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Bounds on the outer-independent double Italian domination number

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Abstract: An outer-independent double Italian dominating function (OIDIDF) on a graph G with vertex set $V(G)$ is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ such that if $f(v) \in \{0, 1\}$ for a vertex $v \in V(G)$ then $\sum_{u\in N[v]} f(u) \geq 3$, and the set $\{u \in V(G)|f(u) = 0\}$ is independent. The weight of an OIDIDF f is the value $w(f) = \sum_{v\in V(G)} f(v)$. The minimum weight of an OIDIDF on a graph G is called the outer-independent double Italian domination number $\gamma_{oidI}(G)$ of G. We present sharp lower bounds for the outer-independent double Italian domination number of a tree in terms of diameter, vertex covering number and the order of the tree.

Keywords: Roman domination, outer-independent double Italian domination, tree.

AMS Subject classification: 05C69

1. Introduction

For definitions and notations not given here we refer to [14]. We consider simple connected graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V$ is $\deg(v) = d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$,

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respectively. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex*. A *strong support vertex* is a support vertex adjacent to more than one leaf. We denote the sets of all leaves and all support vertices of G by $L(G)$ and $S(G)$, respectively. The *diameter* of a graph G, denoted by $diam(G)$, is the greatest *distance* between two vertices of G. A subset D of $V(G)$ is a *dominating set* in G if $\bigcup_{v\in D} N[v] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set I of vertices is *independent* if no pair of vertices of I are adjacent. The maximum cardinality of an independent set in G is called the *independent number* $\alpha(G)$ of G. A *vertex cover* of a graph G is a set D of vertices such that each edge of G has at least one end point in D. The minimum cardinality of a vertex cover is denoted by $\beta(G)$. We write P_n for the path of order n, C_n for the cycle of length n, K_n for the complete graph of order n and $K_{p,q}$ for the complete bipartite graph whose partite sets have cardinalities p and q, respectively. For a subset D of vertices in a graph G, we denote by $G[D]$ the subgraph of G induced by D. The *corona* $H \circ K_1$ is the graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex v' and a pendant edge vv' are added. We denote by $S_{a,b}$ a double star in which one center is adjacent to a leaves and the other center is adjacent to b leaves.
Cockayne et al. [10] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (See [1, 2, 4, 6–9, 22]). One of the generalizations of Roman domination, namely Italian domination has been introduced by Chellali et al. in [5], Klostermeyer and MacGillivray [16], and Henning and Klostermeyer [15]. An *Italian dominating function* (IDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex $v \in V(G)$ with $f(v) = 0$ has at least two neighbors assigned 1 under f or one neighbor assigned 2 under f. The *weight* of an IDF f is the value $w(f) = \sum_{v\in V(G)} f(v)$. The minimum weight of an IDF on a graph G is called the *Italian domination number* $\gamma_I(G)$ of G. We note that Italian domination is a generalization of *Roman domination*. Mojdeh and Volkmann [17] considered an extension of Italian domination as follows. For a graph G, a *double Italian dominating function* (DIDF) is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ having the property that for every vertex $u \in V$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. The weight of a DIDF f is the sum $w(f) = f(V) = \sum_{v\in V} f(v)$, and the minimum weight of a DIDF in a graph G is the *double Italian domination number*, denoted by $\gamma_{dI}(G)$. For a DIDF f, one can denote $f = (V_0, V_1, V_2, V_3)$, where $V_i = \{v \in V : f(v) = i\}$, for $i = 0, 1, 2, 3$. This concept was further studied in [3, 11, 13, 19–21].
In this paper we continue the study of double Italian domination in graphs by considering those double Italian dominating functions f such that $\{v \in V(G) \mid f(v) = 0\}$ is an independent set. A DIDF $f = (V_0, V_1, V_2, V_3)$ is called an *outer-independent double Italian dominating function* (OIDIDF) if V_0 is an independent set. The minimum weight of an OIDIDF on a graph G is called the *outer-independent double Italian domination number* of G and is denoted by $\gamma_{oidI}(G)$. The definitions lead to $\gamma_{oidI}(G) \geq \gamma_{dI}(G)$. We establish various bounds on the outer-independent double Italian domination number. In Section 2 we prove some preliminary results as well as several general bounds for the outer-independent double Italian domination number. In Section 3, we establish various lower bounds on the outer-independent

double Italian domination number in a tree in terms of order, diameter and vertex cover number. We also characterize extremal trees achieving equality for the given bounds. We make use of the following.

Theorem 1 ([12, 18]). *For a graph G of even order n and no isolated vertices, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are the cycle C_4 or the corona $H \circ K_1$ for any connected graph H.*

2. Preliminaries and general results

We begin with the following observation.

Observation 2. If $f = (V_0, V_1, V_2, V_3)$ is a γ_{oidI}-function on a graph G, then
(i) each vertex of V_3 (if any), has a private neighbor in V_0.
(ii) $V_1 \cup V_2 \cup V_3$ is both an outer independent dominating set and a vertex cover in G.
(iii) If G is connected, then $\beta(G) \leq \gamma_{oidI}(G) \leq 3\beta(G)$, and if $\delta \geq 2$, then $\gamma_{oidI}(G) \leq 2\beta(G)$.
(iv) If $\delta(G) > 0$, then $\gamma_{oidI}(G) \leq \gamma(G) + n \leq \frac{3n}{2}$, and if $\delta \geq 2$, then $\gamma_{oidI}(G) \leq n$.

Proof. We prove parts (iii) and (iv).
(iii) The inequality $\beta(G) \leq \gamma_{oidI}(G)$ follows from (ii). To prove $\gamma_{oidI}(G) \leq 3\beta(G)$, let S be a maximum independent set in G. Then the function f defined with $f(u) = 0$ if $u \in S$ and $f(u) = 3$ if $u \notin S$ is an OIDIDF on G, since G is connected. Thus $\gamma_{oidI}(G) \leq 3|V(G) - S| = 3(n - \alpha(G)) = 3\beta(G)$. Now assume that $\delta \geq 2$. Let S be a maximum independent set of G. Then the function f defined by $f(u) = 0$ if $u \in S$ and $f(u) = 2$ otherwise, is an OIDIDF on G. So $\gamma_{oidI}(G) \leq w(f) = 2(|V| - |S|) = 2(n - \alpha) = 2\beta(G)$.
(iv) Given a minimum dominating set D of G, the function f defined by $f(u) = 2$ if $u \in D$ and $f(u) = 1$ otherwise, is an OIDIDF on G, implying that $\gamma_{oidI}(G) \leq |D| + n$. Now the result follows by Ore's Theorem. If $\delta \geq 2$, then it is enough to consider a function which assigns 1 to every vertex of the graph. □

Proposition 1. *For any graph G with at least one edge, there exists a $\gamma_{oidI}(G)$-function $f = (V_0, V_1, V_2, V_3)$ such that $V_0 \neq \emptyset$.*

Proof. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(G)$-function. If $V_0 \neq \emptyset$, then we have done. Thus assume that $V_0 = \emptyset$, and by Observation 2 (i), we may assume that $V_3 = \emptyset$. If $V_1 = \emptyset$, then $V(G) = V_2$, and so replacing $f(u)$ by 1 for one non-isolated vertex u yields an OIDIDF on G with the weight less than $w(f)$, a contradiction. Thus, $V_1 \neq \emptyset$. We consider the following two cases.
Case 1. No vertex of V_1 is adjacent to a vertex of V_2. Then each vertex of V_1 is adjacent to at least two other vertices of V_1. If $H = G[V_1]$, then we note that $\delta(H) \geq 2$. Assume that $\delta(H) \geq 3$. If $v \in V_1$, then the function g defined by $g(v) = 0$ and $g(x) = f(x)$ otherwise is an OIDIDF on G of weight less then $w(f)$, a contradiction. If $\delta(H) = 2$,

then let $v \in V_1$ with $d(v) = 2$. If v has a neighbor u of degree at least three, then let $w \neq u$ be the other neighbor of v. Then the function g defined by $g(v) = 0$, $g(w) = 2$ and $g(x) = f(x)$ otherwise is a desired $\gamma_{oidI}(G)$-function. In the remaining case H contains a cycle C as a component. If $C = v_1 v_2 \dots v_{2k} v_1$ is an even cycle, then the function g with $g(v_{2i-1}) = 2$, $g(v_{2i}) = 0$ for $1 \le i \le k$ and $g(x) = f(x)$ otherwise is a desired $\gamma_{oidI}(G)$-function. If $C = v_1 v_2 \dots v_{2k+1} v_1$ is an odd cycle, then the function g with $g(v_{2i-1}) = 2$, $g(v_{2i}) = 0$ for $1 \le i \le k$, $g(v_{2k+1}) = 1$ and $g(x) = f(x)$ otherwise is a desired $\gamma_{oidI}(G)$-function.
Case 2. There is a vertex $v \in V_1$ such that v is adjacent to a vertex $w \in V_2$. If $d(w) = 1$, then the function g defined by $g(v) = 3$, $g(w) = 0$ and $g(x) = f(x)$ otherwise, is the desired function. Let now that $d(w) \ge 2$. Assume that $d(u) \ge 2$ for every $u \in N(w)$. Then the function g defined by $g(w) = 1$ and $g(x) = f(x)$ for $x \neq w$ is an OIDIDF on G of weight less then $w(f)$, a contradiction. Finally assume that there exists a vertex $z \in N(w)$ with $d(z) = 1$. Then the function g defined by $g(z) = 0$, $g(w) = 3$ and $g(x) = f(x)$ otherwise is a desired $\gamma_{oidI}(G)$-function. □

If C_n is a cycle of length n, then it was shown in [17] that $\gamma_{dI}(C_n) = n$. Using this result, the inequality $\gamma_{oidI}(C_n) \ge \gamma_{dI}(C_n)$, and Observation 2 (iv) (or the proof of Case 1 in Proposition 1), we obtain the next Observation.

Observation 3. If C_n is a cycle of length n, then $\gamma_{oidI}(C_n) = n$.

We close this section by giving Nordhaus-Gaddum type inequalities for the outer-independent double Italian number. We first define a family $\mathcal{G}$ of graphs G such that G is obtained from a complete graph K_p, $(p \ge 4)$, an empty graph $\overline{K_s}$, where $s \ge \left\lceil \frac{3p}{p-3} \right\rceil$ and a new vertex u, by joining u to every vertex of K_p and joining each vertex of $\overline{K_s}$ to at least three vertices of K_p such that each vertex of K_p is non-adjacent to at least three vertices of $\overline{K_s}$. It is clear from the construction of G that $G \in \mathcal{G}$ if and only if $\overline{G} \in \mathcal{G}$.

Theorem 4. *Let G be a graph G of order n. Then $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \le 3n + 1$, with equality if and only if $G \in \{K_1, K_2, \overline{K_2}\}$.*

Proof. Clearly, $\gamma_{oidI}(K_1) + \gamma_{oidI}(\overline{K_1}) = 4$ and $\gamma_{oidI}(K_2) + \gamma_{oidI}(\overline{K_2}) = 7$. Let now $n \ge 3$.
If $\delta(G) > 0$ and $\delta(\overline{G}) > 0$, then it follows from Observation 2 (iv) that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \le \frac{3n}{2} + \frac{3n}{2} = 3n < 3n + 1.$$

Now assume that $\delta(G) = 0$ or $\delta(\overline{G}) = 0$, say $\delta(G) = 0$. Let I be the set of isolated vertices of G, and let $H = G - I$. We deduce from Observation 2 (iv) that

$$\gamma_{oidI}(G) \le 2|I| + \frac{3n(H)}{2} = 2|I| + 2n(H) - \frac{n(H)}{2} = 2n - \frac{n(H)}{2}.$$

Since $n \geq 3$ and $\overline{G}$ has a vertex of degree $n-1$, we note that $\gamma_{oidI}(\overline{G}) \leq 2+(n-1) = n+1$. If $n(H) \geq 2$, then the last two inequalities lead to

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n - \frac{n(H)}{2} + (n+1) \leq 3n < 3n+1.$$

Finally, let $n(H) = 0$. Then $G = \overline{K_n}$ and $\overline{G} = K_n$. As $n \geq 3$, we obtain

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n + n = 3n < 3n+1.$$

□

Theorem 5. *Let G be a graph G of order $n \geq 3$. Then $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 3n$, with equality if and only if $G \in \{K_3, \overline{K_3}\}$.*

Proof. If $n = 3$, then it easy to check that $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = 3n = 9$ if and only if $G \in \{K_3, \overline{K_3}\}$. Let now $n \geq 4$.
If $\delta(G) > 0$ and $\delta(\overline{G}) > 0$, then it follows from Observation 2 (iv) that

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq \gamma(G) + n + \gamma(\overline{G}) + n.$$

If G or $\overline{G}$ has a component which is neither the cycle C_4 nor the corona $H \circ K_1$ for any connected graph H, then by Theorem 1, $\gamma(G) < \frac{n}{2}$ or $\gamma(\overline{G}) < \frac{n}{2}$, and thus the last inequality leads to $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq \gamma(G) + n + \gamma(\overline{G}) + n \leq 3n-1$. Next assume that G or $\overline{G}$, say G has a C_4 as a component. Then we deduce from Observation 2 (iv) that $\gamma_{oidI}(G) \leq 4 + \frac{3(n-4)}{2}$ and therefore

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 4 + \frac{3(n-4)}{2} + \frac{3n}{2} = 3n-2.$$

Now assume that G or $\overline{G}$, say G has a corona $Q = H \circ K_1$ as a component. Let $V(H) = \{v_1, v_2, \ldots, v_k\}$. If $k \geq 2$, then the function g with $g(x) = 2$ for $x \in V(Q) \setminus V(H)$, $g(v_i) = 1$ for $1 \leq i \leq k-1$ and $g(v_k) = 0$ is an OIDIDF on Q with weight $\frac{3n(Q)}{2} - 1$. Again Observation 2 (iv) leads to $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 3n-1$. Finally, assume that $G = pK_2$ for an integer $p \geq 2$. Then $\overline{G}$ is the complete graph minus a perfect matching, and since $n \geq 4$, we observe that $\delta(\overline{G}) \geq 2$ and so $\gamma_{oidI}(\overline{G}) \leq n$ by Observation 2 (iv). Hence we obtain

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq \frac{3n}{2} + n \leq 3n-1.$$

Now assume that $\delta(G) = 0$ or $\delta(\overline{G}) = 0$, say $\delta(G) = 0$. Let I be the set of isolated vertices of G, and let $F = G - I$. We deduce from Observation 2 (iv) that

$$\gamma_{oidI}(G) \leq 2|I| + \frac{3n(F)}{2} = 2|I| + 2n(F) - \frac{n(F)}{2} = 2n - \frac{n(F)}{2}.$$

Since $n \geq 4$ and $\overline{G}$ has a vertex of degree $n-1$, we note that $\gamma_{oidI}(\overline{G}) \leq 2+(n-1) = n+1$. If $n(F) \geq 3$, then the last two inequalities lead to

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n - \frac{n(F)}{2} + (n+1) < 3n.$$

If $n(F) = 2$, then $\overline{G}$ is the complete graph minus an edge, and since $n \geq 4$, we observe that $\delta(\overline{G}) \geq 2$ and so $\gamma_{oidI}(\overline{G}) \leq n$. As above, we obtain the desired bound. Finally, let $n(F) = 0$. Then $G = \overline{K_n}$ and $\overline{G} = K_n$. As $n \geq 4$, we obtain

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \leq 2n + n - 1 = 3n - 1.$$

□

Theorem 6. *Let G be a graph of order n. Then*

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \geq n - 1,$$

with equality if and only if $G \in \mathcal{G}$.

Proof. If G or $\overline{G}$ is the empty graph, then clearly $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) > 2n > n-1$. So assume next that G and $\overline{G}$ are graphs with at least on edge. Let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{oidI}(G)$-function with $V_0 \neq \emptyset$ by Proposition 1, and let $f' = (V_0', V_1', V_2', V_3')$ be a $\gamma_{oidI}(\overline{G})$-function. Then

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = |V_1| + 2|V_2| + 3|V_3| + |V_1'| + 2|V_2'| + 3|V_3'|. \tag{1}$$

Since V_0 is an independent set, it forms a clique in $\overline{G}$ and thus $\gamma_{oidI}(\overline{G}) \geq |V_0| - 1$. Analogously, we have $\gamma_{oidI}(G) \geq |V_0'| - 1$. Therefore (1) leads to

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \geq n + |V_2| + 2|V_3| - 1 \geq n - 1 \tag{2}$$

and

$$\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) \geq n + |V_2'| + 2|V_3'| - 1 \geq n - 1. \tag{3}$$

We now prove the equality part. Assume that $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) = n - 1$. It follows from from (2) and (3) that $V_2 \cup V_3 = V_2' \cup V_3' = \emptyset$. Thus $\gamma_{oidI}(G) = |V_1|$ and $\gamma_{oidI}(\overline{G}) = |V_1'|$. Since V_0 and V_0' are independent sets in G and $\overline{G}$, respectively, we have $|V_0 \cap V_0'| \leq 1$. If $V_0 \cap V_0' = \emptyset$, then $|V_1| + |V_0| + |V_1'| + |V_0'| = 2n$, and so $\gamma_{oidI}(G) + \gamma_{oidI}(\overline{G}) + |V_0| + |V_0'| = 2n$. Then $n - 1 + |V_0| + |V_0'| = 2n$ and so

$|V_0| + |V_0'| = n + 1 > n$, a contradiction. Thus $|V_0 \cap V_0'| = 1$. Let $V_0 \cap V_0' = \{u\}$, $I = V_0 - \{u\}$ and $J = V_0' - \{u\}$. Clearly $I \cap J = \emptyset$ and $I \subseteq V_1'$ and $J \subseteq V_1$. Then

$$\begin{aligned} n - 1 &= |V_1| + |V_1'| \\ &\geq |I| + |J| \\ &= |V_0| + |V_0'| - 2 \\ &= n - |V_1| + n - |V_1'| - 2 \\ &= 2n - (|V_1| + |V_1'|) - 2 \\ &= n - 1. \end{aligned}$$

Thus $I = V_1'$ and $J = V_1$. Note that $J \cup \{u\} = V_0'$ is an independent set in $\overline{G}$ and so $G[J] = K_p$ is a complete graph in G. The vertex u is adjacent to all vertices of J in G and I is an independent set in G. Let $|I| = s$. Then $G[I] = \overline{K_s}$. Since f is a $\gamma_{oidI}(G)$-function, each vertex from $\overline{K_s}$ has at least three neighbors in $V_1 = J = V(K_p)$. If a vertex $v \in V(K_p) = J$ is adjacent to $k \geq s - 2$ vertices in $\overline{K_s}$, then v has at most two neighbors in $V(\overline{K_s}) = I = V_1'$. Then $f'(v) \neq 0$ and so $f'(v) = 1$ and therefore $v \in V_1' = I$. This implies that $v \in I \cap J$, a contradiction. We deduce that every vertex in $J = V(K_p)$ has at most $s - 3$ neighbors in $\overline{K_s}$. Now we find the minimum cardinality of I. Note that each vertex in K_p is adjacent to at most $s - 3$ vertices in $\overline{K_s}$. Thus there exist at most $p(s - 3)$ edges between K_p and $\overline{K_s}$. On the other hand each vertex in $\overline{K_s}$ is adjacent to at least 3 vertices in K_p. Then there exist at least $3s$ edges between K_p and $\overline{K_s}$. Therefore $3s \leq p(s-3)$, and thus $s \geq \frac{3p}{p-3}$. Consequently, $G \in \mathcal{G}$.

Conversely, assume that $G \in \mathcal{G}$. Let f and f' be the functions on G and $\overline{G}$ respectively as follows. $f(v) = 1$ if $v \in V(K_p)$ and $f(x) = 0$ otherwise, $f'(v) = 1$ if $v \in \overline{K_s}$ and $f'(x) = 0$ otherwise. Then f and f' are OIDIDF on G and $\overline{G}$ respectively. So $n-1 \leq \gamma_{oidI}(G)+\gamma_{oidI}(\overline{G}) \leq w(f)+w(f') = p+s = n-1$. Then $\gamma_{oidI}(G)+\gamma_{oidI}(\overline{G}) = n - 1$. □

3. Lower bounds for trees

We first determine the outer independent double Italian domination number of paths.

Proposition 2. *For a path P_n, $\gamma_{oidI}(P_3) = 3$ and $\gamma_{oidI}(P_n) = n + 1$ if $n \geq 4$.*

Proof. The proof is straightforward for $n \leq 6$, thus assume that $n \geq 7$. Let $P_n =: v_1v_2 \dots v_n$. For odd n we define a function f by $f(v_{2i-1}) = 2, 1 \leq i \leq \frac{n+1}{2}$ and $f(v_j) = 0$ otherwise, and for even n we define a function f by $f(v_1) = 1, f(v_{2i}) = 2$, $1 \leq i \leq \frac{n}{2}, f(v_j) = 0$ otherwise. Then f is an OIDIDF on P_n, and so $\gamma_{oidI}(P_n) \leq w(f) = n+1$. Now we use an induction proof on n to show that $\gamma_{oidI}(P_n) \geq n + 1$. For the base step, it is easy to see that $\gamma_{oidI}(P_7) = 8$. Assume that for n' with $7 \leq n' < n$, we have $\gamma_{oidI}(P_{n'}) = n' + 1$. Let f be a γ_{oidI}-function for P_n. If $f(v_n) = 0$, then $f(v_{n-1}) = 3$ and $f(v_{n-2}) \leq 1$. If $f(v_{n-2}) = 1$, then we define the OIDIDF g on P_n by $g(v_{n-2}) = 0$,

$g(v_{n-3}) = f(v_{n-3}) + 1$ and $g(x) = f(x)$ otherwise. Using the induction hypothesis, we obtain $(n-3)+1 = \gamma_{oidI}(P_{n-3}) \le w(g) - 3 = w(f) - 3 = \gamma_{oidI}(P_n) - 3$. Thus $\gamma_{oidI}(P_n) \ge n+1$. Thus assume that $f(v_{n-2}) = 0$. Then the induction hypothesis implies that $(n-3)+1 = \gamma_{oidI}(P_{n-3}) \le w(f) - 3 = \gamma_{oidI}(P_n) - 3$. Thus $\gamma_{oidI}(P_n) \ge n - 2 + 3 = n + 1$. If $f(v_n) = 1$, then $f(v_{n-1}) = 2$, and by the induction hypothesis, $(n-1)+1 = \gamma_{oidI}(P_{n-1}) \le w(f) - 1 = \gamma_{oidI}(P_n) - 1$. Thus assume that $f(v_n) = 2$. Then $f(v_{n-1}) \le 1$. If $f(v_{n-1}) = 1$, then we define the function g by $g(v_{n-1}) = 0$ and $g(v_{n-2}) = f(v_{n-2}) + 1$ and $g(x) = f(x)$ otherwise. Using the induction hypothesis, we obtain $(n-2)+1 = \gamma_{oidI}(P_{n-2}) \le w(g) - 2 = w(f) - 2 = \gamma_{oidI}(P_n) - 2$. Thus $\gamma_{oidI}(P_n) \ge n+1$. If $f(v_{n-1}) = 0$, then we again consider P_{n-2}, and as before we obtain that $\gamma_{oidI}(P_n) \ge n+1$. If $f(v_n) = 3$, then by Observation 1 (i) ,$f(v_{n-1}) = 0$. Furthermore, we observe that $f(v_{n-2}) \ge 1$. Now the function g defined by $g(v_n) = 2$ and $g(x) = f(x)$ otherwise is an OIDIDF on P_n of weight less than $w(f)$, a contradiction. □

Lemma 1. *If v is a leaf in a tree T, then $\gamma_{oidI}(T - v) \le \gamma_{oidI}(T)$.*

Proof. Let v be a leaf of a tree T, f a $\gamma_{oidI}(T)$-function and $u \in N(v)$. If $f(v) = 0$ then f is an OIDIDF on $T - v$ and so $\gamma_{oidI}(T-v) \le w(f) = \gamma_{oidI}(T)$. If $f(v) \in \{2,3\}$, then we define a function g by $g(u) = \max\{f(u), f(v)\}$ and $g(x) = f(x)$ if $x \ne u$. Then $g\mid_{V(T)-\{v\}}$ is an OIDIDF on $T - v$ and so $\gamma_{oidI}(T-v) \le w(f) = \gamma_{oidI}(T)$. Thus assume that $f(v) = 1$. This leads to $f(u) = 2$. Now the function $f\mid_{V(T)-\{v\}}$ is an OIDIDF on $T - v$, and so $\gamma_{oidI}(T-v) \le \gamma_{oidI}(T)$. □

Let $\mathcal{T}$ be the family of trees T such that T is a double star $S_{1,b}$, where $b \ge 1$ or T is obtained from a double star $S_{a,b}$,$a \ge 1$ and $b \ge 1$ by subdivision of the central edge of $S_{a,b}$ at least once.

Theorem 7. *If T is a tree of diameter $d \ne 2$, then $\gamma_{oidI}(T) \ge d+2$, with equality if and only if $T \in \mathcal{T}$.*

Proof. If T is a tree of diameter 1 then $\gamma_{oidI}(T) = 3$ and the result is obtained. Now we consider $d \ge 3$. Let P be a diametrical path of T which is a copy of P_{d+1}. By Proposition 2, we have $\gamma_{oidI}(P_{d+1}) = d+2$. Now applying Lemma 1 for finite times yields that $\gamma_{oidI}(T) \ge \gamma_{oidI}(P_{d+1}) = d+2$.
We next prove the equality part. Assume that $\gamma_{oidI}(T) = d+2$. Let f be a $\gamma_{oidI}(T)$-function and let $v_1v_2\dots v_dv_{d+1}$ be a diametrical path in T such that $\sum_{x\in\{v_1,v_2,\dots,v_d,v_{d+1}\}} f(x)$ is maximum. Let $P = T[\{v_1, v_2, \dots, v_d, v_{d+1}\}]$. Note that $P \equiv P_{d+1}$.

Claim 1. $f\mid_{V(P)}$ is an OIDIDF for P.

Proof of Claim 1. Suppose that $f\mid_{V(P)}$ is not an OIDIDF for P. Then there exists at least one vertex $x \in V(P)$ such that $f(x) \in \{0,1\}$ and $f(x) + f(y) + f(z) < 3$, where $N_P(x) = \{y, z\}$. Let X be the set of such vertices of P. Then the function g defined on P by $g(x) = f(x) + 1$ if $x \in X$, and $g(x) = f(x)$ otherwise, is an OIDIDF for P, implying that $\gamma_{oidI}(P) \leq w(g) = w(f|_{V(P)}) + |X|$. Therefore $w(f|_{V(P)}) \geq d+2-|X|$. Also every $x \in X$ is adjacent to a vertex $t \in V(T) - V(P)$ and $\sum_{u\in N[t]} f(u) \geq 2$. Then $\gamma_{oidI}(T) = w(f) \geq w(f\ |_{V(P)}) + 2|X|$. Therefore $w(f\ |_{V(P)}) \leq d + 2 - 2|X|$, and so $2|X| \leq |X|$, a contradiction. Thus, $f\ |_{V(P)}$ is an OIDIDF for P. $\diamondsuit$

From Claim 1, we deduce that $f(v) = 0$ for all vertices $v \in V(T) - V(P)$, and so any vertex outside P is adjacent to a vertex of P. We show that $\deg_T(v_i) = 2$ for $3 \leq i \leq d-1$, and if $deg(v_2) \geq 3$ and $deg(v_d) \geq 3$ then $d \geq 4$.

Assume that $\deg(v_i) \geq 3$ for some $i \in \{3, ..., d-1\}$. Then clearly $f(v_i) = 3$. Next we show that $f(v_{i-1}) = f(v_{i+1}) = 0$. Assume that $f(v_{i-1}) \geq 1$ and $f(v_{i+1}) \geq 1$. Then changing $f(v_i)$ to 2 produce an OIDIDF for P with weight less than $\gamma_{oidI}(P)$, a contradiction. Assume next, without loss of generality, that $f(v_{i-1}) \geq 1$ and $f(v_{i+1}) = 0$. Since V_0 is an independent set, $f(v_{i+2}) \geq 1$ $(5 \leq i+2 \leq d+1)$. Changing $f(v_i)$ to 2 produce an OIDIDF for P with weight less than $\gamma_{oidI}(P)$, a contradiction. Consequently, $f(v_{i-1}) = f(v_{i+1}) = 0$. Since V_0 is an independent set, $f(v_{i-2}) \geq 1$ $(1 \leq i-2 \leq d-3)$ and $f(v_{i+2}) \geq 1$ $(5 \leq i+2 \leq d+1)$. Then changing $f(v_i)$ to 2 produce an OIDIDF for P with weight less than $\gamma_{oidI}(P)$, a contradiction. We conclude that $\deg_T(v_i) = 2$ for $i = 3, ..., d-1$. If $deg(v_2) \geq 3$ and $deg(v_d) \geq 3$ then $f(v_2) = f(v_d) = 3$. If $d \leq 3$, then it can be seen that $diam(T) = 3$, $\gamma_{oidI}(T) = 6$ and $\gamma_{oidI}(T) \neq d+2$, a contradiction. We deduce that $T \in \mathcal{T}$.

Conversely, assume that $T \in \mathcal{T}$. If $T = S_{1,b}$, where $b \geq 1$, then it is easy to see that $\gamma_{oidI}(T) = 5$ and $diam(T) = 3$ and the result follows. Thus assume that T is obtained from a double star $S_{a,b}$,$a \geq 1$ and $b \geq 1$ by subdivision of the central edge uv of $S_{a,b}$ $k \geq 1$ times, and let x_1, ..., x_k be the new vertices which are obtained by subdivision of uv, where u is adjacent to x_1 and v is adjacent to x_k. It is sufficient to present an OIDIDF of weight $d+2$. If $k = 1$, then $d = 4$ and the function f defined by $f(u) = f(v) = 3$ and $f(x) = 0$ otherwise is an OIDIDF for T of weight 6, as desired. If $k = 2$, then $d = 5$ and the function f defined by $f(u) = f(v) = 3$, $f(x_2) = 1$ and $f(x) = 0$ otherwise is an OIDIDF for T of weight 7, as desired. Thus assume that $k \geq 3$. Clearly, $d = k+3$. If k is odd, then the function f defined by $f(u) = f(v) = 3$, $f(x_1) = f(x_k) = 0$ and $f(x_{2i}) = 2$ and $f(x_{2i+1}) = 0$,$1 \leq i \leq \frac{k-1}{2}$ is an OIDIDF for T of weight $k+5$, as desired. If k is even, then the function f defined by $f(u) = f(v) = 3$, $f(x_1) = 1$, $f(x_{2i+1}) = 2$,$1 \leq i \leq \frac{k-2}{2}$ and $f(x) = 0$ otherwise is an OIDIDF for T of weight $k+5$, as desired. $\square$

Theorem 8. *Let T be a tree of order $n \geq 2$. Then $\gamma_{oidI}(T) \geq 2\beta(T) + 1$, and this bound is sharp.*

Proof. We use an induction method on the order $n = |V(T)|$. The base step is easy to see for $n \leq 4$. Thus assume that $n \geq 5$. Assume that $\gamma_{oidI}(T') \geq 2\beta(T')+1$ for any tree T' of order n' with $4 \leq n' < n$. Now consider the tree T. If $diam(T) = 2$, then T is a star and so $\gamma_{oidI}(T) = 3 \geq 2\beta(T) + 1$. If $diam(T) = 3$, then T is a double star in which $\gamma_{oidI}(T) \in \{5, 6\}$ and it can be seen that $\gamma_{oidI}(T) \geq 2\beta(T) + 1$. Thus assume that $diam(T) \geq 4$. Clearly $n \geq 5$. If T has a strong support vertex u, and v is a leaf adjacent to u, then we consider the tree $T' = T - v$. It can be seen that $\beta(T') = \beta(T)$ and by Lemma 1, $\gamma_{oidI}(T') \leq \gamma_{oidI}(T)$. According to the induction hypothesis, we obtain $2\beta(T) + 1 = 2\beta(T') + 1 \leq \gamma_{oidI}(T') \leq \gamma_{oidI}(T)$. Thus assume that T does not have a strong support vertex. We consider a diametrical path of T. Let r and v be two leaves with $d(r, v) = diam(T)$. We root T at r. Let w be the parent of v and x be the parent of w. Since w is not a strong support vertex, $\deg(w) = 2$. Let f be a $\gamma_{oidI}(T)$-function. There are the following two cases depending to the value of $f(w)$.

Case 1. $f(w) \geq 1$. Then $f(w) + f(v) = 3$, and we may assume that $f(w) = 3$ and $f(v) = 0$. If $f(x) \geq 2$, then replacing $f(w)$ by 0 and $f(v)$ by 2 yields an OIDIDF on T with weight less than $w(f)$, a contradiction. Thus $f(x) \leq 1$.
Assume that $f(x) = 1$. Let g be a function defined by $g(x) = g(v) = 2$, $g(w) = 0$ and $g(u) = f(u)$ otherwise. Then $g' = g \mid_{V(T')}$ is an OIDIDF for $T' = T - \{v, w\}$. Also $\beta(T') = \beta(T) - 1$. By the induction hypothesis, $2\beta(T') + 1 \leq \gamma_{oidI}(T') \leq w(g') = w(f) - 2 = \gamma_{oidI}(T) - 2$. Then $2(\beta(T) - 1) + 1 \leq \gamma_{oidI}(T) - 2$, and the result follows.
Next assume that $f(x) = 0$. Clearly $f(t) \geq 1$ for every $t \in N(x)$. Assume that $deg(x) \geq 3$. Let g be a function defined by $g(x) = 1$, $g(v) = 2$, $g(w) = 0$ and $g(u) = f(u)$ otherwise. Then $g' = g \mid_{V(T')}$ is an OIDIDF for $T' = T - \{v, w\}$, and as before we obtain the result. Thus assume that $\deg(x) = 2$. Let y be the father of x. Since $f(x) = 0$, we have $f(y) \geq 1$. If $f(y) \geq 2$, then $g \mid_{V(T')}$, where g is a function defined by $g(x) = 1$, $g(v) = 2$, $g(w) = 0$ and $g(u) = f(u)$ otherwise, is an OIDIDF for $T - \{v, w\}$, and as before we obtain the result. Thus assume that $f(y) = 1$.
Assume that $\deg(y) = 2$. Let z be the father of y. Then $f(z) \geq 2$, and $g \mid_{V(T')}$, where g is defined by $g(y) = g(w) = 0$, $g(x) = g(v) = 2$ and $g(u) = f(u)$ otherwise, is an OIDIDF for $T - \{v, w\}$, and as before we obtain the result. Thus assume that $\deg(y) \geq 3$. Assume that y is a support vertex and y' is the leaf adjacent to y. Clearly $f(y') = 2$. Then the function g defined by $g(y) = 3, g(y') = 0$ and $g(u) = f(u)$ otherwise, is an OIDIDF on T with $w(g) = w(f)$. Then $g' = g \mid_{V(T')}$ is an OIDIDF for $T' = T - \{v, w, x\}$. Also $\beta(T') = \beta(T) - 1$. The induction hypothesis implies that $2\beta(T) - 1 = 2(\beta(T) - 1) + 1 = 2\beta(T') + 1 \leq \gamma_{oidI}(T') \leq w(g') = w(f) - 3 = \gamma_{oidI}(T) - 3$ and therefore $\gamma_{oidI}(T) \geq 2\beta(T) + 2 \geq 2\beta(T) + 1$. Thus assume that y is not a support vertex. Let y' be a child of y different from x. Clearly $\deg(y') \geq 2$. If y' has a child y'' which y'' is a support vertex and y''' is the child of y'', then y''' plays the role of v in the diametrical path, and so we may assume that $\deg(y') = \deg(y'') = 2$, $f(y') = f(y''') = 0$ and $f(y'') = 3$. Let g be a function defined by $g(y) = 3$, $g(u) = f(u)$ if $u \neq y$, $T_1 = T - \{v, w, y'', y'''\}$ and $g_1 = g \mid_{V(T_1)}$. Then $w(g_1) = w(f) - 4$ and we note that $\beta(T_1) = \beta(T) - 2$. So by the induction hypothesis $2\beta(T_1) + 1 \leq \gamma_{oidI}(T_1) \leq w(g_1) = w(f) - 4 = \gamma_{oidI}(T) - 4$. Then $2\beta(T) + 1 \leq \gamma_{oidI}(T)$ as desired. Thus assume that y' is a support vertex. Let y'' be the child of y'. Clearly $\deg(y') = 2$. Assume that $f(y') = 0$. Then $f(y'') = 2$. Let $T'' = T - \{y', y''\}$. Since every vertex cover contains y' or y'', $\beta(T'') = \beta(T) - 1$. Also $f \mid_{V(T'')}$ is an OIDIDF on T''. So $\gamma_{oidI}(T'') \leq w(f \mid_{V(T'')}) = \gamma_{oidI}(T) - 2$. By the induction hypothesis $\gamma_{oidI}(T'') \geq 2\beta(T'') + 1 = 2\beta(T) - 1$ and so the result follows. Thus

assume that $f(y') \geq 1$. Note that $f(y') + f(y'') = 3$. Then the function g defined by $g(y'') = g(y) = 2$, $g(y') = 0$ and $g(u) = f(u)$ otherwise is an OIDIDF on T with $w(g) = w(f)$. Now letting $T'' = T - \{y', y''\}$, $g\mid_{V(T'')}$ is an OIDIDF on T'', and as before the result follows.

Case 2. $f(w) = 0$. Then $f(v) = 2$. Then $g\mid_{V(T')}$, where $T' = T - \{v, w\}$, is an OIDIDF for T' and as before we obtain the result.
To see the sharpness, consider a star or a path P_n with even n. □

Theorem 9. *If T is a tree of order $n \geq 2$ with ℓ leaves, then $\gamma_{oidI}(T) \geq \frac{n+5-\ell(T)}{2}$, with equality if and only if T is a star of order at least three.*

Proof. For the inequality part we use an induction proof on the order. For the base step of the induction, if $n \leq 3$ then $\ell = 2$ and $\gamma_{oidI}(T) = 3$ and so the result follows. Thus assume that $n \geq 4$. Assume that $\gamma_{oidI}(T') \geq \frac{n'+5-\ell(T')}{2}$ for every tree T' of order n' with $3 \leq n' < n$, and T is a tree of order n. If $diam(T) = 2$, then T is a star with $\gamma_{oidI}(T) = 3$ and $\ell(T) = n-1$, and so $3 \geq \frac{n+5-(n-1)}{2} = 3$. If $diam(T) = 3$, then T is a double star with $\gamma_{oidI}(T) \in \{5, 6\}$ and $\ell(T) = n - 2$. The it can be seen that $\gamma_{oidI}(T) \geq \frac{n+5-(n-2)}{2}$. Thus we assume that $diam(T) \geq 4$.
Assume that T has a strong support vertex u, and let v be a leaf adjacent to u. Then it follows from Lemma 1 and the induction hypothesis that

$$\gamma_{oidI}(T) \geq \gamma_{oidI}(T - v) \geq \frac{n - 1 + 5 - (\ell(T) - 1)}{2} = \frac{n + 5 - \ell(T)}{2}.$$

Thus assume that T does not have a strong support vertex.
Let $v_1 v_2 \ldots v_k$ be a diametrical path in T, where v_1 and v_k are leaves and $k \geq 5$. Since T has no strong support vertex, we find that $\deg(v_2) = \deg(v_{k-1}) = 2$. Let f be a $\gamma_{oidI}(T)$-function.
If $f(v_2) = 2$, then $f(v_1) = 1$. Let $T' = T - v_1$ and $f' = f\mid_{V(T')}$. Then $\gamma_{oidI}(T') \leq w(f') = w(f) - 1 = \gamma_{oidI}(T) - 1$. By the induction hypothesis, $\frac{n'+5-\ell(T')}{2} \leq \gamma_{oidI}(T') \leq \gamma_{oidI}(T) - 1$. Since $\ell(T') = \ell(T)$, we have $\frac{n-1+5-\ell(T)}{2} \leq \gamma_{oidI}(T) - 1$. Thus $\frac{n+5-\ell(T)}{2} + \frac{1}{2} \leq \gamma_{oidI}(T)$, and therefore the result follows.
If $f(v_2) = 1$, then $f(v_1) = 2$. Then replace $f(v_2)$ by 2 and $f(v_1)$ by 1, and we obtain the desired bound as before.
Next assume that $f(v_2) = 0$. Then $f(v_1) = 2$ and $f(v_3) \geq 1$. Let $T'' = T - \{v_1, v_2\}$ and $f'' = f\mid_{V(T'')}$. Then f'' is an OIDIDF for T''. Note that $\ell(T'') \leq \ell(T)$. By the induction hypothesis, $\frac{n''+5-\ell(T'')}{2} \leq \gamma_{oidI}(T'') \leq w(f'') = w(f) - 2 = \gamma_{oidI}(T) - 2$. Then $\frac{n-2+5-\ell(T)}{2} \leq \gamma_{oidI}(T) - 2$. Thus $\frac{n+5-\ell(T)}{2} + 1 \leq \gamma_{oidI}(T)$, and the result follows.
It remains to assume that $f(v_2) = 3$. Then $f(v_1) = 0$. If $f(v_3) \geq 2$, then we consider T'' and f'' as in the previous case, and obtain the result. If $f(v_3) = 1$ then we define the function g with $g(v_3) = 2$ and $g(u) = f(u)$ otherwise. Let $T'' = T - \{v_1, v_2\}$ and $g'' = g\mid_{V(T'')}$. Then $w(g'') = w(f) - 2$, and as before, we obtain the result. Thus assume that $f(v_3) = 0$. We define the function g with $g(v_3) = 2$ and $g(u) = f(u)$

otherwise. Let $T'' = T - \{v_1, v_2\}$ and $g'' = g\mid_{V(T'')}$. Then $w(g'') = w(f) - 1$ and as before, we obtain the result.

We now prove the equality part. Clearly for any star of order $n \geq 3$ the equality holds. To show the other side, let T be a tree of order $n \geq 2$ with ℓ leaves and $\gamma_{oidI}(T) = \frac{n+5-\ell(T)}{2}$. We use an induction on n to show that T is a star. The base step it obvious for $n \in \{2, 3\}$. Thus assume that $n \geq 4$. Assume that every tree T' of order n' with $3 \leq n' < n$ and $\gamma_{oidI}(T') = \frac{n'+5-\ell(T')}{2}$ is a star. Let T be a tree of order $n \geq 4$ and $\gamma_{oidI}(T) = \frac{n+5-\ell(T)}{2}$. Let u be a vertex of T with $\deg(u) = \Delta(T)$ and f be a $\gamma_{oidI}(T)$-function. According to Proposition 2, $\Delta \geq 3$.

Claim 1. u is a support vertex.

Proof of Claim 1. Suppose that u is not a support vertex. Then T is not a star. We first show that every support vertex of T has degree 2. Assume that T has a support vertex w with $\deg(w) \geq 3$, and v is a leaf adjacent to w. Let $T' = T - v$. Then clearly T' is not a star. Since T' is not a star, it follows from the induction hypothesis that $\gamma_{oidI}(T') \neq \frac{n'+5-\ell(T')}{2}$. By the first part of the theorem, $\gamma_{oidI}(T') > \frac{n'+5-\ell(T')}{2} = \frac{n-1+5-\ell(T)+1}{2} = \frac{n+5-\ell(T)}{2} = \gamma_{oidI}(T)$. Thus, $\gamma_{oidI}(T') > \gamma_{oidI}(T)$, a contradiction to Lemma 1. Thus assume that every support vertex of T has degree 2. Let w be a support vertex, v be a leaf adjacent to w, and $x \in N(w) - \{v\}$. Let f be a $\gamma_{oidI}(T)$-function, $T' = T - v$. Since $\ell(T') = \ell(T)$ we obtain from the first part of the theorem and Lemma 1 that

$$\gamma_{oidI}(T') \leq \gamma_{oidI}(T) = \frac{n' + 1 + 5 - \ell(T')}{2} = \frac{n' + 5 - \ell(T')}{2} + \frac{1}{2} \leq \gamma_{oidI}(T') + \frac{1}{2}.$$

Thus we obtain that $\gamma_{oidI}(T') = \gamma_{oidI}(T)$.

Suppose that $f(w) \geq 1$. If $f(x) = 0$, then $f(t) \geq 1$ for every vertex $t \in N(x)$, since f is an OIDIDF. Then we change $f(w)$ and $f(v)$, if necessary, to $f(w) = 2$ and $f(v) = 1$. Then $f' = f\mid_{V(T')}$ is an OIDIDF on T' with $w(f') = w(f) - 1$. Then $\gamma_{oidI}(T') \leq w(f') = w(f) - 1 = \gamma_{oidI}(T) - 1 < \gamma_{oidI}(T)$, a contradiction with $\gamma_{oidI}(T') = \gamma_{oidI}(T)$. Thus $f(x) \geq 1$. If $f(w) + f(v) \geq 3$, we can change $f(w)$ and $f(v)$, if necessary, to $f(w) = 2$ and $f(v) = 1$ and as before we get a contradiction. Thus assume that $f(w) = 0$. Then $f(v) = 2$. We note that $f(x) \geq 1$. Let $T'' = T - \{v, w\}$. Then $\gamma_{oidI}(T'') \leq w(f\mid_{V(T'')}) = w(f) - 2 = \gamma_{oidI}(T) - 2$, and so $\gamma_{oidI}(T'') < \gamma_{oidI}(T)$. Assume that $\deg(x) \geq 3$. Then $\ell(T'') = \ell(T) - 1$. Now we have

$$\gamma_{oidI}(T'') \leq \gamma_{oidI}(T) = \frac{n'' + 2 + 5 - \ell(T'') - 1}{2} = \frac{n'' + 5 - \ell(T'')}{2} + \frac{1}{2} \leq \gamma_{oidI}(T'') + \frac{1}{2}.$$

This implies that $\gamma_{oidI}(T'') = \gamma_{oidI}(T)$, a contradiction with $\gamma_{oidI}(T'') < \gamma_{oidI}(T)$. Thus $\deg(x) = 2$ and so $\ell(T'') = \ell(T)$. Note that T'' is not a star, since u is not adjacent to a leaf. Thus by the contrapositive direction of the induction hypothesis we have $\gamma_{oidI}(T'') \neq \frac{n''+5-\ell(T'')}{2}$. We deduce that

$$\begin{aligned}\gamma_{oidI}(T'') &\leq \gamma_{oidI}(T)\\ &= \frac{n+5-\ell(T)}{2}\\ &= \frac{n''+2+5-\ell(T'')}{2}\\ &= \frac{n''+5-\ell(T'')}{2}+1\\ &< \gamma_{oidI}(T'')+1.\end{aligned}$$

Then we obtain that $\gamma_{oidI}(T'') = \gamma_{oidI}(T)$, a contradiction with $\gamma_{oidI}(T'') < \gamma_{oidI}(T)$. This completes the proof of Claim 1. $\diamondsuit$

Thus u is a support vertex. Let v be a leaf adjacent to u. Let $T' = T - v$. According to Lemma 1 and the hypothesis,

$$\gamma_{oidI}(T') \leq \gamma_{oidI}(T) = \frac{n+5-\ell(T)}{2} = \frac{n'+1+5-(\ell(T')+1)}{2} = \frac{n'+5-\ell(T')}{2}.$$

By the first part of the theorem, $\gamma_{oidI}(T') \geq \frac{n'+5-\ell(T')}{2}$. Thus, $\gamma_{oidI}(T') = \frac{n'+5-\ell(T')}{2}$. By the induction hypothesis T' is a star. So u is the center of T', and consequently T is a star. $\square$

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