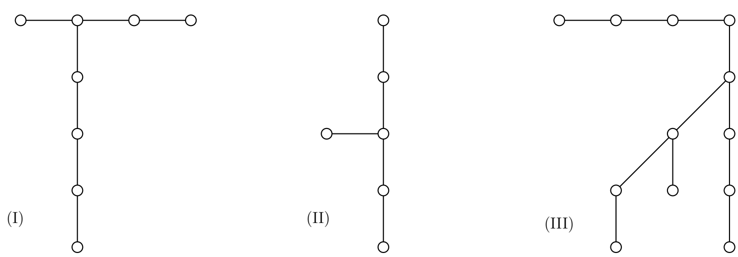


Figure 3. The tree (I) can only be obtained from $P_4$ by a sequence of operations $O_1, O_3$; the tree (II) can only be obtained from $P_4$ by the Operation $O_2$ and the tree (III) can only be obtained from $P_4$ by the Operation $O_4$ (using the tree $T_{1,1,0}$).

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