

Signed total Italian k -domination in graphs

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Received: 18 July 2020; Accepted: 11 November 2020

Published Online: 13 November 2020

Abstract: Let $k \geq 1$ be an integer, and let G be a finite and simple graph with vertex set $V(G)$. A signed total Italian k -dominating function (STIkDF) on a graph G is a function $f : V(G) \rightarrow \{-1, 1, 2\}$ satisfying the conditions that $\sum_{x \in N(v)} f(x) \geq k$ for each vertex $v \in V(G)$, where $N(v)$ is the neighborhood of v , and each vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. The weight of an STIkDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The signed total Italian k -domination number $\gamma_{stI}^k(G)$ of G is the minimum weight of an STIkDF on G . In this paper we initiate the study of the signed total Italian k -domination number of graphs, and we present different bounds on $\gamma_{stI}^k(G)$. In addition, we determine the signed total Italian k -domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed total Roman k -domination number $\gamma_{stR}^k(G)$, introduced and investigated by Volkmann [9, 12].

Keywords: Signed total Italian k -dominating function, Signed total Italian k -domination number, Signed total Roman k -dominating function, Signed total Roman k -domination number.

AMS Subject classification: 05C69

1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let G be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the *order* and the *size* of the graph G , respectively. The *open neighborhood* of a vertex v is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $d_G(v) = d(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. For a set $X \subseteq V(G)$, its *open neighborhood* is the set $N_G(X) = N(X) =$

$\bigcup_{v \in X} N(v)$, and its *closed neighborhood* is the set $N_G[X] = N[X] = N(X) \cup X$. The *complement* of a graph G is denoted by \overline{G} . A *leaf* of a graph G is a vertex of degree 1, while a *support vertex* of G is a vertex adjacent to a leaf. Denote by $S(G)$ the set of support vertices of G . An edge incident with a leaf is called a *pendant edge*. Let P_n , C_n and K_n be the path, cycle and complete graph of order n . A complete bipartite graph with partite sets X and Y , where $|X| = p$ and $|Y| = q$ is denoted by $K_{p,q}$. Specifically, $K_{1,q}$ is called a *star*. Let $S(r, s)$ be the *double star* with exactly two adjacent vertices u and v that are not leaves such that u is adjacent to $r \geq 1$ leaves and v is adjacent to $s \geq 1$ leaves. A *spider graph* is a tree with one vertex of degree at least 3 and all others of degree at most two.

All along this paper we will assume that k is a positive integer. In this paper we continue the study of signed total Roman and signed total Italian dominating functions in graphs and digraphs. For a subset $S \subseteq V(G)$ of vertices of a graph G and a function $f: V(G) \rightarrow \mathbb{R}$, we define $f(S) = \sum_{x \in S} f(x)$.

If $k \geq 1$ is an integer, then Volkmann [9] defined the *signed total Roman k -dominating function* (STRkDF) on a graph G as a function $f: V(G) \rightarrow \{-1, 1, 2\}$ such that $f(N(v)) = \sum_{x \in N(v)} f(x) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Roman k -domination number* $\gamma_{stR}^k(G)$ of G is the minimum weight of an STRkDF on G . The special case $k = 1$ was introduced and investigated by Volkmann [12] and [7]. A $\gamma_{stR}^k(G)$ -function is a signed total Roman k -dominating function on G of weight $\gamma_{stR}^k(G)$. Amjadi and Soroudi [1], Dehgardi and Volkmann [3] and Volkmann [8] studied the signed total Roman domination number in digraphs.

A *signed total Italian k -dominating function* (STIkDF) on a graph G is defined as a function $f: V(G) \rightarrow \{-1, 1, 2\}$ having the property $f(N(v)) \geq k$ for every $v \in V(G)$, and each vertex u with $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$ or to two vertices w and z with $f(w) = f(z) = 1$. Note that in the case $k \geq 2$, the second condition is superfluous. The weight of an STIkDF f is $\omega(f) = \sum_{v \in V(G)} f(v)$. The *signed total Italian k -domination number* $\gamma_{stI}^k(G)$ of G is the minimum weight of an STIkDF on G . The special case $k = 1$ was introduced and investigated by Volkmann [6]. A $\gamma_{stI}^k(G)$ -function is a signed total Italian k -dominating function on G of weight $\gamma_{stI}^k(G)$. For an STIkDF f on G , let $V_i = V_i(f) = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. A signed total Italian k -dominating function $f: V(G) \rightarrow \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$.

The signed total Italian k -domination number exists when $\delta \geq \frac{k}{2}$. The definitions lead to $\gamma_{stI}^k(G) \leq \gamma_{stR}^k(G)$. Therefore each lower bound of $\gamma_{stI}^k(G)$ is also a lower bound of $\gamma_{stR}^k(G)$. Note that the signed total Italian k -domination number is the total version of the weak signed Roman k -domination number, see Volkmann [10, 11].

For an integer $q \geq 1$, Kulli [5] called a subset D of vertices of a graph G a *total q -dominating set* if every vertex $x \in V(G)$ has at least q neighbors in D . The *total q -domination number* $\gamma_{tq}(G)$ is the minimum cardinality of a total q -dominating set

of G . The special case $q = 1$ is the usual *total domination number* $\gamma_t(G)$, introduced by Cockayne, Dawes and Hedetniemi [2].

Our purpose in this work is to initiate the study of the signed total Italian k -domination number. We present basic properties and sharp bounds on $\gamma_{stI}^k(G)$. In particular, we show that many lower bounds on $\gamma_{stR}^k(G)$ are also valid for $\gamma_{stI}^k(G)$. Some of our results are extensions of well-known properties of the signed total Roman k -domination number and the signed total Italian domination number $\gamma_{stI}(G) = \gamma_{stI}^1(G)$, given by Volkmann [6, 9, 12].

2. Preliminary results

In this section we present basic properties of the signed total Italian k -dominating functions and the signed total Italian k -domination numbers.

Proposition 1. Let $k \geq 1$ be an integer, and let G be a graph of order n with $\delta(G) \geq \lceil k/2 \rceil$. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on G , then

- (a) $|V_{-1}| + |V_1| + |V_2| = n$.
- (b) $\omega(f) = |V_1| + 2|V_2| - |V_{-1}|$.
- (c) If $\delta(G) \geq \lceil (k + s)/2 \rceil$ with $s \in \{0, 1\}$, then $V_1 \cup V_2$ is a total $\lceil (k + s)/2 \rceil$ -dominating set of G .

Proof. Since (a) and (b) are immediate, we only prove (c). Suppose on the contrary, that there exists a vertex v with at most $\lceil (k + s)/2 \rceil - 1$ neighbors in $V_1 \cup V_2$. Then v has at least

$$\delta(G) - (\lceil (k + s)/2 \rceil - 1) \geq \lceil (k + s)/2 \rceil - (\lceil (k + s)/2 \rceil - 1) = 1$$

neighbors in V_{-1} . Hence the definition implies the contradiction

$$k \leq f(N(v)) \leq 2(\lceil (k + s)/2 \rceil - 1) - 1 \leq \frac{2(k + s + 1)}{2} - 3 = k + s - 2 \leq k - 1.$$

Consequently, $V_1 \cup V_2$ is a total $\lceil (k + s)/2 \rceil$ -dominating set of G . □

Corollary 1. If G is a graph of order n and minimum degree $\delta \geq \lceil \frac{k+s}{2} \rceil$ with $s \in \{0, 1\}$, then $\gamma_{stI}^k(G) \geq 2\gamma_{t\lceil \frac{k+s}{2} \rceil} - n$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}^k(G)$ -function. Then it follows from Proposition 1 that

$$\gamma_{stI}^k(G) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_{t\lceil \frac{k+s}{2} \rceil} - n.$$

□

The graphs qK_2 and qK_3 show that Corollary 1 is sharp for $k = 1$ and $k = 2$. The proof of the next proposition is identically with the proof of Proposition 2 in [9] and is therefore omitted.

Proposition 2. Let $k \geq 1$ be an integer, and let G be a graph of order n with minimum degree $\delta \geq k/2$ and maximum degree Δ . If $f = (V_{-1}, V_1, V_2)$ is an STIkDF on G , then

- (i) $(2\Delta - k)|V_2| + (\Delta - k)|V_1| \geq (\delta + k)|V_{-1}|$.
- (ii) $(2\Delta + \delta)|V_2| + (\Delta + \delta)|V_1| \geq (\delta + k)n$.
- (iii) $(\Delta + \delta)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2|$.
- (iv) $\omega(f) \geq (\delta - 2\Delta + 2k)n/(2\Delta + \delta) + |V_2|$.

3. Bounds on the signed total Italian k -domination number

We start with a general upper bound, and we characterize all extremal graphs.

Theorem 1. Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{stI}^k(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2}$, and each vertex of G is adjacent to a vertex of minimum degree.

Proof. Define the function $g : V(G) \rightarrow \{-1, 1, 2\}$ by $g(x) = 2$ for each vertex $x \in V(G)$. Since $\delta(G) \geq \lceil \frac{k}{2} \rceil$, the function g is an STIkDF on G of weight $2n$ and thus $\gamma_{stI}^k(G) \leq 2n$.

Now let k be even, $\delta(G) = \frac{k}{2}$, and assume that each vertex of G is adjacent to a vertex of minimum degree. Let f be an STIkDF on G , and let $x \in V(G)$ be an arbitrary vertex. Then x has a neighbor v with $d(v) = \frac{k}{2}$. Therefore the condition $f(N(v)) \geq k$ implies $f(x) = 2$. Thus f is of weight $2n$, and we obtain $\gamma_{stI}^k(G) = 2n$.

Conversely, assume that $\gamma_{stI}^k(G) = 2n$. If $k = 2p + 1$ is odd, then $\delta(G) \geq p + 1$. Define the function $h : V(G) \rightarrow \{-1, 1, 2\}$ by $h(w) = 1$ for an arbitrary vertex w and $h(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then

$$h(N(v)) = \sum_{x \in N(v)} f(x) \geq 2(p + 1) - 1 = 2p + 1 = k$$

for each vertex $v \in V(G)$. Thus the function h is an STIkDF on G of weight $2n - 1$, and we obtain the contradiction $\gamma_{stI}^k(G) \leq 2n - 1$.

Let now k even, and assume that there exists a vertex w such that $d(x) \geq \frac{k}{2} + 1$ for each $x \in N(w)$. Define the function $h_1 : V(G) \rightarrow \{-1, 1, 2\}$ by $h_1(w) = 1$ and $h_1(x) = 2$ for each vertex $x \in V(G) \setminus \{w\}$. Then $h_1(N(w)) \geq k$, $h_1(N(x)) \geq 2(\frac{k}{2} + 1) - 1 = k + 1$ for each $x \in N(w)$ and $h_1(N(y)) \geq k$ for each $y \notin N[w]$. Hence the function h_1 is an STIkDF on G of weight $2n - 1$, a contradiction to the assumption $\gamma_{stI}^k(G) = 2n$. This completes the proof. □

The proof of Theorem 1 also leads to the next result.

Theorem 2. Let G be a graph of order n with $\delta(G) \geq \lceil \frac{k}{2} \rceil$. Then $\gamma_{stR}^k(G) \leq 2n$, with equality if and only if k is even, $\delta(G) = \frac{k}{2}$, and each vertex of G is adjacent to a vertex of minimum degree.

Let $k \geq 2$ be an even integer. If H is a $\frac{k}{2}$ -regular graph of order n , then Theorems 1 and 2 imply that $\gamma_{stI}^k(H) = \gamma_{stR}^k(H) = 2n$. The next corollaries follow from Theorems 1 and 2 immediately.

Corollary 2. Let G be a connected graph of order $n \geq 2$ with $\delta(G) = 1$. Then $\gamma_{stI}^2(G) = 2n$ if and only if $n = 2$.

Corollary 3. Let G be a connected graph of order $n \geq 2$ with $\delta(G) = 1$. Then $\gamma_{stR}^2(G) = 2n$ if and only if $n = 2$.

Observation 3. If G is a graph of order n with $\delta(G) \geq k$, then $\gamma_{stI}^k(G) \leq \gamma_{stR}^k(G) \leq n$.

Proof. Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k$, the function f is an STRkDF on G of weight n and thus $\gamma_{stI}^k(G) \leq \gamma_{stR}^k(G) \leq n$. \square

As an application of Proposition 2 (iii), we obtain a lower bound on the signed total Italian k -domination number for r -regular graphs.

Corollary 4. If G is an r -regular graph of order n with $r \geq \frac{k}{2}$, then

$$\gamma_{stI}^k(G) \geq \frac{kn}{r}.$$

Example 1. If H is a k -regular graph of order n , then it follows from Corollary 4 that $\gamma_{stI}^k(H) \geq n$ and thus $\gamma_{stI}^k(H) = n$, according to Observation 3.

Example 1 shows that Observation 3 and Corollary 4 are both sharp.

Theorem 4. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma_{stI}^k(G) \geq k + 2 + \delta(G) - n.$$

If in addition $\delta(G) - k$ is odd, then $\gamma_{stI}^k(G) \geq k + 3 + \delta(G) - n$.

Proof. Let f be a $\gamma_{stI}^k(G)$ -function. Then there exists a vertex w with $f(w) \geq 1$. It follows from the the definitions that

$$\begin{aligned} \gamma_{stI}^k(G) &= \sum_{x \in V(G)} f(x) = f(w) + \sum_{x \in N(w)} f(x) + \sum_{x \in V(G) - N[w]} f(x) \\ &\geq 1 + k - (n - d(w) - 1) = k + 2 + d(w) - n \geq k + 2 + \delta(G) - n. \end{aligned}$$

Now assume that $\delta(G) - k$ is odd. If there exists a vertex w with $f(w) = 2$ or a vertex v with $f(v) = 1$ and $d(v) \geq \delta(G) + 1$, then the inequality chain above leads to $\gamma_{stI}^k(G) \geq k + 3 + \delta(G) - n$. So assume that $f(x) \in \{-1, 1\}$ for each vertex $x \in V(G)$, and each vertex y with $f(y) = 1$ has degree $d(y) = \delta(G)$. Let now u be a vertex of minimum degree with $f(u) = 1$. Since $\delta(G) - k$ is odd, we observe that

$$\sum_{x \in N(u)} f(x) \geq k + 1$$

and thus

$$\begin{aligned} \gamma_{stI}^k(G) &= \sum_{x \in V(G)} f(x) = f(u) + \sum_{x \in N(u)} f(x) + \sum_{x \in V(G) - N[u]} f(x) \\ &\geq 1 + k + 1 - (n - d(u) - 1) = k + 3 + d(u) - n = k + 3 + \delta(G) - n. \end{aligned}$$

This completes the proof. □

Example 2. If $k \geq 1$ and $n \geq 2$ are integers such that $2n - 2 \geq k$, then it holds:

- (i) If $k \geq n$, then $\gamma_{stI}^k(K_n) = k + 2$.
- (ii) If $k \leq n - 1$ and $n - k$ is odd, then $\gamma_{stI}^k(K_n) = k + 1$.
- (iii) If $k \leq n - 1$ and $n - k$ is even, then $\gamma_{stI}^k(K_n) = k + 2$.

Proof. Let f be a $\gamma_{stI}^k(K_n)$ -function.

(i) Since $k \geq n$, there exists a vertex w with $f(w) \geq 2$. This implies $\gamma_{stI}^k(K_n) = f(w) + f(N(w)) \geq 2 + k$.

For $\gamma_{stI}^k(K_n) \leq k + 2$, let the function $g : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to $2n - k - 2$ vertices the value 1 and to the remaining $k + 2 - n$ vertices the value 2. Then g is an STIkDF on K_n of weight $\omega(g) = k + 2$ and so $\gamma_{stI}^k(K_n) \leq k + 2$. Thus $\gamma_{stI}^k(K_n) = k + 2$ in this case.

(ii) Since there exists a vertex w with $f(w) \geq 1$, we note that $\gamma_{stI}^k(K_n) = f(w) + f(N(w)) \geq 1 + k$.

For $\gamma_{stI}^k(K_n) \leq k + 1$, let the function $g : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to $\frac{n+k+1}{2}$ vertices the value 1 and to the remaining $\frac{n-k-1}{2}$ vertices the value -1. Then g is an STIkDF on K_n of weight $\omega(g) = k + 1$ and so $\gamma_{stI}^k(K_n) \leq k + 1$. Thus $\gamma_{stI}^k(K_n) = k + 1$ in the second case.

(iii) If there exists a vertex w with $f(w) = 2$, then it follows that $\gamma_{stI}^k(K_n) = f(w) + f(N(w)) \geq 2 + k$. If $f(x) \in \{-1, 1\}$ for all $x \in V(K_n)$, then $f(N(x))$ is even when n is odd and $f(N(x))$ is odd when n is even. Since $f(N(x)) \geq k$, we observe that $f(N(x)) \geq k + 1$ when $n - k$ is even. If w is a vertex with $f(w) = 1$, we therefore deduce that $\gamma_{stI}^k(K_n) = f(w) + f(N(w)) \geq 1 + k + 1 = k + 2$.

For $\gamma_{stI}^k(K_n) \leq k + 2$, let the function $g : V(K_n) \rightarrow \{-1, 1, 2\}$ assign to $\frac{n+k+2}{2}$ vertices the value 1 and to the remaining $\frac{n-k-2}{2}$ vertices the value -1. Then g is an STIkDF on K_n of weight $\omega(g) = k + 2$ and so $\gamma_{stI}^k(K_n) \leq k + 2$. Thus $\gamma_{stI}^k(K_n) = k + 2$ in the last case. □

Example 2 (ii) and (iii) show that Theorem 4 is sharp for $k \leq n - 1$. If $\Delta(G) \geq \delta(G) + 3$, then the next lower bound is an improvement of Theorem 4.

Proposition 3. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma_{stI}^k(G) \geq k + \Delta(G) - n.$$

Proof. Let $w \in V(G)$ be a vertex of maximum degree, and let f be a $\gamma_{stI}^k(G)$ -function. Then the definitions imply

$$\begin{aligned} \gamma_{stI}^k(G) &= \sum_{x \in V(G)} f(x) = \sum_{x \in N(w)} f(x) + \sum_{x \in V(G) - N(w)} f(x) \\ &\geq k + \sum_{x \in V(G) - N(w)} f(x) \geq k - (n - \Delta(G)) \\ &= k + \Delta(G) - n, \end{aligned}$$

and the proof of the desired lower bound is complete. □

Corollary 5. Let G be a graph of order n , minimum degree $\delta \geq \frac{k}{2}$ and maximum degree Δ . If $\delta < \Delta$, then

$$\gamma_{stI}^k(G) \geq \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} n \right\rceil.$$

Proof. Multiplying both sides of the inequality in Proposition 2 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2 (iii), we obtain the desired lower bound. □

Since $\gamma_{stR}^k(G) \geq \gamma_{stI}^k(G)$, Corollary 5 leads to the next known lower bound immediately.

Corollary 6. ([9]) Let G be a graph of order n , minimum degree $\delta \geq \frac{k}{2}$ and maximum degree Δ . If $\delta < \Delta$, then

$$\gamma_{stR}^k(G) \geq \left\lceil \frac{2\delta + 3k - 2\Delta}{2\Delta + \delta} n \right\rceil.$$

Examples 12 and 13 in [9] demonstrate that Corollary 6 is sharp and therefore Corollary 5 too. The special case $k = 1$ of Corollaries 4 and 5 can be found in [6].

A set $S \subseteq V(G)$ is a 2 -packing of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The 2 -packing number $\rho(G)$ of G is defined by

$$\rho(G) = \max\{|S| : S \text{ is a } 2\text{-packing of } G\}.$$

Analogously to Theorem 14 in [9], one can prove the next lower bound on the signed total Italian k -domination number.

Theorem 5. If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma_{stI}^k(G) \geq \rho(G)(k + \delta(G)) - n.$$

Corollary 7. ([9]) If G is a graph of order n with $\delta(G) \geq \frac{k}{2}$, then

$$\gamma_{stR}^k(G) \geq \rho(G)(k + \delta(G)) - n.$$

In [9], the author presented an infinite family of graphs achieving equality in Corollary 7. Thus Corollary 7 and Theorem 5 are sharp. Using Corollary 4, one can prove the following Nordhaus-Gaddum type inequality analogously to Theorem 15 in [9].

Theorem 6. If G is an r -regular graph of order n such that $r \geq \frac{k}{2}$ and $n - r - 1 \geq \frac{k}{2} - 1$, then

$$\gamma_{stI}^k(G) + \gamma_{stI}^k(\overline{G}) \geq \frac{4kn}{n - 1}.$$

If n is even, then $\gamma_{stI}^k(G) + \gamma_{stI}^k(\overline{G}) \geq 4k(n - 1)/(n - 2)$.

Let $k \geq 2$ be an even integer, and let H and \overline{H} be k -regular graphs of order $n = 2k + 1$. By Example 1, we have $\gamma_{stI}^k(H) = \gamma_{stI}^k(\overline{H}) = n$. Consequently,

$$\gamma_{stI}^k(H) + \gamma_{stI}^k(\overline{H}) = 2n = \frac{4kn}{n - 1}.$$

Thus the Nordhaus-Gaddum bound of Theorem 6 is sharp for even k .

Theorem 7. Let G be connected graph of order $n \geq 3$ and let $S(G)$ be the set of its support vertices. Then $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq n + |S(G)|$.

Proof. Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = 2$ when x is a support vertex and $f(x) = 1$ otherwise. If v is a leaf, then v is adjacent to a support vertex and so $f(N(v)) = 2$. If v is not a leaf, then $d(v) \geq 2$ and thus $f(N(v)) \geq 2$. Therefore f is an STR2DF on G of weight $n + |S(G)|$ and hence $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq n + |S(G)|$. \square

Example 3. For $n \geq 4$, we have $\gamma_{stR}^2(P_n) = \gamma_{stI}^2(P_n) = n + 2$.

Proof. Let $P_n = v_1v_2 \dots v_n$, and let f be a $\gamma_{stI}^2(P_n)$ -function. According to Theorem 7, we have $\gamma_{stI}^2(P_n) \leq \gamma_{stR}^2(P_n) \leq n + 2$. Conversely, it is easy to see that $f(v_i) \geq 1$ for $1 \leq i \leq n$ and $f(v_2) = f(v_{n-1}) = 2$. Therefore $\gamma_{stR}^2(P_n) \geq \gamma_{stI}^2(P_n) \geq n + 2$ and thus $\gamma_{stR}^2(P_n) = \gamma_{stI}^2(P_n) = n + 2$. \square

Since $\gamma_{stR}^2(P_3) = \gamma_{stI}^2(P_3) = 4$, the condition $n \geq 4$ in Example 3 is necessary.

Example 4. Let H be a spider graph, and let $w \in V(H)$ with $d(w) \geq 3$. If every leaf of H has distance at least two from w , then $\gamma_{stR}^2(H) = \gamma_{stI}^2(H) = n + |S(H)|$.

Proof. According to Theorem 7, we have $\gamma_{stI}^2(H) \leq \gamma_{stR}^2(H) \leq n + |S(H)|$. Conversely, let f be a $\gamma_{stI}^2(H)$ -function. Since every leaf of H has distance at least two from w , we note that $f(x) \geq 1$ for $x \in V(H)$. In addition, we have $f(x) = 2$ for each $x \in S(H)$. Therefore $\gamma_{stR}^2(H) \geq \gamma_{stI}^2(H) \geq n + |S(H)|$ and thus $\gamma_{stR}^2(H) = \gamma_{stI}^2(H) = n + |S(H)|$. \square

Examples 3 and 4 demonstrate that Theorem 7 is sharp.

Example 5. If $n \geq 2$, then $\gamma_{stI}^2(K_{1,n-1}) = \gamma_{stR}^2(K_{1,n-1}) = 4$.

Proof. Let $G = K_{1,n-1}$, and let f be a $\gamma_{stI}^2(G)$ -function. If $n = 2$, then the result is obviously. Let now $n \geq 3$, and let w be the central vertex of the star G . Clearly, $f(w) = 2$. Hence it follows that

$$\gamma_{stR}^2(G) \geq \gamma_{stI}^2(G) = f(w) + f(N(w)) \geq 2 + 2 = 4.$$

For the converse inequality $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq 4$, let v_1, v_2, \dots, v_{n-1} be the leaves of G . If $n - 1 = 2p$ is even, define $g : V(G) \rightarrow \{-1, 1, 2\}$ by $g(w) = 2$, $g(v_i) = 1$ for $1 \leq i \leq p + 1$ and $g(v_i) = -1$ for $p + 2 \leq i \leq 2p$. Then g is an STR2DF on G of weight 4 and thus $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq 4$ in this case. If $n = 2p + 1$ is odd, then define $h : V(G) \rightarrow \{-1, 1, 2\}$ by $h(w) = h(v_{2p+1}) = 2$, $h(v_i) = 1$ for $1 \leq i \leq p$ and $h(v_i) = -1$ for $p + 1 \leq i \leq 2p$. Then h is an STR2DF on G of weight 4 and thus $\gamma_{stI}^2(G) \leq \gamma_{stR}^2(G) \leq 4$ also in this case. \square

Example 5 shows that $\gamma_{stR}^2(H) = \gamma_{stI}^2(H) = n + |S(H)|$ is not valid for each spider graph H . Therefore the condition that every leaf of H has distance at least two from w is important in Example 4.

4. Special classes of graphs

Let C_n be a cycle of length $n \geq 3$. In [6], the author has shown that $\gamma_{stI}(C_n) = n/2$ when $n \equiv 0 \pmod{4}$, $\gamma_{stI}(C_n) = (n + 3)/2$ when $n \equiv 1, 3 \pmod{4}$ and $\gamma_{stI}(C_n) = (n + 6)/2$ when $n \equiv 2 \pmod{4}$. Now we determine $\gamma_{stI}^k(C_n)$ as well as $\gamma_{stR}^k(C_n)$ for $2 \leq k \leq 4$.

Corollary 4 and Observation 3 immediately lead to $\gamma_{stI}^2(C_n) = \gamma_{stR}^2(C_n) = n$. In addition, Theorems 1 and 2 imply $\gamma_{stI}^4(C_n) = \gamma_{stR}^4(C_n) = 2n$.

Example 6. For $n \geq 3$, we have $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil + 1$ when $n \equiv 2 \pmod{4}$ and $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ otherwise.

Proof. Let $C_n = v_1v_2 \dots v_nv_1$. Assume first that $n \equiv 0, 1, 3 \pmod{4}$. Applying Corollary 4, we obtain $\gamma_{stR}^3(C_n) \geq \gamma_{stI}^3(C_n) \geq \lceil \frac{3n}{2} \rceil$.

If $n = 4t$ for an integer $t \geq 1$, then define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$. Then g is an STR3DF on C_n of weight $\omega(g) = 6t = \frac{3n}{2}$ and so $\gamma_{stI}^3(C_n) \leq \gamma_{stR}^3(C_n) \leq \lceil \frac{3n}{2} \rceil$ and thus $\gamma_{stI}^3(C_n) = \gamma_{stR}^3(C_n) = \lceil \frac{3n}{2} \rceil$ in this case.

If $n = 4t + 1$ for an integer $t \geq 1$, then define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$ and $g(v_{4t+1}) = 2$. Then g is an STR3DF on C_n of weight $\omega(g) = 6t + 2 = \lceil \frac{3n}{2} \rceil$ and thus $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ in this case too.

If $n = 4t + 3$ for an integer $t \geq 0$, then define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ for $0 \leq i \leq t$, $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$ and $g(v_{4t+3}) = 1$. Then g is an STR3DF on C_n of weight $\omega(g) = 6t + 5 = \lceil \frac{3n}{2} \rceil$ and thus $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil$ in the third case.

Let now $n = 4t + 2$ for an integer $t \geq 1$. If f is a $\gamma_{stI}^3(C_n)$ -function, then we observe that $f(v_i) \geq 1$ for each $1 \leq i \leq n$ and $f(v_i) + f(v_{i+1}) + f(v_{i+2}) + f(v_{i+3}) \geq 6$, where the indices are taken modulo n . Assume, without loss of generality, that $f(v_1) = f(v_2) = 2$ and $f(v_3) = f(v_4) = 1$. Then it follows that $f(v_5) = f(v_6) = 2$, and we obtain

$$\begin{aligned} \omega(f) &= f(v_1) + f(v_2) + f(v_3) + f(v_4) + f(v_5) + f(v_6) \\ &\quad + \sum_{i=1}^{t-1} (f(v_{4i+3}) + f(v_{4i+4}) + f(v_{4i+5}) + f(v_{4i+6})) \\ &\geq 10 + 6(t - 1) = 6t + 4. \end{aligned}$$

Therefore $\omega(f) \geq 6t + 4 = \lceil \frac{3n}{2} \rceil + 1$ and thus $\gamma_{stR}^3(C_n) \geq \gamma_{stI}^3(C_n) \geq \lceil \frac{3n}{2} \rceil + 1$ in this case. Conversely, if we define $g : V(C_n) \rightarrow \{-1, 1, 2\}$ by $g(v_{4i+1}) = g(v_{4i+2}) = 2$ for $0 \leq i \leq t$ and $g(v_{4i+3}) = g(v_{4i+4}) = 1$ for $0 \leq i \leq t - 1$, then g is an STR3DF on C_n of weight $\omega(g) = 6t + 4 = \lceil \frac{3n}{2} \rceil + 1$ and thus $\gamma_{stR}^3(C_n) = \gamma_{stI}^3(C_n) = \lceil \frac{3n}{2} \rceil + 1$ in the last case. □

Example 7. If $k \geq 1$ and $q \geq p \geq 2$ are integers such that $p \geq k/2$, then $\gamma_{stI}^k(K_{p,q}) = 2k$.

Proof. Let $G = K_{p,q}$, and let g be a $\gamma_{stI}^k(G)$ -function. In addition, let X and Y be a bipartition of the complete bipartite graph G with $|X| = p$ and $|Y| = q$. If $x \in X$ and $y \in Y$, then

$$\gamma_{stI}^k(G) = g(V(G)) = g(N(x)) + g(N(y)) \geq k + k = 2k.$$

For the converse inequality we distinguish different cases.

a) Assume that $k \geq q$. Let the function $f : V(G) \rightarrow \{-1, 1, 2\}$ assign to $2p - k$ vertices of X the value 1, to the remaining $k - p$ vertices of X the value 2, to $2q - k$ vertices of Y the value 1 and to the remaining $k - q$ vertices of Y the value 2. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and so $\gamma_{stI}^k(G) \leq 2k$. Thus $\gamma_{stI}^k(G) = 2k$ in this case.

b) Assume that $q > k \geq p$ and $q - k$ is even. Let the function $f : V(G) \rightarrow \{-1, 1, 2\}$ assign to $2p - k$ vertices of X the value 1, to the remaining $k - p$ vertices of X the value 2, to $\frac{k+q}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and so $\gamma_{stI}^k(G) \leq 2k$. Thus $\gamma_{stI}^k(G) = 2k$ in the second case.

c) Assume that $q > k \geq p$ and $q - k$ is odd. Let the function $f : V(G) \rightarrow \{-1, 1, 2\}$ assign to $2p - k$ vertices of X the value 1, to the remaining $k - p$ vertices of X the value 2, to one vertex of Y the value 2, to $\frac{k+q-3}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k+1}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and so $\gamma_{stI}^k(G) \leq 2k$. Thus $\gamma_{stI}^k(G) = 2k$ in the third case.

d) Assume that $k < p$, $p - k$ is even and $q - k$ is even. Let the function $f : V(G) \rightarrow \{-1, 1, 2\}$ assign to $\frac{k+p}{2}$ vertices of X the value 1, to the remaining $\frac{p-k}{2}$ vertices of X the value -1, to $\frac{k+q}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and thus $\gamma_{stI}^k(G) = 2k$ in this case too.

e) Assume that $k < p$, $p - k$ is odd and $q - k$ is odd. Let the function $f : V(G) \rightarrow \{-1, 1, 2\}$ assign to one vertex of X the value 2, to $\frac{k+p-3}{2}$ vertices of X the value 1, to the remaining $\frac{p-k+1}{2}$ vertices of X the value -1, to one vertex of Y the value 2, to $\frac{k+q-3}{2}$ vertices of Y the value 1 and to the remaining $\frac{q-k+1}{2}$ vertices of Y the value -1. Then f is an STIkDF on G of weight $\omega(f) = 2k$ and thus $\gamma_{stI}^k(G) = 2k$ also in this case.

The cases $k < p$, $p - k$ even and $q - k$ odd or $k < p$, $p - k$ odd and $q - k$ even are analogously, and are therefore omitted. \square

Example 6 and Example 7 with $p = q$ show again that Corollary 4 is sharp.

Example 8. If $S(r, s)$ is a double star with $r, s \geq 2$, then $\gamma_{stI}^2(S(r, s)) = \gamma_{stR}^2(S(r, s)) = 4$.

Proof. Let u and v be two adjacent vertices of $S(r, s)$ such that u is adjacent to r leaves and v is adjacent to s leaves. If g is a $\gamma_{stI}^2(S(r, s))$ -function, then the definition implies

$$\gamma_{stR}^2(S(r, s)) \geq \gamma_{stI}^2(S(r, s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 4.$$

Conversely, let $r = 2p + 1$ and $s = 2q + 1$ be odd. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight 2 to one leaf of u , the weight -1 to $p + 1$ leaves of u , the weight 1 to $p - 1$ leaves of u , the weight 2 to one leaf of v , the weight -1 to $q + 1$ leaves of v , and the weight 1 to $q - 1$ leaves of v . Then f is an STR2DF on $S(r, s)$ of weight 4 and thus $\gamma_{stI}^2(S(r, s)) \leq \gamma_{stR}^2(S(r, s)) \leq 4$ in this case.

Now let $r = 2p$ and $s = 2q$ be even. Define f by $f(u) = f(v) = 2$. In addition, we assign the weight -1 to p leaves of u , the weight 1 to p leaves of u , the weight -1 to q leaves of v , and the weight 1 to q leaves of v . Then f is an STR2DF on $S(r, s)$ of weight 4 and thus $\gamma_{stI}^2(S(r, s)) \leq \gamma_{stR}^2(S(r, s)) \leq 4$ also in this case.

The case r even and s odd or r odd and s even are similar to the cases above and are therefore omitted. □

Example 3 shows that $\gamma_{stR}^2(S(1, 1)) = \gamma_{stI}^2(S(1, 1)) = 6$. For the sake of completeness, we now determine $\gamma_{stI}^2(S(1, s))$ for $s \geq 2$.

Example 9. *If $S(1, s)$ is a double star with $s \geq 2$, then $\gamma_{stI}^2(S(1, s)) = \gamma_{stR}^2(S(1, s)) = 5$.*

Proof. Let u and v be two adjacent vertices of $S(1, s)$ such that u is adjacent to one leaf w and v is adjacent to $s \geq 2$ leaves. If g is a $\gamma_{stI}^2(S(1, s))$ -function, then we observe that $g(v) = 2$ and $g(w) \geq 1$. Hence the definition leads to

$$\gamma_{stR}^2(S(1, s)) \geq \gamma_{stI}^2(S(1, s)) = \omega(g) = g(N(u)) + g(N(v)) \geq 3 + 2 = 5.$$

Conversely, let $s = 2q + 1$ be odd. Define f by $f(u) = f(v) = 2$ and $f(w) = 1$. In addition, we assign the weight 2 to one leaf of v , the weight -1 to $q + 1$ leaves of v , and the weight 1 to $q - 1$ leaves of v . Then f is an STR2DF on $S(1, s)$ of weight 5 and thus $\gamma_{stI}^2(S(1, s)) \leq \gamma_{stR}^2(S(1, s)) \leq 5$ in this case.

Now let $s = 2q$ be even. Define f by $f(u) = f(v) = 2$ and $f(w) = 1$. In addition, we assign the weight -1 to q leaves of v , and the weight 1 to q leaves of v . Then f is an STR2DF on $S(1, s)$ of weight 5 and thus $\gamma_{stI}^2(S(1, s)) \leq \gamma_{stR}^2(S(1, s)) \leq 5$ also in the second case. □

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