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A generalized form of the Hermite-Hadamard-Fejer type inequalities involving fractional integral for co-ordinated convex functions

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Abstract: Recently, a general class of the Hermit–Hadamard-Fejer inequality on convex functions is studied in [H. Budak, March 2019, 74:29, *Results in Mathematics*]. In this paper, we establish a generalization of Hermit–Hadamard–Fejer inequality for fractional integral based on co-ordinated convex functions. Our results generalize and improve several inequalities obtained in earlier studies.

Keywords: Weighted Hermite–Hadamard inequality, fractional integral, convex function

AMS Subject classification: 26D07, 26D10, 26D15, 26A33

1. Introduction

Hermite-Hadamard inequality is an important tool in convex analysis, nonlinear analysis and probability theory [1, 2, 4].

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well-known in the literature as Hermite–Hadamard's inequality.

Fejer proposed the following inequality which is called *the weighted generalization of Hermite-Hadamard inequality* or *the Hermite-Hadamard-Fejer inequality, see [3, 8, 10].*

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{2}$$

holds, where $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{(a+b)}{2}$.

Many studies in the field of fractional calculus [6] have been done in the present and last centuries. Recently, many researchers have been investigated the validity of (1) and (2) for convex functions via fractional integral [7, 10]. For example, in 2013, the Hermite-Hadamard type inequality (1) for fractional integral was proved by Sarikaya *et al.* [7]. Recently, a fractional version of the Hermite-Hadamard inequality (1) for co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$ was obtained by Sarikaya [9]. One thing that seems missing was the developments of the Hermite-Hadamard-Fejer inequality for co-ordinated convex functions in fractional cases. The Hermite-Hadamard type inequality utilizing co-ordinated convex functions for fractional integrals was proved in[11]
The aim of this paper is to obtain new generalizations of Hermite-Hadamard-Fejer type inequalities for co-ordinated convex functions involving fractional integral in more general forms, thus generalizing the previous results. As a special case, our results include the corresponding results of Sarikaya [7, 9, 11].
The paper is organized as follows: Section 2 recalls basic definitions and preliminaries while Section 3 presents our main results. We begin with the definitions and some basic preliminaries [5, 6, 9].

2. Basic tools

Definition 1. The function $f : [a,\, b] \longrightarrow \mathbb{R}$ is said to be convex if the following inequality holds,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Let us now consider a bidemensional interval $\Omega =: [a,\, b] \times [c,\, d] \in \mathbb{R}^2$ with $a < b$ and $c < d$.

Definition 2. [5] A function $f : \Omega \longrightarrow \mathbb{R}$ is said to be co-ordinated convex on Ω, for all $\lambda, \gamma \in [0,1]$ and $(x,y),\ (z,t) \in \Omega$, if the following inequality holds:

$$\begin{aligned} f(\lambda x + (1-\lambda)z, \gamma y + (1-\gamma)t) \ &\leq\ \lambda\gamma f(x,y) + \gamma(1-\lambda)f(z,y) \\ &+\ \lambda(1-\gamma)f(x,t) + (1-\lambda)(1-\gamma)f(z,t). \end{aligned}$$

Definition 3. Let $f \in L_1[a,b]$. The Riemann–Liouville fractional integral $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \leq 0$ are defined by $J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,\ x > a$ and $J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt,\ x < b$, respectively, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

Definition 4. Let $f \in L_1(\Omega)$. The Riemann–Liouville fractional integral $J^{\alpha,\beta}_{a+,c+}$, $J^{\alpha,\beta}_{a+,d-}$, $J^{\alpha,\beta}_{b-,c+}$ and $J^{\alpha,\beta}_{b-,d-}$ of order $\alpha, \beta > 0$ with $a, c > 0$ are defined by

$$\begin{aligned} J^{\alpha,\beta}_{a+,c+}f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt,\ x>a,\ y>c,\\ J^{\alpha,\beta}_{a+,d-}f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\ x>a,\ y<d,\\ J^{\alpha,\beta}_{b-,c+}f(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}f(t,s)dsdt,\ x<b,\ y>c, \end{aligned}$$

and

$$J^{\alpha,\beta}_{b-,d-}f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}f(t,s)dsdt,\ x<b,\ y<d,$$

respectively. Note that,

$$J^{0,0}_{a+,c+}f(x,y) = J^{0,0}_{a+,d-}f(x,y) = J^{0,0}_{b-,c+}f(x,y) = J^{0,0}_{b-,d-}f(x,y) = f(x,y),$$

and

$$J^{1,1}_{a+,c+}f(x,y) = \int_a^x\int_c^y f(t,s)dsdt.$$

3. Main results

This section includes the Hermit–Hadamard–Fejer type inequalities for co-ordinated convex functions via fractional integral.

Theorem 3. *Let $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be co-ordinated convex on $\Omega = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b, 0 \le c < d$ and $f \in L_1(\Omega)$. If $g : \Omega \longrightarrow \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$, then for all $\lambda, \gamma \in [0,1]$ and $\alpha, \beta > 0$ one has the inequalities:*

$$\begin{aligned} & f\left(\frac{\lambda b+(2-\lambda)a}{2}, \frac{\gamma d+(2-\gamma)c}{2}\right) J^{\alpha,\beta}_{b-,d-}(g)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d)\\ \le\ & \frac{1}{4}[J^{\alpha,\beta}_{a+,c+}(fg)(a+\lambda(b-a),\ c+\gamma(d-c)) + J^{\alpha,\beta}_{a+,d-}(fg)(a+\lambda(b-a),\ \gamma c+(1-\gamma)d)\\ +\ & J^{\alpha,\beta}_{b-,c+}(fg)(\lambda a+(1-\lambda)b,\ c+\gamma(d-c)) + J^{\alpha,\beta}_{b-,d-}(fg)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d)]\\ \le\ & J^{\alpha,\beta}_{b-,d-}(g)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d)\frac{f(a,c)+f(b,c)+f(a,d)+f(b,d)}{4}. \end{aligned} \tag{3}$$

If $\lambda = \gamma = 1$ and $g(x) = 1$ in Theorem 3, then we have the following Hermite–Hadamard type inequalities for Riemann–Liouville fractional integrals on the co–ordinates [9].

Corollary 1. *[9] Let $f:\Omega\subset\mathbb{R}^2\longrightarrow\mathbb{R}$ be co-ordinated convex on $\Omega=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\le a<b, 0\le c<d$ and $f\in L_1(\Omega)$. Then one has the inequalities:*

$$\begin{aligned} f\left(\frac{a+b}{2},\ \frac{c+d}{2}\right) &\le \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(c-d)^{\beta}}\times \\ &\quad \left\{J^{\alpha,\,\beta}_{a+,\,c+}f(a,\,b)+J^{\alpha,\,\beta}_{a+,\,d-}f(b,\,c)+J^{\alpha,\,\beta}_{b-,\,c+}f(a,\,d)+J^{\alpha,\,\beta}_{b-,\,d-}f(a,\,c)\right\} \\ &\le \frac{f(a,\,c)+f(b,\,c)+f(a,\,d)+f(b,\,d)}{4},\quad \alpha,\,\beta>0. \end{aligned} \tag{4}$$

Theorem 4. *Suppose that $f:\Omega=[a,b]\times[c,d]\subset\mathbb{R}^2\longrightarrow\mathbb{R}$ is co-ordinates convex on Ω and $f\in L_1(\Omega)$. If $g:\Omega\longrightarrow\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$, then for all $\lambda,\ \gamma\in[0,\ 1]$ and $\alpha,\ \beta>0$ one has the inequalities:*

$$\begin{aligned} & J^{\alpha,\beta}_{a+,[d-\gamma(d-c)]+}f\left(b,\ \frac{\gamma d+(2-\gamma)c}{2}\right)g(b,d) \\ +\ & J^{\alpha,\beta}_{a+,d-}f\left(b,\ \frac{\gamma d+(2-\gamma)c}{2}\right)g(b,d-\gamma(d-c)) \\ +\ & J^{\alpha,\beta}_{b-,[d-\gamma(d-c)]+}f\left(a,\ \frac{\gamma d+(2-\gamma)c}{2}\right)g(a,d-\gamma(d-c)) \\ +\ & J^{\alpha,\beta}_{b-,d-}f\left(a,\ \frac{\gamma d+(2-\gamma)c}{2}\right)g(a,d-\gamma(d-c)) \\ +\ & J^{\beta,\alpha}_{c+,[b-\lambda(b-a)]+}f\left(\frac{\lambda b+(2-\lambda)a}{2},d\right)g(b,d) \\ +\ & J^{\beta,\alpha}_{c+,b-}f\left(\frac{\lambda b+(2-\lambda)a}{2},d\right)g(b-\lambda(b-a),d) \\ +\ & J^{\beta,\alpha}_{d-,[b-\lambda(b-a)]+}f\left(\frac{\lambda b+(2-\lambda)a}{2},c\right)g(b,c)+J^{\beta,\alpha}_{d-,b-}f\left(\frac{\lambda b+(2-\lambda)a}{2},c\right)g(b,c) \\ \le\ & J^{\alpha,\beta}_{a+,c+}(fg)(b,c+\gamma(d-c))+J^{\alpha,\beta}_{a+,d-}(fg)(b,d-\gamma(d-c)) \\ +\ & J^{\alpha,\beta}_{b-,c+}(fg)(a,c+\gamma(d-c))+J^{\alpha,\beta}_{b-,d-}(fg)(a,d-\gamma(d-c)) \\ +\ & J^{\beta,\alpha}_{c+,a+}(fg)(a+\lambda(b-a),d)+J^{\beta,\alpha}_{c+,b-}(fg)(d,b-\lambda(b-a)) \\ +\ & J^{\beta,\alpha}_{d-,a+}(fg)(a+\lambda(b-a),c)+J^{\beta,\alpha}_{d-,b-}(fg)(b-\lambda(b-a),c) \\ \le\ & \frac{1}{2}\left\{J^{\alpha,\beta}_{a+,c+}[f(b,c)+f(b,d)]g(b,c+\gamma(d-c))\right\} \\ +\ & \frac{1}{2}\left\{J^{\alpha,\beta}_{a+,d-}[f(b,c)+f(b,d)]g(b,d-\gamma(d-c))\right\} \\ +\ & \frac{1}{2}\left\{J^{\alpha,\beta}_{b-,c+}[f(a,c)+f(a,d)]g(a,c+\gamma(d-c))\right\} \\ +\ & \frac{1}{2}\left\{J^{\alpha,\beta}_{b-,d-}[f(a,c)+f(a,d)]g(a,d-\gamma(d-c))\right\} \\ +\ & \frac{1}{2}\left\{J^{\beta,\alpha}_{c+,a+}[f(a,d)+f(b,d)](fg)(a+\lambda(b-a),d)\right\} \\ +\ & \frac{1}{2}\left\{J^{\beta,\alpha}_{c+,b-}[f(a,d)+f(b,d)](fg)(b-\lambda(b-a),d)\right\} \\ +\ & \frac{1}{2}\left\{J^{\beta,\alpha}_{d-,a+}[f(a,c)+f(b,c)](fg)(a+\lambda(b-a),c)\right\} \\ +\ & \frac{1}{2}\left\{J^{\beta,\alpha}_{d-,b-}[f(a,c)+f(b,c)](fg)(b-\lambda(b-a),c)\right\}. \end{aligned}$$

If $\lambda=\gamma=1$ and $g(x)=1$ in Theorem 4, then the following inequalities hold for Riemann–Liouville fractional integrals on the co–ordinates [9].

Corollary 2. *[9] Let $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ be co-ordinated convex function on $\Omega := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $0 \leq a < b, 0 \leq c < d$ and $f \in L_1(\Omega)$. Then one has the inequalities:*

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}\left[J^\alpha_{a+} f\left(b,\, \frac{d+c}{2}\right) + J^\alpha_{b-} f\left(a,\, \frac{c+d}{2}\right)\right] \\
+ & \frac{\Gamma(\beta+1)}{4(d-c)^\beta}\left[J^\beta_{c+} f\left(\frac{a+b}{2},\, d\right) + J^\beta_{d-} f\left(\frac{a+b}{2},\, c\right)\right] \\
\leq & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha(d-c)^\beta}[J^{\alpha,\,\beta}_{a+,\,c+} f(a,\, b) + J^{\alpha,\,\beta}_{a+,\,d-} f(b,\, c) + J^{\alpha,\,\beta}_{b-,\,c+} f(a,\, d) + J^{\alpha,\,\beta}_{b-,\,d-} f(a,\, c)] \\
\leq & \frac{\Gamma(\alpha+1)}{4(b-a)^\alpha}[J^\alpha_{a+} f(b,\, c) + J^\alpha_{a+} f(b,\, d) + J^\alpha_{b-} f(a,\, c) + J^\alpha_{b-} f(a,\, d)] \\
+ & \frac{\Gamma(\beta+1)}{4(d-c)^\beta}[J^\beta_{c+} f(a,\, d) + J^\beta_{c+} f(b,\, d) + J^\beta_{d-} f(a,\, c) + J^\beta_{d-} f(b,\, c)], \quad \alpha,\, \beta > 0.
\end{aligned}$$

Proof of Theorem 3. For all $s,\, t \in [0,\, 1]$, we consider

$$x := ta + (1-t)[\lambda b + (1-\lambda)a], \qquad y =: tb + (1-t)[\lambda a + (1-\lambda)b], \tag{5}$$

$$z := sc + (1-s)[\gamma d + (1-\gamma)c], \qquad w =: sd + (1-s)[\gamma c + (1-\gamma)d]. \tag{6}$$

By using

$$f\left(\frac{x+y}{2},\, \frac{z+w}{2}\right) \leq \frac{f(x,z) + f(x,w) + f(y,z) + f(y,w)}{4},$$

we obtain,

$$\begin{aligned}
4f\left(\frac{x+y}{2},\, \frac{z+w}{2}\right) &= 4f\left(\frac{\lambda b + (2-\lambda)a}{2},\, \frac{\lambda d + (2-\lambda)c}{2}\right) \\
&\leq f(x,z) + f(x,w) + f(y,z) + f(y,w).
\end{aligned}$$

Or

$$\begin{aligned}
& 4\int_0^1\int_0^1 f\left(\frac{\lambda b + (2-\lambda)a}{2},\, \frac{\gamma d + (2-\gamma)c}{2}\right) G(t,s)dtds \\
\leq & \int_0^1\int_0^1 f\left(ta + (1-t)[\lambda b + (1-\lambda)a],\ sc + (1-s)[\gamma d + (1-\gamma)c]\right) G(t,s)dtds \\
+ & \int_0^1\int_0^1 f\left(ta + (1-t)[\lambda b + (1-\lambda)a],\, sd + (1-s)[\gamma c + (1-\gamma)d]\right) G(t,s)dtds \\
+ & \int_0^1\int_0^1 f\left(tb + (1-t)[\lambda a + (1-\lambda)b],\, sc + (1-s)[\gamma d + (1-\gamma)c]\right) G(t,s)dtds \\
+ & \int_0^1\int_0^1 f\left(tb + (1-t)[\lambda a + (1-\lambda)b],\, sd + (1-s)[\gamma c + (1-\gamma)d]\right) G(t,s)dtds
\end{aligned} \tag{7}$$

where $G(t,s) = t^{\alpha-1}s^{\beta-1}g\left(tb + (1-t)[\lambda a + (1-\lambda)b],\ sd + (1-s)[\gamma c + (1-\gamma)d]\right)$ Set $u := tb + (1-t)[\lambda a + (1-\lambda)b]$ and $v := sd + (1-s)[\gamma b + (1-\gamma)d]$. Then $du = (b - [\lambda a + (1-\lambda)b])dt = \lambda(b-a)dt,\ \lambda a + (1-\lambda)b \leq u \leq b,$

$$t = \frac{u - \lambda a - (1-\lambda)b}{\lambda(b-a)}, \quad 1 - t = \frac{b-u}{\lambda(b-a)},$$

and $dv = (d - [\gamma c + (1-\gamma)d])ds = \gamma(d-c)ds,\ \gamma c + (1-\gamma)d \le v \le d,$

$$s = \frac{v - \gamma c - (1-\gamma)d}{\gamma(d-c)}, \quad 1 - s = \frac{d-v}{\gamma(d-c)}.$$

Therefore, using the above substitution variables into (7), we have

$$\begin{aligned} & 4f\left(\frac{\lambda b + (2-\lambda)a}{2},\ \frac{\gamma d + (2-\gamma)c}{2}\right)\int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} H(u,v)dudv \\ \le & \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} f\left(a+b-u,\ c+d-v\right)H(u,v)dudv \\ + & \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} f\left(a+b-u,\ v\right)H(u,v)dudv \\ + & \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} f\left(u,\ c+d-v\right)H(u,v)dudv \\ + & \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} f\left(u,\ v\right)H(u,v)dudv, \end{aligned}$$

where $H(u,v) = [u - (\lambda a + (1-\lambda)b]^{\alpha-1}[v - (\gamma c + (1-\gamma)d]^{\beta-1} f\left(u,\ c+d-v\right) g\left(u,\ v\right)$. Rewriting the above inequality, we obtain the following inequality

$$\begin{aligned} & 4f\left(\frac{\lambda b + (2-\lambda)a}{2},\ \frac{\gamma d + (2-\gamma)c}{2}\right)\int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} \\ & [u - (\lambda a + (1-\lambda)b]^{\alpha-1}[v - (\gamma c + (1-\gamma)d]^{\beta-1} g(u,\ v)dudv \\ \le & \int_{a}^{a+\lambda(b-a)}\int_{c}^{c+\lambda(d-c)} [a + \lambda(b-a) - u]^{\alpha-1}[c + \gamma(d-c) - v]^{\beta-1} \times \\ & \qquad f\left(u,\ v\right) g\left(a+b-u,\ c+d-v\right) dudv \\ + & \int_{a}^{a+\lambda(b-a)}\int_{\gamma c+(1-\gamma)d}^{d} [a + \lambda(b-a) - u]^{\alpha-1}[v - (\gamma c + (1-\gamma)d]^{\beta-1} \times \\ & \qquad f\left(u,\ v\right) g\left(a+b-u,\ v\right) dudv \\ + & \int_{\lambda a+(1-\lambda)b}^{b}\int_{c}^{c+\lambda(d-c)} [u - (\lambda a + (1-\lambda)b]^{\alpha-1}[c + \gamma(d-c) - v]^{\beta-1} \times \\ & \qquad f\left(u,\ v\right) g\left(u,\ c+d-v\right) dudv \\ + & \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} [u - (\lambda a + (1-\lambda)b]^{\alpha-1}[v - (\gamma c + (1-\gamma)d]^{\beta-1} \times \\ & \qquad f\left(u,\ v\right) g\left(u,\ v\right) dudv. \end{aligned}$$

Since g is symmetric respect to $\frac{a+b}{2},\ \frac{c+d}{2}$, then $g(a+b-u,\ c+d-v) = g(u,\ v)$.

Also, $\lambda a + (1-\lambda)b = b - \lambda(b-a)$. Therefore

$$
\begin{aligned}
& 4f\left(\frac{\lambda b + (2-\lambda)a}{2},\ \frac{\gamma d + (2-\gamma)c}{2}\right)\int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d} \\
& [u-(\lambda a+(1-\lambda)b]^{\alpha-1}[v-(\gamma c+(1-\gamma)d]^{\beta-1}g(u,\,v)dudv \\
\leq & \int_{a}^{a+\lambda(b-a)}\int_{c}^{c+\lambda(d-c)}[a+\lambda(b-a)-u]^{\alpha-1}[c+\gamma(d-c)-v]^{\beta-1}\times \\
& \qquad f\left(u,\ v\right)g\left(u,\ v\right)dudv \\
+ & \int_{a}^{a+\lambda(b-a)}\int_{\gamma c+(1-\gamma)d}^{d}[a+\lambda(b-a)-u]^{\alpha-1}[v-(\gamma c+(1-\gamma)d]^{\beta-1}\times \\
& \qquad f\left(u,\ v\right)g\left(u,\ v\right)dudv \\
+ & \int_{\lambda a+(1-\lambda)b}^{b}\int_{c}^{c+\gamma(d-c)}[u-(\lambda a+(1-\lambda)b]^{\alpha-1}[c+\gamma(d-c)-v]^{\beta-1}\times \\
& \qquad f\left(u,\ v\right)g\left(u,\ v\right)dudv \\
+ & \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d}[u-(\lambda a+(1-\lambda)b]^{\alpha-1}[v-(\gamma c+(1-\gamma)d]^{\beta-1}\times \\
& \qquad f\left(u,\ v\right)g\left(u,\ v\right)dudv.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& 4f\left(\frac{\lambda b + (2-\lambda)a}{2},\ \frac{\gamma d + (2-\gamma)c}{2}\right)J^{\alpha,\,\beta}_{b^-,\,d^-}g(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d) \\
\leq\ & J^{\alpha,\,\beta}_{a^+,\,c^+}(fg)(a+\lambda(b-a),\ c+\gamma(d-c)) \\
+\ & J^{\alpha,\,\beta}_{a^+,\,d^-}(fg)(a+\lambda(b-a),\ \gamma c+(1-\gamma)d) \\
+\ & J^{\alpha,\,\beta}_{b^-,\,c^+}(fg)(\lambda a+(1-\lambda)b,\ c+\gamma(d-c)) \\
+\ & J^{\alpha,\,\beta}_{b^-,\,d^-}(fg)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d).
\end{aligned} \tag{8}
$$

To prove the right-side inequality (3), we need Definition 2, (5) and (6). Therefore,

$$
\begin{aligned}
& f(x,z)+f(y,z)+f(x,w)+f(y,w)= \\
& f(ta+(1-t)[\lambda b+(1-\lambda)a],\ sc+(1-s)[\gamma d+(1-\gamma)c]) \\
+\ & f(tb+(1-t)[\lambda a+(1-\lambda)b],\ sc+(1-s)[\gamma d+(1-\gamma)c]) \\
+\ & f(ta+(1-t)[\lambda b+(1-\lambda)a],\ sd+(1-s)[\gamma c+(1-\gamma)d) \\
+\ & f(tb+(1-t)[\lambda a+(1-\lambda)b],\ sd+(1-s)[\gamma c+(1-\gamma)d) \\
\leq\ & tsf(a,c)+\lambda(1-t)sf(b,c)+(1-\lambda)(1-t)sf(a,c) \\
+\ & \gamma t(1-s)f(a,d)+t(1-s)(1-\gamma)f(a,c)+\lambda\gamma(1-t)(1-s)f(b,d) \\
+\ & \lambda(1-t)(1-s)(1-\gamma)f(b,c)+(1-t)(1-s)(1-\lambda)(1-\gamma)f(a,c) \\
+\ & (1-s)\gamma(1-\lambda)f(a,d). \\
+\ & tsf(b,c)+\gamma t(1-s)f(b,d)+t(1-s)(1-\gamma)f(b,c) \\
+\ & \lambda s(1-t)f(a,c)+s(1-t)(1-\lambda)f(b,c)+(1-t)(1-s)\lambda\gamma f(a,d) \\
+\ & (1-t)(1-s)\lambda(1-\gamma)f(a,c)+(1-t)(1-s)(1-\lambda)(1-\gamma)f(b,c). \\
+\ & (1-t)(1-s)(1-\lambda)\gamma f(b,d) \\
+\ & tsf(a,d)+t(1-s)\gamma f(a,c)+t(1-s)(1-\gamma)f(a,d) \\
+\ & s(1-t)\lambda f(b,d)+s(1-t)(1-\lambda)f(a,d) \\
+\ & (1-s)(1-t)\gamma\lambda f(b,c)+(1-s)(1-t)\lambda(1-\gamma)f(b,d)
\end{aligned}
$$

$$\begin{aligned}
&+ (1-s)(1-t)(1-\lambda)\gamma f(a,c) + (1-s)(1-t)(1-\lambda)(1-\gamma)f(a,d)\\
&+ stf(b,d) + \gamma t(1-s)f(b,c)\\
&+ t(1-s)(1-\gamma)f(b,d) + s(1-t)\lambda f(a,d)\\
&+ s(1-t)(1-\lambda)f(b,d) + (1-t)(1-s)\lambda\gamma f(a,c)\\
&+ (1-t)(1-s)\lambda(1-\gamma)f(a,d) + (1-t)(1-s)\gamma(1-\lambda)f(b,c)\\
&+ (1-t)(1-s)(1-\lambda)(1-\gamma)f(b,d).\\
&= [ts + t(1-s)(1-\gamma) + (1-t)(1-\lambda)s + (1-t)(1-\lambda)(1-s)(1-\gamma)\\
&+ (1-t)s\lambda + (1-t)\lambda(1-s)(1-\gamma) + t(1-s)\gamma\\
&+ (1-t)(1-\lambda)(1-s)\gamma + (1-t)\lambda(1-s)\gamma]f(a,\, c)\\
&+ [(1-t)s\lambda + (1-t)\lambda(1-s)(1-\gamma) + ts + t(1-s)(1-\gamma)\\
&+ (1-t)(1-\lambda)s + (1-t)(1-\lambda)(1-s)(1-\gamma) + (1-t)\lambda(1-s)\gamma\\
&+ t(1-s)\gamma + (1-t)(1-\lambda)(1-s)\gamma]f(b,\, c)\\
&+ [t(1-s)\gamma + (1-t)(1-\lambda)(1-s)\gamma + (1-t)\lambda(1-s)\gamma\\
&+ ts + t(1-s)(1-\gamma) + (1-t)(1-\lambda)s + (1-t)(1-\lambda)(1-s)(1-\gamma)\\
&+ (1-t)\lambda s + (1-t)\lambda(1-s)(1-\gamma)]f(a,\, d)\\
&+ [(1-t)\lambda(1-s)\gamma + t(1-s)\gamma + (1-t)(1-\lambda)(1-s)\gamma\\
&+ (1-t)\lambda s + (1-t)\lambda(1-s)(1-\gamma) + ts + t(1-s)(1-\gamma)\\
&+ (1-t)(1-\lambda)s + (1-t)(1-\lambda)(1-s)(1-\gamma)]f(b,\, d).
\end{aligned}$$

We can easily to show that each factor of $f(a,c)$, $f(b,c)$, $f(a,d)$, $f(b,d)$ from the above equations is equal to 1. Hence,

$$\begin{aligned}
& f(ta + (1-t)[\lambda b + (1-\lambda)a],\ sc + (1-s)[\gamma d + (1-\gamma)c])\\
+\ & f(tb + (1-t)[\lambda a + (1-\lambda)b],\ sc + (1-s)[\gamma d + (1-\gamma)c])\\
+\ & f(ta + (1-t)[\lambda b + (1-\lambda)a],\ sd + (1-s)[\gamma c + (1-\gamma)d)\\
+\ & f(tb + (1-t)[\lambda a + (1-\lambda)b],\ sd + (1-s)[\gamma c + (1-\gamma)d)\\
\leq\ & f(a,\, c) + f(b,\, c) + f(a,\, d) + f(b,\, d).
\end{aligned} \tag{9}$$

Multiplying both sides of (9) by

$$G(t,s) := t^{\alpha-1}s^{\beta-1}g\,(tb + (1-t)[\lambda a + (1-\lambda)b],\ sd + (1-s)[\gamma c + (1-\gamma)d])$$

and integrating with respect to (t,s) over $[0,\,1]\times[0,\,1]$, we obtain

$$\begin{aligned}
& \overbrace{\int_0^1\int_0^1 F_1(t,s)G(t,s)dtds}^{L_1} + \overbrace{\int_0^1\int_0^1 F_(t,s)G(t,s)dtds}^{L_2}\\
+\ & \overbrace{\int_0^1\int_0^1 F_3(t,s)G(t,s)}^{L_3} + \overbrace{\int_0^1\int_0^1 F_4(t,s)G(t,s)dtds}^{L_4}\\
\leq\ & \overbrace{f(a,\, c)\int_0^1\int_0^1 G(t,s)dtds}^{R_1} + \overbrace{f(b,\, c)\int_0^1\int_0^1 G(t,s)dtds}^{R_2}\\
+\ & \overbrace{f(a,\, d)\int_0^1\int_0^1 G(t,s)dtds}^{R_3} + \overbrace{f(b,\, d)\int_0^1\int_0^1 G(t,s)dtds}^{R_4},
\end{aligned}$$

where,

$$\begin{aligned} F_1(t,s) &= f(ta+(1-t)[\lambda b+(1-\lambda)a],\ sc+(1-s)[\gamma d+(1-\gamma)c]) \\ F_2(t,s) &= f(tb+(1-t)[\lambda a+(1-\lambda)b],\ sc+(1-s)[\gamma d+(1-\gamma)c]) \\ F_3(t,s) &= f(ta+(1-t)[\lambda b+(1-\lambda)a],\ sd+(1-s)[\gamma c+(1-\gamma)d) \end{aligned}$$

and

$$F_4(t,s) = f(tb+(1-t)[\lambda a+(1-\lambda)b],\ sd+(1-s)[\gamma c+(1-\gamma)d).$$

It should be noted that integrating L_1, L_2, L_3, L_4 and R_1, R_2, R_3, R_4 are similar to (7). If we set $u := tb+(1-t)[\lambda a+(1-\lambda)b]$ and $v := sd+(1-s)[\gamma b+(1-\gamma)d]$, then $du = (b-[\lambda a+(1-\lambda)b])dt = \lambda(b-a)dt,\ \lambda a+(1-\lambda)b \le u \le b,$

$$t = \frac{u-\lambda a-(1-\lambda)b}{\lambda(b-a)}, \quad 1-t = \frac{b-u}{\lambda(b-a)},$$

and $dv = (d-[\gamma c+(1-\gamma)d])ds = \gamma(d-c)ds,\ \gamma c+(1-\gamma)d \le v \le d,$

$$s = \frac{v-\gamma c-(1-\gamma)d}{\gamma(d-c)}, \quad 1-s = \frac{d-v}{\gamma(d-c)}.$$

Hence,

$$\begin{aligned} R_1 &= \Gamma(\alpha)\Gamma(\beta)J^{\alpha,\beta}_{a+,\,c+}(fg)(a+\lambda(b-a),\ c+\gamma(d-c)), \\ R_2 &= \Gamma(\alpha)\Gamma(\beta)J^{\alpha,\beta}_{a+,\,d-}(fg)(a+\lambda(b-a),\ \gamma c+(1-\gamma)d), \\ R_3 &= \Gamma(\alpha)\Gamma(\beta)J^{\alpha,\beta}_{b-,\,c+}(fg)(\lambda a+(1-\lambda)b,\ c+\gamma(d-c)), \\ R_4 &= \Gamma(\alpha)\Gamma(\beta)J^{\alpha,\beta}_{b-,\,d-}(fg)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_0^1\int_0^1 t^{\alpha-1}s^{\beta-1}g\left(tb+(1-t)[\lambda a+(1-\lambda)b],\ sd+(1-s)[\gamma c+(1-\gamma)d]\right)dtds \\ =& \int_{\lambda a+(1-\lambda)b}^{b}\int_{\gamma c+(1-\gamma)d}^{d}[u-(\lambda a+(1-\lambda)b]^{\alpha-1}[v-(\gamma c+(1-\gamma)d]^{\beta-1}g(u,\ v)dudv \\ =& \Gamma(\alpha)\Gamma(\beta)J^{\alpha,\beta}_{b-,\,d-},g(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d). \end{aligned} \tag{10}$$

Now if we substitute (10), (10), (10), (10) and (10) into L_1, L_2, L_3, L_4 and R_1, R_2, R_3, R_4, we obtain

$$\begin{aligned} & J^{\alpha,\beta}_{a+,\,c+}(fg)(a+\lambda(b-a),\ c+\gamma(d-c)) + J^{\alpha,\beta}_{a+,\,d-}(fg)(a+\lambda(b-a),\ \gamma c+(1-\gamma)d) \\ +\ & J^{\alpha,\beta}_{b-,\,c+}(fg)(\lambda a+(1-\lambda)b,\ c+\gamma(d-c)) + J^{\alpha,\beta}_{b-,\,d-}(fg)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d). \\ \le\ & \{f(a,\ c)+f(b,\ c)+f(a,\ d)+f(b,\ d)\}\, J^{\alpha,\beta}_{b-,\,d-},g(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d). \end{aligned} \tag{11}$$

Combining (8) and (11), we have

$$\begin{aligned}
& f\left(\frac{\lambda b+(2-\lambda)a}{2},\ \frac{\gamma d+(2-\gamma)c}{2}\right) J^{\alpha,\,\beta}_{b^-,\,d^-}\, g(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d) \\
\leq\ & \frac{1}{4}[J^{\alpha,\,\beta}_{a^+,\,c^+}(fg)(a+\lambda(b-a),\ c+\gamma(d-c))+J^{\alpha,\,\beta}_{a^+,\,d^-}(fg)(a+\lambda(b-a),\ \gamma c+(1-\gamma)d) \\
+\ & J^{\alpha,\,\beta}_{b^-,\,c^+}(fg)(\lambda a+(1-\lambda)b,\ c+\gamma(d-c))+J^{\alpha,\,\beta}_{b^-,\,d^-}(fg)(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d)] \\
\leq\ & J^{\alpha,\,\beta}_{b^-,\,d^-}\, g(\lambda a+(1-\lambda)b,\ \gamma c+(1-\gamma)d)\,\frac{f(a,\,c)+f(b,\,c)+f(a,\,d)+f(b,\,d)}{4}. \qquad (12)
\end{aligned}$$

This completes the proof.

Now, we prove Theorem 4 in full description as follows.
Proof of Theorem 4. Since $f:\Omega\longrightarrow\mathbb{R}$ is a convex function on the co-ordinates, then it follows that the mapping $h_x:[c,d]\longrightarrow\mathbb{R}$, $h_x(y)=f(x,\,y)$ is a convex function on $[c,d]$ for all $x\in[a,b]$. Then

$$\begin{aligned}
& h_x\left(\frac{\gamma d+(2-\gamma)c}{2}\right) J^{\beta}_{[d-\gamma(d-c)]^+}k_x(d)+J^{\beta}_{d^-}k_x(d-\gamma(d-c))] \\
\leq\ & J^{\beta}_{c^+}(h_xk_x)(c+\gamma(d-c))+J^{\beta}_{d^-}(h_xk_x)(d-\gamma(d-c)) \\
\leq\ & \frac{h_x(c)+h_x(d)}{2}\left\{J^{\beta}_{c^+}(k_x)(c+\gamma(d-c))+J^{\beta}_{d^-}(k_x)(d-\gamma(d-c))\right\}, \quad x\in[a,\,b].
\end{aligned}$$

Indeed we have,

$$\begin{aligned}
& f\left(x,\frac{\gamma d+(2-\gamma)c}{2}\right)\int_{d-\gamma(d-c)}^{d}(d-y)^{\beta-1}g(x,y)dy \\
+\ & f\left(x,\frac{\gamma d+(2-\gamma)c}{2}\right)\int_{d-\gamma(d-c)}^{d}[y-(d-\gamma(d-c))]^{\beta-1}g(x,y)dy \\
\leq\ & \int_{c}^{c+\gamma(d-c)}[c+\gamma(d-c)-y]^{\beta-1}f(x,\,y)g(x,\,y)dy \\
+\ & \int_{d-\gamma(d-c)}^{d}[y-(d-\gamma(d-c))]^{\beta-1}f(x,\,y)g(x,\,y)dy \\
\leq\ & \frac{f(x,\,c)+f(x,\,d)}{2}\left\{\int_{c}^{c+\gamma(d-c)}g_1(x,\,y)dy+\int_{d-\gamma(d-c)}^{d}g_2(x,\,y)\right\}dy, \qquad (13)
\end{aligned}$$

where $g_1(y)=[c+\gamma(d-c)-y]^{\beta-1}g(x,y)$ and $g_2(y)=[y-(d-\gamma(d-c))]^{\beta-1}g(x,y)$ Multiplying both sides of (13) by $(b-x)^{\alpha-1}$ and $(x-a)^{\alpha-1}$, and integrating with respect to x over $[a, b]$, respectively, we have

$$\begin{aligned}
&\int_a^b \int_{d-\gamma(d-c)}^{d} f\left(x, \frac{\gamma d+(2-\gamma)c}{2}\right)(b-x)^{\alpha-1}(d-y)^{\beta-1}g(x,y)dydx \\
+ &\int_a^b \int_{d-\gamma(d-c)}^{d} (b-x)^{\alpha-1} f\left(x, \frac{\gamma d+(2-\gamma)c}{2}\right)[y-(d-\gamma(d-c))]^{\beta-1}g(x,y)dydx \\
\leq &\int_a^b \int_{c}^{c+\gamma(d-c)} (b-x)^{\alpha-1}[c+\gamma(d-c)-y]^{\beta-1}f(x,\, y)g(x,\, y)dydx \\
+ &\int_a^b \int_{d-\gamma(d-c)}^{d} (b-x)^{\alpha-1}[y-(d-\gamma(d-c))]^{\beta-1}f(x,\, y)g(x,\, y)dydx \\
\leq &[\int_a^b \int_{c}^{c+\gamma(d-c)} (b-x)^{\alpha-1}[c+\gamma(d-c)-y]^{\beta-1}\frac{f(x,\, c)+f(x,\, d)}{2}g(x,\, y)dy \\
+ &\int_a^b \int_{d-\gamma(d-c)}^{d} (b-x)^{\alpha-1}[y-(d-\gamma(d-c))]^{\beta-1}\frac{f(x,\, c)+f(x,\, d)}{2}g(x,\, y)dydx] \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \int_{d-\gamma(d-c)}^{d} f\left(x, \frac{\gamma d+(2-\gamma)c}{2}\right)(x-a)^{\alpha-1}(d-y)^{\beta-1}g(x,y)dydx \\
+ &\int_a^b \int_{d-\gamma(d-c)}^{d} (x-a)^{\alpha-1} f\left(x, \frac{\gamma d+(2-\gamma)c}{2}\right)[y-(d-\gamma(d-c))]^{\beta-1}g(x,y)dydx \\
\leq &\int_a^b \int_{c}^{c+\gamma(d-c)} (x-a)^{\alpha-1}[c+\gamma(d-c)-y]^{\beta-1}f(x,\, y)g(x,\, y)dydx \\
+ &\int_a^b \int_{d-\gamma(d-c)}^{d} (x-a)^{\alpha-1}[y-(d-\gamma(d-c))]^{\beta-1}f(x,\, y)g(x,\, y)dydx \\
\leq &\int_a^b \int_{c}^{c+\gamma(d-c)} (x-a)^{\alpha-1}[c+\gamma(d-c)-y]^{\beta-1}\frac{f(x,\, c)+f(x,\, d)}{2}g(x,\, y)dy \\
+ &\int_a^b \int_{d-\gamma(d-c)}^{d} (x-a)^{\alpha-1}[y-(d-\gamma(d-c))]^{\beta-1}\frac{f(x,\, c)+f(x,\, d)}{2}g(x,\, y)dydx. \qquad (15)
\end{aligned}$$

By similar argument applied for the mapping $h_y,\, k_y : [a,\, b] \longrightarrow \mathbb{R}$, $h_y(x) = f(x,\, y)$ and $k_y(x) = g(x,\, y)$, we have

$$\begin{aligned}
&h_y\left(\frac{\lambda b+(2-\lambda)a}{2}\right) J^{\alpha}_{[b-\lambda(b-a)]^+}k_y(b) + J^{\alpha}_{b-}k_y(b-\lambda(b-a))] \\
\leq\; &J^{\alpha}_{a+}(h_y k_y)(a+\lambda(b-a)) + J^{\alpha}_{b-}(h_y k_y)(b-\lambda(b-a)) \\
\leq\; &\frac{h_y(a)+h_y(b)}{2}\left\{J^{\alpha}_{a+}(k_y)(a+\lambda(b-a)) + J^{\alpha}_{b-}(k_y)(b-\lambda(b-a))\right\}, \quad y \in [c,\, d].
\end{aligned}$$

Indeed we have,

$$\begin{aligned}
&f\left(\frac{\lambda b+(2-\lambda)a}{2}, y\right)\int_{c-\lambda(b-a)}^{b}(b-x)^{\alpha-1}g(x,y)dx \\
+\; &f\left(\frac{\lambda b+(2-\lambda)a}{2}, y\right)\int_{b-\lambda(b-a)}^{b}[x-(b-\lambda(b-b))]^{\alpha-1}g(x,y)dx \\
\leq\; &\int_a^{a+\lambda(b-a)} g_3(x,y)f(x,\, y)dx + \int_{b-\lambda(b-a)}^{b} g_4(x,y)f(x,\, y)dx \\
\leq\; &\frac{f(a,\, y)+f(b,\, y)}{2}\left\{\int_a^{a+\lambda(b-a)} g_3(x,\, y)dx + \int_{b-\lambda(b-a)}^{b} g_4(x,\, y)dx\right\}, \qquad (16)
\end{aligned}$$

where $g_3(x,y)=[a+\lambda(b-a)-x]^{\alpha-1}g(x,y)$ and $g_4(x,y)=[x-(b-\lambda(b-a))]^{\alpha-1}g(x,y)$. Multiplying both sides of (16) by $(c-y)^{\beta-1}$ and $(y-c)^{\beta-1}$, and integrating with respect to y over $[c,\, d]$, respectively, we have

$$\begin{aligned}
&\int_c^d \int_{b-\gamma(b-a)}^b f\left(\frac{\lambda b+(2-\lambda)a}{2}, y\right) dydx\\
+&\int_c^d \int_{b-\gamma(b-a)}^b f\left(\frac{\lambda b+(2-\lambda)a}{2}, y\right) dydx\\
\leq&\int_c^d \int_a^{a+\lambda(b-a)} f(x,\, y)dydx+\int_c^d \int_{b-\lambda(b-a)}^b f(x,\, y)dydx\\
\leq&\int_c^d \int_a^{a+\lambda(b-a)} \frac{f(x,\, a)+f(x,\, b)}{2} f(x,\, y)g(x,\, y)dy\\
+&\int_c^d \int_{b-\lambda(b-a)}^b \frac{f(x,\, a)+f(x,\, b)}{2} f(x,\, y)g(x,\, y)dydx, \qquad (17)
\end{aligned}$$

where

$$\begin{aligned}
g_5(x,y)&=(d-y)^{\beta-1}(b-x)^{\alpha-1}g(x,y),\\
g_6(x,y)&=(d-y)^{\beta-1}[y-x-(b-\lambda(b-a)]^{\alpha-1}g(x,y),\\
g_7(x,y)&=(d-y)^{\beta-1}[a+\lambda(b-a)-x]^{\beta-1}g(x,y),\\
g_8(x,y)&=(d-y)^{\beta-1}[x-(b-\lambda(b-a))]^{\alpha-1}g(x,y),\\
g_9(x,y)&=(d-y)^{\beta-1}[a+\lambda(b-a)-x]^{\alpha-1}g(x,y),
\end{aligned}$$

and $g_{10}(x,y)=(d-y)^{\beta-1}[x-(b-\lambda(b-a))]^{\alpha-1}g(x,y)$, and other hand

$$\begin{aligned}
&\int_c^d \int_{b-\gamma(b-a)}^b (y-c)^{\beta-1}(b-x)^{\alpha-1} f\left(\frac{\lambda b+(2-\lambda)a}{2}, y\right) g(x,y)dydx\\
+&\int_c^d \int_{b-\gamma(b-a)}^b (y-c)^{\beta-1}[y-x-(b-\lambda(b-a)]^{\alpha-1} f\left(\frac{\lambda b+(2-\lambda)a}{2}, y\right)\\
&\qquad\times g(x,y)dydx\\
\leq&\int_c^d \int_a^{a+\lambda(b-a)} (y-c)^{\beta-1}[a+\lambda(b-a)-x]^{\beta-1} f(x,\, y)g(x,\, y)dydx\\
+&\int_c^d \int_{b-\lambda(b-a)}^b (y-c)^{\beta-1}[x-(b-\lambda(b-a))]^{\alpha-1} f(x,\, y)g(x,\, y)dydx\\
\leq&\int_c^d \int_a^{a+\lambda(b-a)} (y-c)^{\beta-1}[a+\lambda(b-a)-x]^{\alpha-1} \frac{f(x,\, a)+f(x,\, b)}{2}\times\\
&\qquad f(x,\, y)g(x,\, y)dydx\\
+&\int_c^d \int_{b-\lambda(b-a)}^b (y-c)^{\beta-1}[x-(b-\lambda(b-a))]^{\alpha-1} \frac{f(x,\, a)+f(x,\, b)}{2}\times\\
&\qquad f(x,\, y)g(x,\, y)dydx. \qquad (18)
\end{aligned}$$

Each of the inequalities (14), (15), (17) and (18) is equivalent to the following inequalities, respectively.

$$
\begin{aligned}
& J^{\alpha,\beta}_{a^+,[d-\gamma(d-c)]^+} f\left(b, \frac{\gamma d+(2-\gamma)c}{2}\right) g(b,d) \\
+ \; & J^{\alpha,\beta}_{a^+,d^-} f\left(b, \frac{\gamma d+(2-\gamma)c}{2}\right) g(b,d-\gamma(d-c)) \\
\leq \; & J^{\alpha,\beta}_{a^+,c^+}(fg)(b,c+\gamma(d-c)) + J^{\alpha,\beta}_{a^+,d^-}(fg)(b,d-\gamma(d-c)) \\
\leq \; & \frac{1}{2} J^{\alpha,\beta}_{a^+,c^+}[f(b,c)+f(b,d)]g(b,c+\gamma(d-c)) \\
+ \; & \frac{1}{2} J^{\alpha,\beta}_{a^+,d^-}[f(b,c)+f(b,d)]g(b,d-\gamma(d-c)), \qquad (19)
\end{aligned}
$$

$$
\begin{aligned}
& J^{\alpha,\beta}_{b^-,[d-\gamma(d-c)]^+} f\left(a, \frac{\gamma d+(2-\gamma)c}{2}\right) g(a,d-\gamma(d-c)) \\
+ \; & J^{\alpha,\beta}_{b^-,d^-} f\left(a, \frac{\gamma d+(2-\gamma)c}{2}\right) g(a,d-\gamma(d-c)) \\
\leq \; & J^{\alpha,\beta}_{b^-,c^+}(fg)(a,c+\gamma(d-c)) + J^{\alpha,\beta}_{b^-,d^-}(fg)(a,d-\gamma(d-c)) \\
\leq \; & \frac{1}{2} J^{\alpha,\beta}_{b^-,c^+}[f(a,c)+f(a,d)]g(a,c+\gamma(d-c)) \\
+ \; & \frac{1}{2} J^{\alpha,\beta}_{b^-,d^-}[f(a,c)+f(a,d)]g(a,d-\gamma(d-c)), \qquad (20)
\end{aligned}
$$

$$
\begin{aligned}
& J^{\beta,\alpha}_{c^+,[b-\lambda(b-a)]^+} f\left(\frac{\lambda b+(2-\lambda)a, d}{2}, d\right) g(b,d) \\
+ \; & J^{\beta,\alpha}_{c^+,b^-} f\left(\frac{\lambda b+(2-\lambda)a}{2}, d\right) g(b-\lambda(b-a),d) \\
\leq \; & J^{\beta,\alpha}_{c^+,a^+}(fg)(a+\lambda(b-a),d) + J^{\beta,\alpha}_{c^+,b^-}(fg)(d,b-\lambda(b-a)) \\
\leq \; & \frac{1}{2} J^{\beta,\alpha}_{c^+,a^+}[f(a,d)+f(b,d)](fg)(a+\lambda(b-a),d) \\
+ \; & \frac{1}{2} J^{\beta,\alpha}_{c^+,b^-}[f(a,d)+f(b,d)](fg)(b-\lambda(b-a),d) \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
& J^{\beta,\alpha}_{d^-,[b-\lambda(b-a)]^+} f\left(\frac{\lambda b+(2-\lambda)a, d}{2}, c\right) g(b,c) + J^{\alpha,\beta}_{d^-,b^-} f\left(\frac{\lambda b+(2-\lambda)a}{2}, c\right) g(b,c) \\
\leq \; & J^{\beta,\alpha}_{d^-,a^+}(fg)(a+\lambda(b-a),c) + J^{\beta,\alpha}_{d^-,b^-}(fg)(b-\lambda(b-a),c) \\
\leq \; & \frac{1}{2} J^{\beta,\alpha}_{d^-,a^+}[f(a,c)+f(b,c)](fg)(a+\lambda(b-a),c) \\
+ \; & \frac{1}{2} J^{\beta,\alpha}_{d^-,b^-}[f(a,c)+f(b,c)](fg)(b-\lambda(b-a),c). \qquad (22)
\end{aligned}
$$

By adding inequalities (19)-(22), the proof will be completed.

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