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On Zagreb energy and edge-Zagreb energy

Rakshith Billava Ramanna

Department of Mathematics, Vidyavardhaka College of Engineering, Mysuru 570002, India
ranmsc08@yahoo.co.in

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Abstract: In this paper, we obtain some upper and lower bounds for the general extended energy of a graph. As an application, we obtain few bounds for the (edge) Zagreb energy of a graph. Also, we deduce a relation between Zagreb energy and edge-Zagreb energy of a graph G with minimum degree $\delta \geq 2$. A lower and upper bound for the spectral radius of the edge-Zagreb matrix is obtained. Finally, we give some methods to construct (edge) Zagreb equienergetic graphs and show that there are (edge) Zagreb equienergetic graphs of order $n \geq 9$.

Keywords: Zagreb energy, edge-Zagreb energy, equienergetic graphs

AMS Subject classification: 05C05

1. Introduction

Graphs considered here are simple and undirected. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$, edge set $E(G)$ and vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A(G)$, the adjacency matrix of G. Then the energy of the graph G [15] is defined as $\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|$. Recent research about graph energies can be found in the survey article by Gutman and Ramane [10], and also see [2, 9]. The Zagreb matrix of G, denoted by $Z_1(G)$, is the $n \times n$ matrix whose (i,j)th entry is $d_i + d_j$ if $v_i v_j \in E(G)$, 0 otherwise. This matrix was first considered in [13], where it was named "first Zagreb matrix". In [13] the respective energy was called "first Zagreb energy". The spectrum of $Z_1(G)$ is called the Zagreb spectrum of G and is denoted by $spec(Z_1(G)) = \{\eta_1, \eta_2, \ldots, \eta_n\}$, where $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$. The Zagreb energy of G (same as the first Zagreb energy in [13]) is denoted by $Z\mathcal{E}_1(G)$ and is defined as $Z\mathcal{E}_1(G) = \displaystyle\sum_{i=1}^{n} |\eta_i|$. The concept of Zagreb energy was recently introduced by Rad, Jahanbani and Gutman [13]. Some properties of Zagreb matrix and some upper and lower bounds for the Zagreb energy can be found in [4, 13]. The edge-Zagreb matrix of G, denoted by $Z_2(G)$, is the $n \times n$ matrix whose (i, j)th entry is $d_i d_j$ if $v_i v_j \in E(G)$, 0 otherwise. This matrix was first considered in [13], where it was named "second Zagreb matrix". In [13] the respective energy was called "second Zagreb energy". The spectrum of $Z_2(G)$ is called the edge-Zagreb spectrum of G and is denoted by $spec(Z_2(G)) = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. The edge-Zagreb energy of G (same as the second Zagreb energy in [13]) is defined as $Z\mathcal{E}_2(G) = \sum_{i=1}^{n} |\gamma_i|$. The invariant $Z\mathcal{E}_2(G)$ was introduced by Rad, Jahanbani and Gutman in [13]. Following Zhan, Qiao and Cai [17], we shall refer to it as the "edge-Zagreb energy". In [17], the authors obtained several properties of the edge-Zagreb matrix and some upper bounds for the spectral radius γ_1, and also some upper and lower bounds of the edge-Zagreb energy $Z\mathcal{E}_2(G)$.

Let $TI(G)$ be a degree based topological index of the form

$$TI = \sum_{v_i v_j \in E(G)} \mathcal{F}(d_i, d_j),$$

where $\mathcal{F}$ is an appropriately chosen function such that $\mathcal{F}(d_i, d_j) = \mathcal{F}(d_j, d_i)$.

Some of the well-known degree based topological indices of this kind are first Zagreb index M_1, $\mathcal{F}(d_i, d_j) = d_i + d_j$, second Zagreb index M_2, $\mathcal{F}(d_i, d_j) = d_i d_j$, hyper Zagreb index HM, $\mathcal{F}(d_i, d_j) = (d_i + d_j)^2$, general Randić index R_α, $\mathcal{F}(d_i, d_j) = (d_i d_j)^\alpha$, harmonic index, $\mathcal{F}(d_i, d_j) = \dfrac{2}{d_i + d_j}$, atom-bond connectivity index ABC, $\mathcal{F}(d_i, d_j) = \sqrt{\dfrac{d_i + d_j - 2}{d_i d_j}}$, geometric-arithmetic index GA_1, $\mathcal{F}(d_i, d_j) = \dfrac{2\sqrt{d_i d_j}}{d_i + d_j}$, etc. See [7, 8, 14] for more details. The general extended adjacency matrix associated with the topological index $TI(G)$ is denoted by $T(G)$ and is defined as $T(G) = (t_{ij})_{n \times n}$, where

$$t_{ij} = \begin{cases} \mathcal{F}(d_i, d_j), & \text{if } v_i v_j \in E(G) \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{F}(d_i, d_j) = 1$, then $T(G) = A(G)$, the adjacency matrix of G and if $\mathcal{F}(d_i, d_j) = d_i + d_j$, then $T(G) = Z_1(G)$. Also, $T(G) = Z_2(G)$ if $\mathcal{F}(d_i, d_j) = d_i d_j$. Let $f_1 \geq f_2 \geq \cdots \geq f_n$ be the eigenvalues of the topological matrix T, Then the energy of the general extended adjacency matrix T, $\mathcal{E}_T(G)$, is defined as

$$\mathcal{E}_T(G) = \sum_{i=1}^{n} |f_i|.$$

Studies on general extended adjacency matrix and its energy can be found be in [6]. Two non-isomorphic graphs of same order are said to be equienergetic if their energies are equal. Similarly, two graphs are said to be (edge) Zagreb equienergrtic if their (edge) Zagreb energies are same. More details on equienergetic graphs can be found

in [1, 15]. As usual, we denote by P_n, C_n, K_n and $K_{n,m}$, the path graph, the cycle, the complete graph and the complete bipartite graph of order $n+m$.
In Section 2 of the paper, we obtain some upper and lower bounds for the general extended energy of a graph. As an application, we obtain few bounds for the (edge) Zagreb energy of a graph. Also, we obtain a relation between Zagreb and edge-Zagreb energy of a graph G with minimum degree $\delta \geq 2$. In Section 3, we obtain a lower and upper bound for the spectral radius γ_1 of $Z_2(G)$. In Section 4, we give some methods to construct (edge) Zagreb equienergetic graphs and show that there are (edge) Zagreb equienergetic graphs of order $n \geq 9$.

2. Some bounds for the (edge) Zagreb energy of a graph

Let M be a $m \times n$ matrix and let the singular values of M be denoted by $s_1(M) \geq s_2(M) \geq \cdots \geq s_m(M)$. We need the following two lemmas (see, [16]) to prove one of our results.

Lemma 1. [16] *If A and B are $n \times n$ complex matrices. Then*

$$\sum_{i=1}^{k} s_i(A+B) \leq \sum_{i=1}^{k} s_i(A) + \sum_{i=1}^{k} s_i(B),\ k = 1, 2, \ldots, n.$$

Lemma 2. [16] *If $A_1, A_2, \ldots, A_m$ are $n \times n$ complex matrices. Then*

$$\sum_{i=1}^{k} s_i(A_1 A_2 \ldots A_m) \leq \sum_{i=1}^{k} s_i(A_1) s_i(A_2) \ldots s_i(A_m),\ k = 1, 2, \ldots, n.$$

In the following theorem, we obtain an upper bound for the energy of the general extended adjacency matrix in terms of ordinary energy when $\mathcal{F}(d_i, d_j) = f(d_i) + f(d_j) + g(d_i)g(d_j)$ and $f(d_i)$ and $g(d_i)$ are non-negative real numbers.

Theorem 1. *Let $f(d_i)$ and $g(d_i)$ be non-negative real numbers. If $\mathcal{F}(d_i, d_j) = f(d_i) + f(d_j) + g(d_i)g(d_j)$. Then*

$$\mathcal{E}_T(G) \leq (2f_m + g_m^2)\mathcal{E}(G) \tag{1}$$

where $f_m = \max\{f(d_1), f(d_2), \ldots, f(d_n)\}$ and $g_m = \max\{g(d_1), g(d_2), \ldots, g(d_n)\}$. Equality in (1) is attained if and only if $f(d_1) = f(d_2) = \cdots = f(d_n)$ and $g(d_1) = g(d_2) = \cdots = g(d_n)$.

Proof. Let D_1 be the diagonal matrix with diagonal entries $f(d_1), f(d_2), \ldots, f(d_n)$ and let D_2 be the diagonal matrix with diagonal entries $g(d_1), g(d_2), \ldots, g(d_n)$. Since $\mathcal{F}(d_i, d_j) = f(d_i) + f(d_j) + g(d_i)g(d_j)$, the matrix T can be written as

$$T(G) = D_1 A(G) + A(G) D_1 + D_2 A(G) D_2.$$

Therefore by Lemma 1 and Lemma 2, we get

$$\begin{aligned}\sum_{i=1}^{n} s_i(T(G)) &\le \sum_{i=1}^{n} s_i(D_1A(G)+A(G)D_1)+\sum_{i=1}^{n} s_i(D_2A(G)D_2)\\ &\le \sum_{i=1}^{n} s_i(D_1A(G))+s_i(A(G)D_1)+\sum_{i=1}^{n} s_i(D_2A(G)D_2)\\ &= 2\sum_{i=1}^{n} s_i(D_1A(G))+\sum_{i=1}^{n} s_i(D_2A(G)D_2)\\ &\le 2\sum_{i=1}^{n} s_i(D_1)s_i(A(G))+\sum_{i=1}^{n} s_i(D_2)s_i(A(G))s_i(D_2)\\ &\le 2f_m\sum_{i=1}^{n} s_i(A(G))+g_m^2\sum_{i=1}^{n} s_i(A(G)) \qquad (2)\\ &= 2f_m\mathcal{E}(G)+g_m^2\mathcal{E}(G).\end{aligned}$$

Thus $\mathcal{E}_T(G) \le (2f_m+g_m^2)\mathcal{E}(G)$. Suppose the equality in (1) holds. Then the inequality in (2) must be equality. Thus $f(d_1)=f(d_2)=\cdots=f(d_n)$ and $g(d_1)=g(d_2)=\cdots=g(d_n)$. Conversely, if $f(d_1)=f(d_2)=\cdots=f(d_n)$ and $g(d_1)=g(d_2)=\cdots=g(d_n)$, then $T(G)=(2f_m+g_m^2)A(G)$. Hence $\mathcal{E}_T(G)=(2f_m+g_m^2)\mathcal{E}(G)$. This completes the proof. □

As an application of Theorem 1, we obtain the following corollary which gives an upper bound for (edge) Zagreb energy in terms of ordinary energy.

Corollary 1. *Let G be a graph of order n, Then $Z\mathcal{E}_1(G) \le 2\Delta\mathcal{E}(G)$ and $Z\mathcal{E}_2(G) \le \Delta^2\mathcal{E}(G)$. Equality is attained only if G is a regular graph.*

Proof. Setting $f(d_i)=d_i$, $g(d_i)=0$ and $f(d_i)=0$, $g(d_i)=d_i$ in Theorem 1 we obtain the desired result. □

The following theorem gives an upper bound for the energy of general extended adjacency matrix in terms of $\mathcal{F}(d_i,d_j)$.

Theorem 2. *Let G be a graph on n vertices. Then*

$$\mathcal{E}_T(G) \le \sum_{i=1}^{n}\sqrt{\sum_{v_iv_j\in E(G)}\mathcal{F}^2(d_i,d_j)}.$$

Proof. We have $2T=\sum_{i=1}^{n} T_i$, where T_i is the $n\times n$ matrix obtained from T by letting all the entries equal to 0 expect the ith row and ith column entries. By Lemma 1,

$$2\mathcal{E}_T(G) \le \sum_{i=1}^{n}\mathcal{E}(T_i). \qquad (3)$$

Claim: $\mathcal{E}(T_i) = 2\sqrt{\sum_{v_iv_j\in E(G)} \mathcal{F}^2(d_i,d_j)}$.
From the definition of T_i, we see that the matrix T_i is similar to the matrix $ST_i = \begin{pmatrix} 0 & X \\ X^T & \mathbf{0} \end{pmatrix}$, where the jth entry of the row vector X is $\mathcal{F}(d_i,d_j)$ if $v_iv_j \in E(G)$, 0 otherwise. It is straightforward that the rank of ST_i is either 0 or 2. If $rank(ST_i) = 0$, then we are done. Suppose $rank(ST_i) = 2$. Since $trace(ST_i) = 0$, it follows that μ and $-\mu$ are the only non-zero eigenvalues of ST_i, where μ is an unknown positive number. Thus $2\mu^2 = trace(ST_i^2) = 2\sum_{v_iv_j\in E(G)} \mathcal{F}^2(d_i,d_j)$ and so $\mu = \sqrt{\sum_{v_iv_j\in E(G)} \mathcal{F}^2(d_i,d_j)}$.
Hence $spec(T_i) = \{\pm\sqrt{\sum_{v_iv_j\in E(G)} \mathcal{F}^2(d_i,d_j)}, 0, 0, \ldots, 0\}$. Therefore

$$\mathcal{E}(T_i) = 2\sqrt{\sum_{v_iv_j\in E(G)} \mathcal{F}^2(d_i,d_j)}. \tag{4}$$

From equations (3) and (4), we get the desired result. □

From Theorem 2, we obtain the following two corollaries which give an upper bound for the (edge) Zagreb energy in terms of vertex degrees.

Corollary 2. *Let G be a graph on n vertices. Then*

$$Z\mathcal{E}_1(G) \le \sum_{i=1}^{n}\sqrt{\sum_{v_iv_j\in E(G)} (d_i+d_j)^2}. \tag{5}$$

Proof. Setting $\mathcal{F}(d_i,d_j) = d_i + d_j$ in Theorem 2, we obtain (5). □

Corollary 3. [17] *Let G be a graph on n vertices. Then*

$$Z\mathcal{E}_2(G) \le \sum_{i=1}^{n}\sqrt{\sum_{v_iv_j\in E(G)} (d_id_j)^2}. \tag{6}$$

Proof. Setting $\mathcal{F}(d_i,d_j) = d_id_j$ in Theorem 2, we obtain (6). □

Let M be a Hermitian matrix of order p. We denote the eigenvalues of M by $\theta_1(M) \ge \theta_2(M) \ge \cdots \ge \theta_p(M)$. The following lemmas are useful to prove our next theorem.

Lemma 3. [11] *Let $A = (a_{ij})$ and $B = (b_{ij})$ be symmetric, non-negative matrices of order n. If $A \ge B$, i.e., $a_{ij} \ge b_{ij}$ for all i, j, then $\theta_1(A) \ge \theta_1(B)$.*

Lemma 4. [5] *Let G be a graph of order n. Then $|f_1| = |f_2| = \cdots = |f_n|$ if and only if $G \cong \overline{K_n}$ or $G \cong \frac{n}{2}K_2$.*

Lemma 5. [3] *Let G be a connected graph of order n and size m. Then $\lambda_1 \geq 2m/n$. Equality holds if and only if G is a regular graph.*

Theorem 3. *Let G be a connected graph on n vertices with m edges. Suppose $\mathcal{F}(d_i, d_j) > 0$ and s is the smallest positive element in $T(G)$. Then*

$$\mathcal{E}_T(G) \leq t + \sqrt{(n-1)\left(2 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - t^2\right)}, \tag{7}$$

where $t = \max\left\{\sqrt{\frac{2\sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i,d_j)}{n}}, \frac{2ms}{n}\right\}$. Moreover the equality holds if and only if G is a compete graph or a strongly regular graph with two non-trivial eigenvalues whose absolute values are equal to $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/(n-1)}$.

Proof. From the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\sum_{i=2}^{n} |f_i| \leq \sqrt{(n-1)\sum_{i=2}^{n} f_i^2} &= \sqrt{(n-1)\left(\sum_{i=1}^{n} f_i^2 - f_1^2\right)} \\ &= \sqrt{(n-1)\left(2 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - f_1^2\right)}.\end{aligned} \tag{8}$$

Thus

$$\mathcal{E}_T(G) \leq f_1 + \sqrt{(n-1)\left(2 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - f_1^2\right)}.$$

Let $f(t) := t + \sqrt{(n-1)\left(2\sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i,d_j) - t^2\right)}$. Then f has maximum at $t_0 = \sqrt{\frac{2\sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i,d_j)}{n}}$. Since $T(G) \geq sA(G)$, from Lemma 3 and Lemma 5, we get $f_1 \geq s\lambda_1 \geq \frac{2ms}{n}$. Now, if $f_1 \leq t_0$, then

$$\mathcal{E}(G) \leq t_0 + \sqrt{(n-1)\left(2 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - t_0^2\right)}. \tag{9}$$

Also, if $\dfrac{2ms}{n} \geq t_0$, then

$$\mathcal{E}(G) \leq \frac{2ms}{n} + \sqrt{(n-1)\left(2\sum_{v_iv_j\in E(G)} \mathcal{F}^2(d_i,d_j) - \left(\frac{2ms}{n}\right)^2\right)}. \tag{10}$$

The first part of the proof is done. Suppose the equality in (7) holds. Then the inequality (8) must be equality. Thus $|f_2| = |f_3| = \cdots = |f_n|$. If the equality in (9) holds, then $f_1 = t_0$. Therefore, $nf_1^2 = \sum_{i=1}^n f_i^2$ and so $|f_1| = |f_2| = |f_3| = \cdots = |f_n|$. Thus by Lemma 4, we must have $G \cong K_1$ or $G \cong K_2$. Now, if the equality in (10) holds then $f_1 = \dfrac{2ms}{n}$, i.e., $\lambda_1 = 2m/n$ and so by Lemma 5, G must be a regular graph. Now, since G is regular and $|f_2| = |f_3| = \cdots = |f_n|$, we must have $|\lambda_2| = |\lambda_3| = \cdots = |\lambda_n|$. Hence G is a regular graph with at most 3 distinct eigenvalues. First, if G has exactly one distinct eigenvalue, then $G \cong K_1$ and also if G has exactly two distinct eigenvalues, then $G \cong K_n$ (see, [3]). Next, if G has exactly three distinct eigenvalues, then G is a strongly regular graph (see, [3]) with two non-trivial eigenvalues whose absolute values are equal to $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/(n-1)}$. Conversely, one can easily check that the equality in (7) holds for the graphs K_n and the strongly regular graph with two non-trivial eigenvalues whose absolute values are equal to $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/(n-1)}$. □

Employing Theorem 3 for the (edge) Zagreb energy of a connected graph, we obtain the following upper bounds for the (edge) Zagreb energy.

Corollary 4. *Let G be a connected graph on n vertices with m edges and s be the smallest positive element in $Z_1(G)$. Then*

$$Z\mathcal{E}_1(G) \leq t + \sqrt{(n-1)\left(2HM(G) - t^2\right)},$$

where $t = \max\{\sqrt{2HM/n}, 2ms/n\}$. Moreover the equality holds if and only if G is a complete graph or a strongly regular graph with two non-trivial eigenvalues whose absolute values are equal to $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/(n-1)}$.

Corollary 5. *Let G be a connected graph on n vertices with m edges and s be the smallest positive element in $Z_2(G)$. Then*

$$Z\mathcal{E}_2(G) \leq t + \sqrt{(n-1)\left(2R_2(G) - t^2\right)},$$

where $t = \max\{\sqrt{2R_2(G)/n}, 2ms/n\}$*. Moreover the equality holds if and only if* G *is a complete graph or a strongly regular graph with two non-trivial eigenvalues whose absolute values are equal to* $\sqrt{\left(2m - \left(\frac{2m}{n}\right)^2\right)/(n-1)}$.

Lemma 6. [11] *Let* M *be a symmetric matrix of order* n*, and let* M_k *be its leading* $k \times k$ *submatrix. Then* $\theta_{n-k+i}(M) \leq \theta_i(M_k) \leq \theta_i(M)$ *for* $i = 1, 2, \ldots, k$*.*

Lemma 7. [3] *Let* G *be a graph of order* $n \geq 2$*. Then* $n_0(G) = n - 2$ *if and only if* $G \cong K_{p,q} \cup (n-p-q)K_1$*, where* $p + q \leq n$*.*

Theorem 4. *Let* G *be a connected graph on* n *vertices with* m *edges. Suppose* $\mathcal{F}(d_i, d_j) > 0$ *and* $s > 0$*,* l *be the smallest element and largest element in* $T(G)$*, respectively. Then* $\mathcal{E}_T(G) \geq \min\{\mathcal{E}_T^a, \mathcal{E}_T^b\}$*, where* $\mathcal{E}_T^a = s + \sqrt{4 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - 3s^2}$ *and* $\mathcal{E}_T^b = t\lambda_1 + \sqrt{4 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - 3t^2\lambda_1^2}$*. Equality holds if and only if* $G \cong K_3$ *or* $K_{p,q}$*, where* $p + q = n$*.*

Proof. We have

$$\left(\sum_{i=1}^{n-1} |f_i|\right)^2 = \sum_{i=1}^{n-1} f_i^2 + 2\sum_{i<j} |f_i||f_j| \geq \sum_{i=1}^{n-1} f_i^2 + \left|2\sum_{i<j} f_i f_j\right| \text{ (by triangle inequality)} \tag{11}$$

$$= \sum_{i=1}^{n-1} f_i^2 + \left|\sum_{i=1}^{n-1} f_i^2 - f_n^2\right| = 2\sum_{i=1}^{n-1} f_i^2 - f_n^2$$

$$= 2\sum_{i=1}^{n} f_i^2 - 3f_n^2. \tag{12}$$

From (12), we get

$$\mathcal{E}_T(G) \geq |f_n| + \sqrt{2\sum_{i=1}^{n} f_i^2 - 3f_n^2}$$

$$= |f_n| + \sqrt{4 \sum_{v_iv_j \in E(G)} \mathcal{F}^2(d_i, d_j) - 3f_n^2}. \tag{13}$$

By Lemma 6, $f_n \leq \theta_2(M_2)$, where M_2 is a principal submatrix of the matrix T. Thus $f_n \leq -s$. Also, by Lemma 3, we have $|f_n| \leq f_1 \leq l\lambda_1$.

Let $f(t) := t + \sqrt{4\sum_{v_iv_j\in E(G)}\mathcal{F}^2(d_i,d_j) - 3t^2}$. Then f has maximum at $t_0 = \sqrt{(1/3)\sum_{v_iv_j\in E(G)}\mathcal{F}^2(d_i,d_j)}$. Thus if $s \leq |f_n| \leq t_0$, by (13), we get

$$\mathcal{E}_T(G) \geq s + \sqrt{4\sum_{v_iv_j\in E(G)}\mathcal{F}^2(d_i,d_j) - 3s^2}$$

and if $t_0 \leq |f_n| \leq l\lambda_1$, then

$$\mathcal{E}_T(G) \geq l\lambda_1 + \sqrt{4\sum_{v_iv_j\in E(G)}\mathcal{F}^2(d_i,d_j) - 3l^2\lambda_1^2}.$$

Suppose $\mathcal{E}_T(G) = \min\{\mathcal{E}_T^a, \mathcal{E}_T^b\}$. Then the equality in (11) holds and so $f_2 = f_3 = \ldots = f_{n-2} = 0$.
Case 1: If $\mathcal{E}_T(G) = \mathcal{E}_T^a$. Then $|f_n| = s$. Suppose $n = 2, 3$. Then $G \cong K_2$ or K_3. Let $n \geq 4$. Assume that G has P_3 as its induced subgraph then by Lemma 6, $-s\sqrt{2} \geq f_3(P_3) \geq f_n = -s$, a contradiction. Thus P_3 is not an induced subgraph of G. Hence $G \cong K_n$. Therefore, $0 > f_2 = -s$, a contradiction.
Case 2: If $\mathcal{E}_T(G) = \mathcal{E}_T^b$, then $f_1 = |f_n| = l\lambda_1$. Since $f_1 = l\lambda_1$, we must have $T(G) = lA(G)$. Suppose $n = 2, 3$. Then $G \cong K_2$ or $K_{1,2}$. Let $n \geq 4$. Since $T(G) = lA(G)$, $f_1 = |f_n|, f_2 = f_3 = \cdots = f_{n-2} = 0$, we have $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1} = 0$. Hence by Lemma 7, we get $G \cong K_{p,q}$, where $p + q = n$.
Conversely, one can easily check that the equality holds when $G \cong K_3$ or $K_{p,q}$, where $p + q = n$. □

From Theorem 4, we have the following lower bounds for the edge (Zagreb) energy of a connected graph.

Corollary 6. *Let G be a connected graph on n vertices with m edges. Let $s > 0$ and l be the smallest element and largest element in $Z_1(G)$, respectively. Then $Z\mathcal{E}_1(G) \geq \min\{Z\mathcal{E}_1^a, Z\mathcal{E}_1^b\}$, where $Z\mathcal{E}_1^a = s + \sqrt{4HM(G) - 3s^2}$ and $Z\mathcal{E}_1^b = t\lambda_1 + \sqrt{4HM - 3t^2\lambda_1^2}$. Equality holds if and only if $G \cong K_3$ or $K_{p,q}$, where $p + q = n$.*

Corollary 7. *Let G be a connected graph on n vertices with m edges. Let $s > 0$ and l be the smallest element and largest element in $T(G)$, respectively. Then $Z\mathcal{E}_2(G) \geq \min\{Z\mathcal{E}_2^a, Z\mathcal{E}_2^b\}$, where $Z\mathcal{E}_2^a = s + \sqrt{4R_2(G) - 3s^2}$ and $Z\mathcal{E}_2^b = t\lambda_1 + \sqrt{4R_2(G) - 3t^2\lambda_1^2}$. Equality holds if and only if $G \cong K_3$ or $K_{p,q}$, where $p + q = n$.*

In the following theorem, we give a comparison between Zagreb energy and edge-zagreb energy of a graph G with minimum degree $\delta \geq 2$.

Theorem 5. *Let G be a graph on n vertices with minimum degree $\delta \geq 2$. Then*

$$Z\mathcal{E}_2(G) \geq \sqrt{\frac{2}{n}} Z\mathcal{E}_1(G).$$

Proof. We have

$$\begin{aligned} Z\mathcal{E}_2^2(G) = \left(\sum_{i=1}^{n} |\gamma_i|\right)^2 &= \sum_{i=1}^{n} \gamma_i^2 + 2\sum_{i<j} |\gamma_i||\gamma_j| \\ &\geq \sum_{i=1}^{n} \gamma_i^2 + 2|\sum_{i<j} \gamma_i\gamma_j| \quad \text{(by triangle inequality)} \\ &= 2\sum_{i=1}^{n} \gamma_i^2 \left(\text{because } \sum_{i=1}^{n} \gamma_i^2 = -2\sum_{i<j} \gamma_i\gamma_j\right) \\ &= 4 \sum_{v_iv_j \in E(G)} (d_id_j)^2 \\ &\geq 4 \sum_{v_iv_j \in E(G)} (d_i + d_j)^2. \end{aligned} \tag{14}$$

Now, from Cauchy-Schwarz inequality, we get

$$\begin{aligned} Z\mathcal{E}_1^2(G) = \left(\sum_{i=1}^{n} |\eta_i|\right)^2 &\leq n\sum_{i=1}^{n} \eta_i^2 \\ &= 2n \sum_{v_iv_j \in E(G)} (d_i + d_j)^2. \end{aligned} \tag{15}$$

Thus from (14) and (15), we obtain

$$Z\mathcal{E}_2(G) \geq \sqrt{\frac{2}{n}} Z\mathcal{E}_1(G).$$

□

3. A lower and upper bound for the spectral radius of edge -Zagreb matrix

In [4], a novel lower and upper bound for the spectral radius of Zagreb matrix is given in terms of minimum degree δ, maximum degree Δ, order n and size m of the graph. Motivated by this, we give a lower and upper bound for the spectral radius of edge-Zagreb matrix in terms of minimum degree δ, maximum degree Δ, order n and size m of the graph. We need the following lemma to prove our bounds for the spectral radius γ_1 of the edge-Zagreb matrix.

Lemma 8. [11] *Let $A = (a_{i,j})$ be an $n \times n$ irreducible non-negative matrix with spectral radius θ_1 and let $R_i(A) = \sum_{j=1}^{n} a_{ij}$ be the ith row sum of A. Then*

$$\min\{R_i(A) : 1 \leq i \leq n\} \leq \theta_1 \leq \max\{R_i(A) : 1 \leq i \leq n\}.$$

Moreover, if the row sums of A are not all equal, then both of the inequalities are strict.

Theorem 6. *Let G be a graph of order n, m edges with minimum degree δ and maximum degree Δ. Then*

$$\delta\left(2m - (n-1-\delta)\Delta - \delta\right) \leq \gamma_1(G) \leq \Delta\left(2m - (n-1-\Delta)\delta - \Delta\right), \tag{16}$$

with both equalities hold if and only if G is a regular graph.

Proof. Let $G_1, G_2, \ldots, G_k$ be the connected components of G. Let n_i and m_i be the order and size of the component G_i $(1 \leq i \leq k)$. Then $\gamma_1(G) = \max\{\gamma_1(G_1), \gamma_1(G_2), \ldots, \gamma_1(G_k)\}$. Suppose $\gamma_1(G) = \gamma_1(G_l)$. Let δ_l and Δ_l be the minimum degree and maximum degree of a vertex in G_l. Also, let $\overline{d_i}$ be the average degree of the vertices adjacent to the vertex v_i in G.
Lower bound:
Since $d_i\overline{d_i} \geq 2m_l - d_i - (n_l - d_i - 1)\Delta_l$, from Lemma 8 we have

$$\begin{aligned}
\gamma_1(G_l) &\geq \min_{v_i \in V(G_l)} \sum_{v_iv_j \in E(G_l)} d_id_j \\
&= \min_{v_i \in V(G_l)} d_i^2\overline{d_i} \\
&\geq \min_{v_i \in V(G_l)} d_i(2m_l - d_i - (n_l - d_i - 1)\Delta_l) \\
&= \min_{v_i \in V(G_l)} d_i(2m_l - (n_l - 1)\Delta_l + (\Delta_l - 1)d_i) \\
&\geq \delta_l(2m_l - (n_l - 1 - \delta_l)\Delta_l - \delta_l) \\
&\geq \delta_l\left(2m_l - (n_l - 1 - \delta_l)\Delta - \delta_l - \sum_{\substack{i \neq l \\ i=1}}^{k}(n_i\Delta - 2m_i)\right) \\
&= \delta_l\left(2m - (n-1)\Delta + (\Delta - 1)\delta_l\right) \\
&\geq \delta\left(2m - (n-1-\delta)\Delta - \delta\right).
\end{aligned}$$

Suppose the left inequality in (16) holds. Then all the inequalities in the above argument must be equalities. Thus $\delta_l = \delta$, $\Delta_l = \Delta$ and $n_i\Delta = 2m_i$, for $1 \leq i \leq k, i \neq l$. Also, if $d_1, d_2, \ldots, d_l$ are the degrees of the vertices in G_l. Then by Lemma 8, we must have $d_1^2\overline{d_1} = d_2^2\overline{d_2} = \ldots = d_l^2\overline{d_l}$. Note that $\delta_l^2\overline{d}_n \leq \delta_l^2\Delta_l \leq \delta_l\Delta_l^2 \leq \Delta_l^2\overline{d}_1$. Therefore $\delta_l^2\Delta_l = \Delta_l^2\delta_l$. Hence $\Delta_l = \delta_l$. i.e., G is a regular graph.

Upper bound:
Since $d_i\overline{d_i} \leq 2m_l - d_i - (n_l - d_i - 1)\delta_l$, from Lemma 8 we have

$$\begin{aligned}
\gamma_1(G_l) &\leq \max_{v_i \in V(G_l)} \left(\sum_{v_i v_j \in E(G_l)} d_i d_j \right) \\
&\leq \max_{v_i \in V(G_l)} d_i^2 \overline{d_i} \\
&\leq \max_{v_i \in V(G_l)} d_i(2m_l - d_i - (n_l - d_i - 1)\delta_l) \\
&= \max_{v_i \in V(G_l)} d_i(2m_l - (n_l - 1)\delta_l + (\delta_l - 1)d_i) \\
&\leq \Delta_l(2m_l - (n_l - 1 - \Delta_l)\delta_l - \Delta_l) \\
&\leq \Delta_l \left(2m_l - (n_l - 1 - \Delta_l)\delta - \Delta_l + \sum_{\substack{i \neq l \\ i=1}}^{k} (2m_i - n_i\delta) \right) \\
&= \Delta_l \left(2m - (n-1)\delta + (\delta - 1)\Delta_l\right) \\
&\leq \Delta \left(2m - (n - 1 - \Delta)\delta - \Delta\right).
\end{aligned}$$

Similar to the case of lower bound, we can conclude that the right equality in (16) holds only if G is regular. □

4. Some (edge) Zagreb equienergetic graphs

Let G and H be two graphs with vertex set $V(G) = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n_2}\}$, respectively. The direct product of G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_i, v_j) and (u_k, v_l) are adjacent if and only if $u_i u_k \in E(G)$ and $v_j v_l \in E(H)$. For details, see [12]. Let $A = (a_{ij})$ be a $n \times m$ matrix and $B = (b_{ij})$ be a $p \times q$ matrix. Then the Kronecker product $A \otimes B$ of A and B is the $np \times mq$ matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$. Let A and B be square matrices of order n and m, respectively. If the eigenvalues of A are λ_i ; $i = 1, 2, \ldots, n$ and the eigenvalues of B are μ_j $j = 1, 2, \ldots, m$, then the spectrum of $A \otimes B$ consists of the products $\lambda_i \mu_j$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ [3].

Proposition 1. *Let G be a graph of order n_1 and H be a r-regular graph of order n_2. Then $Z\mathcal{E}_1(G \times H) = rZ\mathcal{E}_1(G)\mathcal{E}(H)$.*

Proof. Let $spec(Z_1(G)) = \{\eta_1, \eta_2, \ldots, \eta_{n_1}\}$ and $spec(A(H)) = \{\lambda_1, \lambda_2, \ldots, \lambda_{n_2}\}$. Since H is r-regular graph, from the definition of $G \times H$, we get $Z_1(G \times H) = Z_1(G) \otimes rA(H)$. Therefore, the $spec(Z_1(G \times H))$ consists of $r\eta_i\lambda_j$, for $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_2$. Thus $Z\mathcal{E}_1(G \times H) = rZ\mathcal{E}_1(G)\mathcal{E}(H)$. □

Corollary 8. *Let G be a graph and let H_1 and H_2 be two r-regular equienergetic graphs. Then the graphs $G \times H_1$ and $G \times H_2$ are Zagreb equienergetic.*

The following proposition gives the edge-Zagreb energy of $G \times H$. We omit the details of the proof as it is similar to Proposition 1.

Proposition 2. *Let G and H be graphs. Then $Z\mathcal{E}_2(G \times H) = Z\mathcal{E}_2(G)Z\mathcal{E}_2(H)$.*

Corollary 9. *Let G be a graph and let H_1 and H_2 be edge-Zagreb equienergetic graphs. Then the graphs $G \times H_1$ and $G \times H_2$ are edge-Zagreb equienergetic.*

Let $J_{n_1 \times n_2}$ be a $n_1 \times n_2$ matrix having all its entries equal to 1.

Lemma 9. [1] *For $i = 1, 2$, let M_i be a normal matrix of order n_i having all its row sums equal to r_i. Suppose $r_i, \theta_{i2}, \theta_{i3}, \ldots, \theta_{in_i}$ are the eigenvalues of M_i, then for any two constants a and b, the eigenvalues of*

$$M := \begin{bmatrix} M_1 & aJ_{n_1 \times n_2} \\ bJ_{n_2 \times n_1} & M_2 \end{bmatrix}$$

are θ_{ij} for $i = 1, 2$, $j = 2, 3, \ldots, n_i$ and the two roots of the quadratic equation $(x - r_1)(x - r_2) - abn_1n_2 = 0$.

Theorem 7. *Let G_1 be a r_1-regular graph of order n_1 and let G_2 be a r_2-regular graph of order n_2. Then the spectrum of $Z_1(G_1 \vee G_2)$ consists of $2(r_1+n_2)\lambda_i(G_1)$ and $2(r_2+n_1)\lambda_j(G_2)$ for $i = 2, 3, \ldots, n_1$ and $j = 2, 3, \ldots, n_2$, and the two roots of the quadratic equation $(x - 2(r_1 + n_2)r_1)(x - 2(r_2 + n_1)r_2) - (n_1 + n_2 + r_1 + r_2)^2 n_1n_2$.*

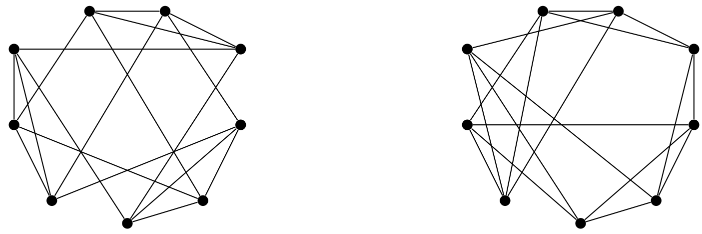
Proof. Since G_1 and G_2 are regular graphs, the Zagreb matrix of $G_1 \vee G_2$ can be obtained as follows:

$$Z_1(G_1 \vee G_2) = \begin{pmatrix} 2(r_1 + n_2)A(G_1) & (n_1 + n_2 + r_1 + r_2)J_{n_1 \times n_2} \\ (n_1 + n_2 + r_1 + r_2)J_{n_2 \times n_1} & 2(r_2 + n_1)A(G_2) \end{pmatrix}.$$

Setting $a = b = (n_1 + n_2 + r_1 + r_2)$ in Lemma 9, we arrive at the desired result. □

Theorem 8. *There exists a pair of Zagreb equienergetic graphs of order $n \geq 9$.*

Proof. Let H_1 and H_2 be graphs as depicted in Figure 1. Then H_1 and H_2 are equienergetic 4- regular graphs of order 9 with energy 16. Employing Theorem 7 for the graphs $K_m \vee H_1$ and $K_m \vee H_2$, we see that $Z\mathcal{E}_1(K_m \vee H_1) = Z\mathcal{E}_1(K_m \vee H_2)$. This completes the proof. □



H_1 H_2

Figure 1. **Equienergetic 4-regular graphs on 9 vertices with energy 16.**

Theorem 9. *Let G_1 be a r_1-regular graph of order n_1 and let G_2 be a r_2-regular graph of order n_2. Then the spectrum of $Z_2(G_1 \vee G_2)$ consists of $(r_1+n_2)^2\lambda_i(G_1)$ and $(r_2+n_1)^2\lambda_j(G_2)$ for $i = 2, 3, \ldots, n_1$ and $j = 2, 3, \ldots, n_2$, and the two roots of the quadratic equation $(x - (r_1+n_2)^2 r_1)(x-(r_2+n_1)^2 r_2) - (n_2+r_1)^2(n_1+r_2)^2 n_1 n_2$.*

Proof. Since G_1 and G_2 are regular graphs, the Zagreb matrix of $G_1 \vee G_2$ can be obtained as follows:

$$Z_2(G_1 \vee G_2) = \begin{pmatrix} (r_1+n_2)^2 A(G_1) & (n_1+r_2)(n_2+r_1)J_{n_1 \times n_2} \\ (n_1+r_2)(n_2+r_1)J_{n_2 \times n_1} & (r_2+n_1)^2 A(G_2) \end{pmatrix}.$$

Setting $a = b = (n_1+r_2)(n_2+r_1)$ in Lemma 9, we arrive at the desired result. □

The proof of the following theorem is similar to that of Theorem 8.

Theorem 10. *There exists a pair of edge-Zagreb equienergetic graphs of order $n \geq 9$.*

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