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The upper domatic number of powers of graphs

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Abstract: Let A and B be two disjoint subsets of the vertex set V of a graph G. The set A is said to dominate B, denoted by $A \to B$, if for every vertex $u \in B$ there exists a vertex $v \in A$ such that $uv \in E(G)$. For any graph G, a partition $\pi = \{V_1, V_2, \ldots, V_p\}$ of the vertex set V is an *upper domatic partition* if $V_i \to V_j$ or $V_j \to V_i$ or both for every $V_i, V_j \in \pi$, whenever $i \neq j$. The *upper domatic number* $D(G)$ is the maximum order of an upper domatic partition. In this paper, we study the upper domatic number of powers of graphs and examine the special case when power is 2. We also show that the upper domatic number of k^{th} power of a graph can be viewed as its k-upper domatic number.

Keywords: Domatic number, k-domatic number, upper domatic partition, upper domatic number, k-upper domatic number

AMS Subject classification: 05C69, 05C70

1. Introduction

Let $G = (V, E)$ be a graph of order $n = |V(G)|$ and size $m = |E(G)|$. Throughout this paper, we consider only finite, simple and undirected graphs. For definitions and notations, we refer [3, 5, 7]. The *distance* between two vertices x and y in a graph G, $d(x, y)$ is the length of a shortest $x - y$ path in the graph G. The *eccentricity* of a vertex $v \in V(G)$ is the maximum distance from v to any vertex in G. The *radius* of G, $rad(G)$ is the minimum eccentricity among all the vertices of G whereas the *diameter* of G, $diam(G)$ is the maximum eccentricity among all the vertices of G. The *open neighborhood* of a vertex $v \in V$, $N(v)$ is the set of all vertices that are adjacent to v, while the *closed neighborhood* of a vertex $v \in V$, is $N[v] = N(v) \cup \{v\}$. The *degree*

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of a vertex v is the cardinality of $N(v)$. The maximum and minimum degrees of the vertices in G is denoted by $\Delta(G)$ and $\delta(G)$ respectively. A *clique* in a graph G is a subgraph where any two vertices are adjacent to each other. The maximum number of vertices in a clique of a graph G is called the *clique number* of the graph G, $\omega(G)$. For two disjoint subsets A and B of the vertex set V of a graph G, the set A is said to *dominate* B, denoted by $A \to B$, if for every vertex $u \in B$ there exists a vertex $v \in A$ such that u and v are adjacent. A subset S of the vertex set of G is said to be a dominating set of a graph G, if $S \to V(G)\backslash S$.

E. J. Cockayne and S. T. Hedetniemi initiated a study on partitioning the vertex set into maximum number of dominating sets in [1]. A partition $\pi = \{V_1, V_2, \ldots, V_p\}$ is a *domatic partition* if each $V_i \in \pi$ is a dominating set. The maximum order of such a partition is called the *domatic number*. Based on the domination between any two sets in the partition, T. W. Haynes and others studied another generalisation of the domatic number, called the upper domatic number [4]. For a graph G, a partition $\pi = \{V_1, V_2, \ldots, V_p\}$ of the vertex set V is an *upper domatic partition* if V_i dominates V_j or V_j dominates V_i or both for every $V_i, V_j \in \pi$, whenever $i \neq j$. The *upper domatic number* $D(G)$ is the maximum order of an upper domatic partition of G. Any upper domatic partition of order $D(G)$ is referred to as a *D-partition* of G.

In the present study, we explore the upper domatic number of powers of graphs. For any positive integer k, the k^{th} power of a graph G is the graph G^k having vertex set $V(G^k) = V(G)$ such that two vertices $u, v \in V(G^k)$ are adjacent if and only if $d(u, v) \leq k$ in G.

One immediate observation is that we can interpret $D(G^k)$ using the concept of k-domination. Given two subsets A and B of the vertex of a graph G, the set A is said to *k-dominate* B, denoted by $A \xrightarrow{k} B$, if for every vertex $u \in B$ there exists a vertex $v \in A$ such that $d(u, v) \leq k$. A subset S of the vertex set of G is said to be a k-dominating set of a graph G, if every vertex in $V(G)$ is at a distance at most k from a vertex in S. The *k-upper domatic partition* is a vertex partition $\pi = \{V_1, V_2, \ldots, V_p\}$, where for any $V_i, V_j \in \pi$, the set $V_i \xrightarrow{k} V_j$ or $V_j \xrightarrow{k} V_i$ or both, when $i \neq j$. The maximum order of a k-upper domatic partition of a graph G is called the *k-upper domatic number*, $D_k(G)$. Any k-upper domatic partition of order $D_k(G)$ is referred to as a *D_k-partition* of G. The following theorem shows that for any graph G, the value of $D_k(G)$ is the same as that of $D(G^k)$.

Theorem 1. The k-upper domatic number of a graph G is equal to the upper domatic number of the k^{th} power of G.

Proof. Let $\pi_k = \{V_1, V_2, \ldots, V_p\}$ be a D_k-partition of G. Then by the definition of k-upper domatic partition, $V_i \xrightarrow{k} V_j$ or $V_j \xrightarrow{k} V_i$ for all $V_i, V_j \in \pi$, where $i \neq j$. Without loss of generality, let us assume that $V_i \xrightarrow{k} V_j$. Then, for every vertex u in V_j there is a vertex $v \in V_i$, such that $d(u, v) \leq k$ implying that $uv \in E(G^k)$ and $V_i \to V_j$ in G^k. Therefore, π_k is an upper domatic partition in G^k and $D(G^k) \geq D_k(G)$.

Similarly, assume that $\pi = \{V_1, V_2, \ldots, V_q\}$ is a D-partition of G^k. Then by definition of upper domatic partition, for each pair of distinct vertex subsets V_i and V_j belonging to π, either $V_i \to V_j$ or $V_j \to V_i$. Assume without loss of generality that $V_i \to V_j$. Then every vertex in V_j is adjacent to some vertex in V_i of G^k. Further, every vertex in V_j is at a distance at most k from a vertex in V_i in G. Hence, π is a k-upper domatic partition of G, thus proving the theorem. □

2. Bounds of $D(G^k)$

First we examine the bounds of $D(G^k)$. The first result is a lower bound for $D(G^k)$ when $k < \text{diam}(G)$.

Proposition 1. If G is a connected graph, then $D(G^k) \geq k+1$ whenever $1 \leq k < \text{diam}(G)$.

Proof. Since $\text{diam}(G) \geq k+1$, there exists a path $(v_1, v_2, \ldots, v_{k+1})$ in G. Then the vertex partition $\pi = \{\{v_1\}, \{v_2\}, \ldots, \{v_k\}, V(G)\backslash\{v_1, v_2, \ldots, v_k\}\}$ is an upper domatic partition of G^k with order $k+1$. Therefore, $D(G^k) \geq k+1$. □

Next, we characterize graphs G such that $D(G^k) = n$ in terms of diameter of graph G.

Proposition 2. Let G be a connected graph of order n and diameter $\text{diam}(G)$. Then $D(G^k) = n$ if and only if $k \geq \text{diam}(G)$, where k is a positive integer.

Proof. If $\text{diam}(G) \leq k$, then for each vertex $u \in V(G)$ and any $v \in V(G)\backslash\{u\}$, $d(u,v) \leq k$. Thus, in G^k the vertex u will dominate every vertex $v \in V(G)\backslash\{u\}$ and $D(G^k) = n$.
Conversely, if $D(G^k) = n$, then by the definition of upper domatic number, for all $u, v \in V(G)$, the vertices u and v are adjacent in G^k, and further $d(u,v) \leq k$ in G, implying that $\text{diam}(G) \leq k$. □

Since $D(G^k) = n$ whenever $k \geq \text{diam}(G)$, we primarily focus on $k \leq \text{diam}(G)$. Now we examine the relation between the upper domatic number of a graph and its subgraph.

Theorem 2. *[4]* For a graph G and a subgraph H, if H has D-partition with a dominating set, then $D(G) \geq D(H)$.

Proposition 3. For a graph G, if H is its spanning subgraph, then $D(G) \geq D(H)$.

Proof. Let $\pi = \{V_1, V_2, \ldots, V_p\}$ be the D-partition of H. If $V_i \to V_j$ in H for $V_i, V_j \in \pi$, then $V_i \to V_j$ in G as well. Therefore, the partition π is an upper domatic partition of G. □

If we consider two positive integers k_1 and k_2, where $k_1 < k_2$, then G^{k_1} is a spanning subgraph of G^{k_2} which leads to the following result.

Proposition 4. For any graph G and positive integers k_1 and k_2, if $k_1 < k_2$, then $D(G^{k_1}) \leq D(G^{k_2})$.

An example for a graph with the property that $D(G^{k_1}) = D(G^{k_2})$ when $k_1 < k_2$ is that of $G = P_4$. It can be verified that $D(P_4^2) = D(P_4) = 3$.
Proposition 4 gives a relation between upper domatic numbers of k^{th} powers of a graph, for different values of k. The following result is an upper bound for $D(G)$ in terms of maximum degree of the graph [4].

Theorem 3. *[4]* If G is a graph with maximum degree $\Delta(G)$, then $D(G) \leq \Delta(G) + 1$.

In Proposition 4, we have seen that for $k_1 < k_2$, $D(G^{k_1}) \leq D(G^{k_2})$. For any graph G, if $k_1 < \text{diam}(G)$, then there exists a k_2 such that $D(G^{k_1}) < D(G^{k_2})$. We provide a value for k_2, where the inequality is strong.

Theorem 4. For a graph G, if $k < \text{diam}(G)$, then $D(G^k) < D(G^{2k+1})$.

Proof. Let G be a graph and $k < \text{diam}(G)$. If $2k + 1 \geq \text{diam}(G)$, then by Proposition 2, $D(G^{2k+1}) = n > D(G)$. Let us consider that $2k+1 < \text{diam}(G)$. By Theorem 3, we know that $D(G^k) \leq \Delta(G^k) + 1$. Let u be a vertex of maximum degree in G^k. For any two vertices $v, w \in N(u)$, the distance between v and w is at most $2k$ in G. Since $\text{diam}(G) > 2k + 1$, there exists at least one vertex, say x, at distance $k + 1$ from u. The vertices in $N[u] \cup \{x\}$ induces a clique of order $\Delta(G^k) + 2$ in G^{2k+1}, implying that $D(G^{2k+1}) \geq \Delta(G^k) + 2 > D(G^k)$. □

Theorem 5. For a graph G, $D(G^k) \geq \Delta(G^{\lfloor \frac{k}{2} \rfloor}) + 1$.

Proof. The result is obvious if $k \geq \text{diam}(G)$. Taking $k < \text{diam}(G)$, let u be the vertex in $G^{\lfloor \frac{k}{2} \rfloor}$ with maximum degree. The distance between any two vertices in the open neighbourhood of u is at most k. Hence, $N[u]$ induces a clique in G^k and $D(G^k) \geq \Delta(G^{\lfloor \frac{k}{2} \rfloor}) + 1$. □

The bounds for the upper domatic number of a graph in terms of the clique number is obtained in [6].

Theorem 6. *[6]* If $\omega(G)$ is the clique number of a graph G of order n, then $\omega(G) \leq D(G) \leq \dfrac{n + \omega(G)}{2}$.

Characterization of graphs for a given value of $D(G)$ can be found in [4].

Theorem 7. *[4]* For any graph G of order n,

1. $D(G) = 1$ if and only if $G = \overline{K_n}$,
2. $D(G) = 2$ if and only if G is a galaxy with at least one edge,
3. $D(G) \geq 3$ if and only if G contains a K_3 or a P_3.

We now characterise the graphs whose k^{th} power have upper domatic number equal to one for any k.

Theorem 8. For any graph G and positive integer k, $D(G^k) = 1$ if and only if G is $\overline{K_n}$.

Proof. It is obvious that $D(\overline{K_n}^k) = 1$. Conversely, if $D(G^k) = 1$ and $G \neq \overline{K_n}$, then $D(G) \geq 2$. But by Proposition 4, $D(G^k) \geq D(G)$, which contradicts the assumption that $G \neq \overline{K_n}$. □

Now we proceed to characterise the graphs with $D(G^k) = 2$ when $k \geq 2$.

Theorem 9. For any positive integer $k \geq 2$, $D(G^k) = 2$ if and only if $G = n_1K_2 \cup n_2K_1$, where n_1 is a positive integer and n_2 is a non-negative integer.

Proof. It can be easily observed that $D(K_2^k) = 2$ for any value of k. Hence $D(G^k) = 2$, where k is any positive integer. Conversely, if $G \neq n_1K_2 \cup n_2K_1$, then by Theorem 8, $G \neq nK_1$. Hence G contains at least a path P_3 with vertices u_1, u_2, u_3. When $k \geq 2$, $\pi = \{\{u_1\}, \{u_2\}, V(G) \backslash \{u_1, u_2\}\}$ is an upper domatic partition of G^k, implying that $D(G^k) \geq 3$, which is a contradiction. □

Corollary 1. If G is a connected graph of order greater than 2 then $D(G^k) \geq 3$ for $k \geq 2$.

In Proposition 1, we observed that $k+1$ is a lower bound for the upper domatic number of k^{th} power of a graph when $1 \leq k \leq n-1$. We now give a characterization of connected graphs with $D(G^k) = k+1$. The following theorem that gives the exact value of $D(P_n^k)$ is required for further discussion.

Theorem 10. *[6]* The upper domatic number of the k^{th} power of path P_n is

$$D(P_n^k) = \begin{cases} k+1, & \text{if } n = k+1, \\ l+k+1, & \text{if } k+2 \leq n \leq 3k \text{ where } l = \left\lfloor \frac{n-k-1}{2} \right\rfloor, \\ 2k+1, & \text{if } n \geq 3k+1. \end{cases}$$

Theorem 11. For any positive integer $k > 1$, G is a connected graph with $D(G^k) = k+1$ if and only if $G = P_{k+2}$ or G is of order $k+1$.

Proof. If G is a connected graph of order $k+1$, then $\operatorname{diam}(G) \leq k$, so that $D(G^k) = k+1$. On the other hand, when $G = P_{k+2}$, $D(G^k) = k+1$ by Theorem 10.
Conversely, assume that G is a connected graph of order n such that $D(G^k) = k+1$. It is immediate that $n \geq k+1$. We wish to prove that G is of order $k+1$ when $G \neq P_{k+2}$. If possible let $n > k+1$. Then $\operatorname{diam}(G) = k+1$, otherwise $D(G^k) \geq k+2$, contradicting the assumption. Consider the diametral path of G say P which is of order $k+2$. Let $V(P) = \{v_1, v_2, \dots, v_{k+2}\}$ where v_i is adjacent to v_{i+1}, $1 \leq i \leq k+1$. The graph G being connected, there exists a vertex $v \notin V(P)$ which is adjacent to v_i, $1 \leq i \leq k+2$. Suppose v is adjacent to v_i, $2 \leq i \leq k+1$. Then v is at most at distance k from v_1 or v_{k+2} and consequently the k^{th} power of $G[V(P) \cup \{v\}]$ contains a clique of order $k+2$, suggesting that $D(G^k) \geq k+2$ by Theorem 6. If v is adjacent to one of the end vertices of P say v_1, then $V(P) \cup \{v\}$ contains a P_{k+3}. Then $\big\{\{v_1\}, \{v_2\} \dots \{v_{k+1}\}, \{v, v_{k+2}\}\big\}$ is a D-partition of cardinality $k+2$. Hence by Theorem 2, $D(G^k) \geq k+2$, contradicting the assumption that $D(G^k) = k+1$. Thus, G is of order $k+1$ if $G \neq P_{k+2}$. □

As a special case of Theorem 11, when $k = 2$ we have the following result.

Corollary 2. Let G be a connected graph. Then, $D(G^2) = 3$ if and only if $G \in \{P_3, K_3, P_4\}$.

The following characterization of graphs with $D(G^2) \geq 4$ immediately follows from Corollary 2.

Corollary 3. $D(G^2) \geq 4$ if and only if G contains $H \in \{K_{1,3}, C_4, P_5\}$ whenever G is a connected graph.

3. Upper domatic number of square of graphs

We now explore few properties of upper domatic number of square of graphs. As seen in Proposition 4, the value of upper domatic number of square of a graph is at least as much as the upper domatic number of the graph. Moreover, the difference between these values of a graph can be arbitrarily large. For $a \geq 2$ and $b \geq 1$, let G be the graph of order $a+b$, with a clique of order a and b pendant vertices adjacent to a single vertex in the clique. It can be easily verified that $D(G) = a$, while $D(G^2) = a+b$, for $a \geq 2$ and $b \geq 1$. It is evident from the previous example that there exists graph G, such that $D(G) < D(G^2)$. Similarly, there are graphs for which its upper domatic number coincides with the upper domatic number of its square. In [4], Haynes et al. proved that $\Delta(G)+1$ is an upper bound for the upper domatic number of a graph and called a graph G with $D(G) = \Delta(G)+1$ *upper domatically full.* We next show that a graph whose upper domatic number coincides with the upper domatic number of its square is upper domatically full.

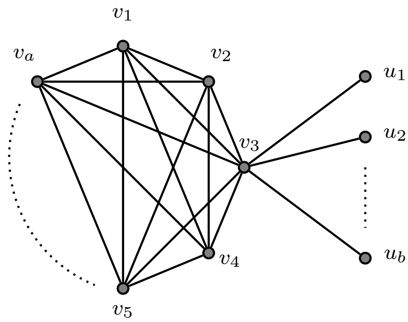

Figure 1. **The graph G.**

Theorem 12. For a graph G, if $D(G) = D(G^2)$, then G is upper domatically full.

Proof. Consider the vertex $v \in V(G)$ of maximum degree and let $v_1, v_2, \ldots, v_{\Delta(G)}$ be the vertices adjacent to v. Then, the set of vertices $\{v, v_1, v_2, \ldots, v_{\Delta(G)}\}$ forms a clique in G^2, proving that $\omega(G^2) \geq \Delta(G) + 1$. Hence, by Theorem 6, $D(G^2) \geq \omega(G^2) \geq \Delta(G) + 1$. Since $\Delta(G) + 1$ is an upper bound for $D(G)$, it follows that $D(G) \leq \Delta(G) + 1 \leq D(G^2)$, thus completing the proof. □

Corollary 4. *If G is a graph with maximum degree $\Delta(G)$, then $D(G^2) \geq \Delta(G) + 1$.*

Theorem 13. For a connected graph G of order $n \geq 4$ and minimum degree $\delta(G) \geq 2$, the upper domatic number of G^2 is at least four.

Proof. Since $\delta(G) \geq 2$, G contains a cycle. If the graph G contains a C_t, where $t \geq 4$, then $D(G^2) \geq 4$. If the graph G contains no cycle of order at least 4, then G contains contains a triangle. Since $n \geq 4$, G contains $K_{1,3}$ and by Corollary 4, $D(G^2) \geq 4$. □

A Nordhaus–Gaddum type bounds for the upper domatic number of square of a graph is presented next.

Theorem 14. For a graph G of n vertices, $n + 1 \leq D(G^2) + D(\overline{G}^2) \leq 2n$.

Proof. By Corollary 4, $D(G^2) \geq \Delta(G) + 1$ and $D(\overline{G}^2) \geq \Delta(\overline{G}) + 1$. Let v be the vertex of maximum degree in G. The degree of vertex v in $\overline{G}$ is $n - \Delta(G) - 1$. Therefore, $D(\overline{G}^2) \geq n - \Delta(\overline{G})$. Hence, $D(G^2) + D(\overline{G}^2) \geq n + 1$. □

The *corona* $G \circ H$ of two graphs G and H is defined as the graph obtained by taking one copy of G and n copies of H, and joining the i^{th} vertex of G with every vertex in the i^{th} copy of H, where n is the order of G.

Theorem 15. If G is a graph of order n, then $D((G \circ K_1)^2) \leq n + 1$. Moreover, $D((G \circ K_1)^2) = n + 1$ if and only if $\text{diam}(G) \leq 2$.

Proof. Let G be a graph of order n, then in a D-partition of $(G \circ K_1)^2$ there is at most one set containing only pendant vertices of the corona. If there are two sets say, X and Y containing only pendant vertices of the corona, then X does not dominate Y in $(G \circ K_1)^2$ and vice versa, as the distance between any two pendant vertices is at least three. Therefore, $D((G \circ K_1)^2) \leq n + 1$.
If $D((G \circ K_1)^2) = n + 1$ and $\text{diam}(G) \geq 3$, then by the same arguments there is exactly one set containing only pendant vertices of the corona, suggesting that in a D-partition π of $(G \circ K_1)^2$ there are n sets containing vertices from $V(G)$. If the n sets contain only vertices from $V(G)$, then $D(G^2) = n$ and $\text{diam}(G) \leq 2$. Hence, some sets contain vertices from both $V(G)$ and pendant vertices. Consider vertices u and v of G which are at diametral distance. Let $A, B \in \pi$ such that $u \in A$, $v \in B$. Without loss of generality assume that $A \to B$. Since $d(u, v) \geq 3$, A contains a pendant vertex. If $C \in \pi$ is the set containing only pendant vertices of the corona, then $A \to C$. Further, the vertices of C are adjacent either to u or to a vertex in $N(u)$. However, these vertices in C are at a distance greater than two from v. Therefore, B does not dominate C in $(G \circ K_1)^2$ and vice versa. Thus, $\text{diam}(G) \leq 2$ whenever $D((G \circ K_1)^2) = n + 1$. The converse can be proved easily. □

Now, we determine the upper domatic number of powers of some common classes of graphs. It is not difficult to obtain the value of $D(G^k)$ when G is complete graph K_n and complete bipartite graph $K_{r,s}$. The upper domatic number of powers of cycles was determined in [6].

Theorem 16. *[6]* The upper domatic number of the k^{th} power of cycle C_n is

$$D(C_n^k) = \begin{cases} \text{n}, & \text{if } 2k \leq n \leq 2k+1, \\ \text{l+k+1}, & \text{if } 2k+1 < n \leq 3k \quad \text{where } l = \left\lfloor \frac{n-k-1}{2} \right\rfloor, \\ 2k+1, & \text{if } n \geq 3k+1. \end{cases}$$

4. Generalized petersen graph

For the integers $a \geq 3$ and $1 \leq b \leq \left\lfloor \dfrac{a-1}{2} \right\rfloor$, the *generalized petersen graph* denoted as $P(a, b)$ is a connected graph with the vertex set $V(P(a, b)) = \{u_1, u_2, \ldots, u_a, v_1, v_2, \ldots, v_a\}$ and the edge set $E(G) = \{u_i u_{(i+1)}, u_i v_i, v_i v_{(i+b)} | \text{ for } 1 \leq i \leq a\}$ where the indices are taken modulo a. The generalized petersen graph is a 3-regular graph of order $2a$.

Theorem 17. The upper domatic number of a generalized petersen graph is four.

Proof. Let $P(a,b)$ be a generalized petersen graph. By the definition of generalized petersen graphs, the set $\{u_1, u_2, \ldots, u_a\}$ induces a cycle. Since the upper domatic number of a cycle is three and the set $\{v_1, v_2, \ldots, v_a\}$ is a dominating set of $P(a,b)$, $D(P(a,b)) \geq 4$. Moreover, $P(a,b)$ is a 3-regular graph and $D(G) \leq \Delta(G)+1$. Hence, $D(G) = 4$. □

We require the following results about $P(a,1)$, which can be verified in order to determine the upper domatic number of $P(a,1)^k$, for $k \geq 2$.

Proposition 5. For the generalized petersen graph $P(a,1)$,

1. $\text{diam}(P(a,1)) = \left\lfloor \frac{a}{2} \right\rfloor + 1$

2. $\Delta(P(a,1)^k) = \begin{cases} 4k-1, & \text{if } 1 \leq k \leq \text{diam}(P(a,1)) - 2, \\ 2a-3, & \text{if } a \text{ is odd and } k = \text{diam}(P(a,1)) - 1, \\ 2a-2, & \text{if } a \text{ is even and } k = \text{diam}(P(a,1)) - 1, \\ 2a-1, & \text{if } k = \text{diam}(P(a,1)). \end{cases}$

3. $\omega(P(a,1)^k) = \begin{cases} 2k, & \text{if } 1 \leq k \leq \text{diam}(P(a,1)) - 2, \\ a, & \text{if } k = \text{diam}(P(a,1)) - 1, \\ 2a, & \text{if } k = \text{diam}(P(a,1)). \end{cases}$

Theorem 18. For $a \geq 3$, the upper domatic number of k^{th} power of $P(a,1)$ is

$$D(P(a,1)^k) = \begin{cases} 2a, & \text{if } k = d+1, \\ \left\lfloor \frac{3a}{2} \right\rfloor, & \text{if } k = d, \\ k+a, & \text{if } \frac{a}{3} \leq k < d, \\ 4k, & \text{if } 1 \leq k < \frac{a}{3}, \end{cases} \quad \text{where } d = \left\lfloor \frac{a}{2} \right\rfloor.$$

Proof. It is important to note that a vertex $u_i \in V(P(a,1))$ is at a distance at most k from each of the vertex for $1 \leq j \leq k$ in the set $\{u_{(i+j)}, u_{(i-j)}, v_{(i+j-1)}, v_{(i-j+1)}\}$, where the indices are taken modulo a. For different values of k, we consider the following cases.

Case 1: If $k = d+1 = \text{diam}(P(a,1))$, then $D(P(a,1)^k) = 2a$ by Proposition 2.

Case 2: If $k = d$, then we partition the vertices of $P(a,1)^k$ into $\left\lfloor \frac{3a}{2} \right\rfloor$ sets in the following manner.

$$V_i = \begin{cases} \{u_i\}, & \text{if } 1 \leq i \leq a, \\ \{v_{i-a}, v_{i+d-a}\}, & \text{if } a+1 \leq i \leq a+d-1, \\ \{v_d, v_a\}, & \text{if } i = a+d \text{ and } a \text{ is even}, \\ \{v_d, v_{a-1}, v_a\}, & \text{if } i = a+d \text{ and } a \text{ is odd}. \end{cases}$$

Since the vertices $\{u_1, u_2, \dots, u_a\}$ induces a clique of order a, the sets $V_1, V_2, \dots, V_a$ dominates each other. The sets $V_{a+1}, V_{a+2}, \dots, V_{a+d}$ are dominating sets of $P(a,1)^k$. Therefore, $D(P(a,1)^k) \geq \left\lfloor \frac{3a}{2} \right\rfloor$. But considering the clique number of $P(a,1)$, by Theorem 6, we have $D(P(a,1)^k) = \left\lfloor \frac{3a}{2} \right\rfloor$.

Case 3: If $\frac{a}{3} \leq k < d$, then we provide the following upper domatic partition of order $k+a$.

$$V_i = \begin{cases} \{u_i\}, & \text{if } 1 \leq i \leq k, \\ \{v_{i-k}\}, & \text{if } k+1 \leq i \leq 2k, \\ \{u_{i-k}, v_{a+2k-i+1}\}, & \text{if } 2k+1 \leq i \leq k+a. \end{cases}$$

The sets $V_1, V_2, \dots, V_{2k}$ induces a clique of $P(a,1)^k$ and the sets $V_{2k+1}, V_{2k+2}, \dots, V_{k+a}$ are dominating sets of $P(a,1)^k$, indicating that $D(P(a,1)^k) \geq k+a$. By Theorem 6 we have $D(P(a,1)^k) \leq \left\lfloor \frac{2a+2k}{2} \right\rfloor = a+k$. Therefore, $D(P(a,1)^k) = k+a$.

Case 4: If $1 \leq k < \frac{a}{3}$, then a vertex partition of a subgraph of $P(a,1)^k$ is given below.

$$V_i = \begin{cases} \{u_i\}, & \text{if } 1 \leq i \leq k, \\ \{v_{i-k}\}, & \text{if } k+1 \leq i \leq 2k, \\ \{u_{i-k}, v_{a+2k-i+1}\}, & \text{if } 2k+1 \leq i \leq 3k, \\ \{v_{i-2k}, u_{a+3k-i+1}\}, & \text{if } 3k+1 \leq i \leq 4k. \end{cases}$$

Similar to Case 3, when $1 \leq k < \frac{a}{3}$, the set $\{u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k\}$ induces a clique of order $2k$. Hence, the sets $V_1, V_2, \dots, V_{2k}$ dominates each other. Moreover, the sets $V_{2k+1}, V_{2k+2}, \dots, V_{4k}$ are dominating sets of $P(a,1)^k$, implying that $D(P(a,1)^k) \geq 4k$. But $\Delta(P(a,1)^k) = 4k-1$. Hence, Theorem 3 implies that $D(P(a,1)^k) = 4k$. □

5. Hypercubes

The n-hypercube graph, denoted by Q_n, is the graph whose vertex set comprises all the 2^n symbols $a_1, a_2, \dots, a_n$ where $a_i \in \{0,1\}$ and two vertices are adjacent if and only if their symbols differ in exactly one coordinate. In [4], the upper domatic number of a hypercube was discussed.

Theorem 19. *[4] For a n-hypercube, $D(Q_n) = n$.*

Note that for any two vertices u and v in a n-hypercube, if $d(u,v) \leq k$, then their symbols differ in at most k positions. The diameter of a n-hypercubes is n. Now, we explore the maximum degree and clique number of k^{th} power of n-hypercubes. The following result can be easily extended from [2].

Theorem 20. For the n-hypercubes Q_n,

1. $\Delta(Q_n^k) = \sum_{i=1}^{k} \binom{n}{i}$,

2. $\omega(Q_n^k) = \begin{cases} \sum_{i=0}^{\frac{k}{2}} \binom{n}{i}, & \text{if } k \text{ is even,} \\ 2\sum_{i=0}^{\frac{k-1}{2}} \binom{n-1}{i}, & \text{if } k \text{ is odd.} \end{cases}$

Theorem 21. For positive integers $n \geq k$, $D(Q_{n+1}^k) \geq D(Q_n^k) + k$.

Proof. In order to prove this theorem, we first show that a domatic partition of order k can be obtained from the vertices of Q_n^{k-1}, for $1 \leq k \leq n$. Let v be a vertex of Q_n^{k-1}. The sets in the domatic partition is defined as $V_i = \{u \in V(Q_n) \mid d(u,v) = i \pmod{k}\}$, where $0 \leq i \leq k-1$. Since, for any vertex $u \in V(Q_n^{k-1})$ there exists a diametral path of Q_n^{k-1} starting from v and containing u, every set V_i is a dominating set. After all, Q_{n+1} contains two vertex disjoint Q_ns as induced subgraphs say H_1 and H_2. Let π_1 be a D-partition of H_1^k and π_2 be the domatic partition of H_2^{k-1}. Every set in π_2 being a dominating set of H_2^{k-1}, any set in π_2 dominates each vertex of H_1 in Q_{n+1}^k. Hence the partition $\pi = \pi_1 \cup \pi_2$ of order $D(Q_n^k) + k$ is an upper domatic partition of Q_{n+1}^k. □

Theorem 22. Let n and k be positive integers. Then $D(Q_{n+1}^{k+1}) \geq 2D(Q_n^k)$.

Proof. Let $\pi = \{V_1, V_2, \ldots, V_p\}$ be a D-partition of Q_n^k. We define V_i' as the set of vertices obtained by concatenating symbol zero before the symbol of every vertex in V_i. Similarly, V_i'' is the set of vertices obtained by concatenating symbol one before the symbol of every vertex in V_i. The partition $\pi' = \{V_1', V_2', \ldots, V_p', V_1'', V_2'', \ldots, V_p''\}$ is an upper domatic partition of Q_{n+1}^{k+1}, therefore, $D(Q_{n+1}^{k+1}) \geq 2D(Q_n^k)$. □

From the recursive relation in Theorem 22, one can easily deduce the following lower bound for the upper domatic number of k^{th} power of n-hypercube.

Corollary 5. If G is a n-hypercube, then $D(Q_n^k) \geq (n-k+2)2^{k-1}$.

Theorem 23. For any positive integer k, $D(Q_k^k) = 2^k$.

Proof. This theorem immediately follows from the fact that diameter of an n-hypercube is n and its order is 2^n. □

Theorem 24. If $n \geq 2$ and $k = n-1$, then $D(Q_n^k) = 3 \times 2^{n-2}$.

Proof. We first show that $D(Q_n^k) \geq 3 \times 2^{n-2}$, when $k = n-1$. This follows immediately from Corollary 5. The clique number of Q_n^k is $\sum_{i=0}^{\frac{n-1}{2}} \binom{n}{i}$ when n is odd and $2\sum_{i=0}^{\frac{n-2}{2}} \binom{n-1}{i}$ when n is even. It is well known that $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ and $\binom{n}{i} = \binom{n}{n-i}$. Therefore the clique number of Q_n^k when k is $n-1$ is 2^{n-1}. By Theorem 6, $D(Q_n^k) \leq \dfrac{2^{n-1}+2^n}{2} = 3 \times 2^{n-2}$. Hence, $D(Q_n^k) = 3 \times 2^{n-2}$. □

Theorem 25. If $k \geq 1$ and $n = 2k+1$, then $D(Q_n^k) = 2^{n-1}$.

Proof. For a vertex v in Q_n^k having the symbol $(a_1, a_2, \ldots, a_n)$, let $\overline{v}$ be the vertex whose symbol is obtained by toggling the value of a_i at the position i. Here n is odd and $k = \left\lfloor \dfrac{n}{2} \right\rfloor$. Any vertex whose symbol differs at a maximum of k positions from that of v is at a distance at most k from v and the vertices whose binary code differs at a minimum of $k+1$ positions from that of v is at a distance at most k from $\overline{v}$. Therefore, for any $v \in Q_n$, the set $\{v, \overline{v}\}$ forms a dominating set of Q_n^k and the vertex set of Q_n^k can be partitioned into 2^{n-1} disjoint dominating sets. Hence, $D(Q_n^k) \geq 2^{n-1}$ when $k \geq 1$ and $n = 2k+1$. By Theorem 20, $\Delta(Q_n^k) = \sum_{i=1}^{k} \binom{n}{i} = 2^{n-1} - 1$. But by Theorem 3, $D(G^k) \leq \Delta(G^k) + 1$. Thus, $D(Q_n^k) = 2^{n-1}$ when $k \geq 1$ and $n = 2k+1$. □

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