Communications in Combinatorics and Optimization
Vol. 5 No. 2, 2020 pp.179-190
DOI: 10.22049/CCO.2020.26647.1127

CCO
Commun. Comb. Optim.

On strongly 2-multiplicative graphs

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Received: 22 August 2019; Accepted: 10 April 2020
Published Online: 14 April 2020

Abstract: A simple connected graph G of order $n \geq 3$ is a strongly 2-multiplicative if there is an injective mapping $f : V(G) \to \{1, 2, \ldots, n\}$ such that the induced mapping $h : \mathcal{A} \to \mathbb{Z}^+$ defined by $h(\mathcal{P}) = \prod_{i=1}^{3} f(v_{j_i})$, where $j_1, j_2, j_3 \in \{1, 2, \ldots, n\}$, and $\mathcal{P}$ is the path homotopy class of paths having the vertex set $\{v_{j_1}, v_{j_2}, v_{j_3}\}$, is injective. Let $\Lambda(n)$ be the number of distinct path homotopy classes in a strongly 2-multiplicative graph of order n. In this paper we obtain an upper bound and also a lower bound for $\Lambda(n)$. Also we prove that triangular ladder, $P_2 \bigodot C_n$, $P_m \bigodot P_n$, the graph obtained by duplication of an arbitrary edge by a new vertex in path P_n and the graph obtained by duplicating all vertices by new edges in a path P_n are strongly 2-multiplicative.

Keywords: graph labeling, strongly 2-multiplicative

AMS Subject classification: 05C78

1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. During the past five decades thousands of research papers on graph labelings and their applications have been published. Many of the methods on graph labelings are motivated by certain practical problems. For more details one may refer the survey article by Gallian [7].
In 2001, Beineke and Hegde [6] have introduced the concept of strongly multiplicative graphs. A graph with n vertices is said to be strongly multiplicative if there is an injection $f : V \to \{1, 2, \ldots, n\}$ such that the induced mapping $f^{\times} : E(G) \to \mathbb{N}$ defined

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by $f^{\times}(e) = f(u)f(v)$, where $e = uv$, is injective. They have proved that certain classes of graphs are strongly multiplicative. They have also obtained an upper bound for $\lambda(n)$, the maximum number of edges for a given strongly multiplicative graph of order n. In [2], Adiga, Ramaswamy and Somashekara have given a lower bound for $\lambda(n)$ and proved that the complete bipartite graph $K_{n,n}$ is strongly multiplicative if and only if $n \leq 4$. In [3], Adiga, Ramaswamy and Somashekara have given a formula for $\lambda(n)$ and proved that every wheel is strongly multiplicative. In [4], Adiga, Ramaswamy and Somashekara have given an upper bound for $\lambda(n)$, improving the upper bound obtained by Beineke and Hegde [6]. Seoud and Zid [9], Adiga and Smitha [5], Acharya, Germina and Ajitha [1], Vaidya and Kanani [11] and Muthusamy, Raajasekar and Basker Babujee [8] are among many others who contributed to the concept of strongly multiplicative graphs.

Motivated by this Somashekara, Veena and Ravi [10] have introduced the concept of strongly k-multiplicative graphs as follows: Consider a simple connected graph G of order n. Let P_1 and P_2 be two paths in G with the same vertex set S. Then we say that P_1 and P_2 are path homotopic with respect to S. We denote this by $P_1 \simeq_S P_2$. One can easily prove that this relation is an equivalence relation. Let $\mathcal{P}$ be the path homotopy class consisting of those paths which are path homotopic to the path P with a given vertex set and let $\mathcal{A}$ denote the set of all distinct path homotopy classes in G.

Definition 1. A simple connected graph G of order n is said to be strongly k-multiplicative if there is an injective mapping $f : V(G) \to \{1, 2, \ldots, n\}$ such that the induced mapping $h : \mathcal{A} \to \mathbb{Z}^+$ defined by $h(\mathcal{P}) = \prod_{i=1}^{k+1} f(v_{j_i})$, where $j_1, j_2, \ldots, j_{k+1} \in \{1, 2, \ldots, n\}$, $k+1 \leq n$ and $\mathcal{P}$ is the path homotopy class of paths having the vertex set $\{v_{j_1}, v_{j_2}, \ldots, v_{j_{k+1}}\}$, is injective.

In particular if k=2 we call G, strongly 2- multiplicative and if $k = 1$, then G is strongly 1-multiplicative or strongly multiplicative.

The following results of Adiga, Ramaswamy and Somashekara [2, 4] will be used in section 2.

Theorem 1. *([2]) If $\lambda(n)$ denotes the maximum number of edges in a strongly multiplicative graph of order n, then*

$$\lambda(n) \geq n(n-2) - \sum_{k=1}^{n-3} \left[\frac{kn}{k+1}\right], n \geq 4$$

where $[x]$ denotes the largest integer less than or equal to x.

Theorem 2. *([4]) If $\lambda(n)$ denotes the maximum number of edges in a strongly multiplicative graph of order n, then*

$$\lambda(n) \leq \frac{n(n+1)}{2} - \sum_{i=2}^{n} \frac{i}{p(i)} + (n-2) - \left[\frac{(n+2)}{4}\right]$$

where $[x]$ denotes the largest integer less than or equal to x.

In [10], Somashekara, Veena and Ravi have proved that certain classes of graphs like path, cycle, ladder etc., are strongly 2-multiplicative. In this paper we obtain an upper bound and also a lower bound for $\Lambda(n)$, where $\Lambda(n)$ is the number of distinct path homotopy classes in a strongly 2-multiplicative graph of order n. It is easy to see that $\Lambda(n) = |\{rst \mid 1 \leq r < s < t \leq n\}|$. Also we prove that triangular ladder, $P_2 \bigodot C_n$, $P_m \bigodot P_n$, the graph obtained by duplication of an arbitrary edge by a new vertex in path P_n and the graph obtained by duplicating all vertices by new edges in a path P_n are strongly 2-multiplicative.

2. Upper bound for $\Lambda(n)$

Let $\delta(3) = \Lambda(3)$ and $\delta(n) = \Lambda(n) - \Lambda(n-1)$, $n \geq 4$, the number of new products one can get by going from $n-1$ to n as the largest factor. We first give an upper bound for $\delta(n)$ in the following Lemma.

Lemma 1. *Let $p(n) = p$ denote the least prime divisor of n. Then*

$$\delta(n) \leq \begin{cases} \frac{n(n-1)}{2} - \sum_{i=2}^{n-1} \frac{i}{p(i)} + (n-3) - \left[\frac{(n+1)}{4}\right], & \text{if } n \text{ is a prime} \\ \frac{n(n-1)}{2} - \frac{3}{2}(\frac{n}{p}-1)(\frac{n}{p}-2), & \text{if } n \text{ is not a prime,} \end{cases}$$

where $[x]$ denotes the largest integer less than or equal to x.

Proof. To find an upper bound for $\delta(n)$, we need to consider, an array of new products having n as one of the factors, as shown below.

$$\begin{array}{lllll} n\cdot 2\cdot 1 & & & & \\ n\cdot 3\cdot 1 & n\cdot 3\cdot 2 & & & \\ n\cdot 4\cdot 1 & n\cdot 4\cdot 2 & n\cdot 4\cdot 3 & & \\ n\cdot 5\cdot 1 & n\cdot 5\cdot 2 & n\cdot 5\cdot 3 & n\cdot 5\cdot 4 & \\ \cdots & & & & \\ \cdots & & & & \\ n\cdot(n-1)\cdot 1 & n\cdot(n-1)\cdot 2 & n\cdot(n-1)\cdot 3 & \cdots & n\cdot(n-1)\cdot(n-2). \end{array} \tag{1}$$

Now, if n is a prime, then the number of distinct products in (1) is same as the number of distinct products in the array

$$\begin{array}{lllll}
2\cdot 1 & & & & \\
3\cdot 1 & 3\cdot 2 & & & \\
4\cdot 1 & 4\cdot 2 & 4\cdot 3 & & \\
5\cdot 1 & 5\cdot 2 & 5\cdot 3 & 5\cdot 4 & \\
\cdots & & & & \\
\cdots & & & & \\
(n-1)\cdot 1 & (n-1)\cdot 2 & (n-1)\cdot 3 & \cdots & (n-1)\cdot(n-2).
\end{array} \tag{2}$$

But the number of distinct products in (2) is the cardinality of the set $\{st \mid 1 \leq s < t \leq n-1\}$. Hence by the Theorem 2, it follows that

$$\delta(n) \leq \frac{n(n-1)}{2} - \sum_{i=2}^{n-1} \frac{i}{p(i)} + (n-3) - \left[\frac{(n+1)}{4}\right].$$

Next, if n is not a prime, let $p(n)$ denote the smallest prime factor of n. If $p(n) = p$, then $p < n$ and the product in the first column, namely $n\cdot 2\cdot 1,\ n\cdot 3\cdot 1, \ldots, n\cdot(\frac{n}{p}-1)\cdot 1$ can be written respectively as $\frac{n}{p}\cdot 2p\cdot 1,\ \frac{n}{p}\cdot 3p\cdot 1, \ldots,\ \frac{n}{p}\cdot p(\frac{n}{p}-1)\cdot 1$. Thus at most $(\frac{n}{p}-1)-1$ products possibly repeat. Also $n\cdot 2\cdot 1,\ n\cdot 3\cdot 1, \ldots, n\cdot(\frac{n}{p}-1)\cdot 1$ can also be written respectively as $\frac{n}{p}\cdot 2\cdot p,\ \frac{n}{p}\cdot 3\cdot p, \ldots,\ \frac{n}{p}\cdot(\frac{n}{p}-1)\cdot p$. Again $(\frac{n}{p}-1)-1$ products possibly repeat. Let $p_1(n)$ denote the smallest prime factor of n, $p_2(n)$ denote the second smallest prime factor of n. Let $p_1(n) = p_1$ and $p_2(n) = p_2$. Then the products in first column namely $n\cdot 2\cdot 1,\ n\cdot 3\cdot 1, \ldots, n\cdot(\frac{n}{p}-1)\cdot 1$ can be written respectively as $\frac{n}{p_1\cdot p_2}\cdot 2p_1\cdot p_2,\ \frac{n}{p_1\cdot p_2}\cdot 3p_1\cdot p_2, \ldots,\ \frac{n}{p_1\cdot p_2}\cdot(\frac{n}{p_1}-1)\cdot p_1\cdot p_2$. Thus at most $(\frac{n}{p}-1)-1$ products possibly repeat. Thus in all at most $3[(\frac{n}{p}-1)-1]$ products possibly repeat. Applying similar argument to second column, we find that at most $3[(\frac{n}{p}-1)-2]$ products possibly repeat in the array. Proceeding like this, we find that in the $(\frac{n}{p}-2)^{th}$ column, $3[(\frac{n}{p}-1)-(\frac{n}{p}-2)]$ products possibly repeat in the array. Since the array (1) contains $\frac{(n-1)(n-2)}{2} < \frac{n(n-1)}{2}$ products, we have

$$\begin{aligned}
\delta(n) &\leq \frac{n(n-1)}{2} - 3\sum_{j=1}^{\frac{n}{p}-2}\left(\left(\frac{n}{p}-1\right)-j\right) \\
&= \frac{n(n-1)}{2} - 3\left[\left(\frac{n}{p}-1\right)\left(\frac{n}{p}-2\right) - \frac{1}{2}\left(\frac{n}{p}-2\right)\left(\frac{n}{p}-1\right)\right] \\
&= \frac{n(n-1)}{2} - 3\left[\frac{1}{2}\left(\frac{n}{p}-1\right)\left(\frac{n}{p}-2\right)\right] \\
&\leq \frac{n(n-1)}{2} - \frac{3}{2}\left(\frac{n}{p}-1\right)\left(\frac{n}{p}-2\right).
\end{aligned}$$

□

Theorem 3. *For any strongly 2-multiplicative graph of order n,*

$$\Lambda(n) \leq \frac{(n-1)n(n+1)-6}{6} - \frac{3}{2} \sum_{\substack{i=3, \\ i,\ not\ a\ prime}}^{n} \left(\frac{i}{p(i)}-1\right)\left(\frac{i}{p(i)}-2\right)$$

$$+ \sum_{\substack{i=3, \\ i,\ a\ prime}}^{n} (i-3) - \sum_{\substack{i=3, \\ i,\ a\ prime}}^{n} \left[\frac{i+1}{4}\right] - \sum_{\substack{i=3, \\ i,\ a\ prime}}^{n} \sum_{k=2}^{i-1} \frac{k}{p(k)}.$$

Proof. Since

$$\delta(3) = \Lambda(3)$$

and

$$\delta(n) = \Lambda(n) - \Lambda(n-1), \quad for \quad n \geq 4$$

we have

$$\Lambda(n) = \sum_{i=3}^{n} \delta(i).$$

Using Lemma 1, we obtain

$$\begin{aligned} \Lambda(n) &= \sum_{i=3}^{n} \delta(i) \\ &\leq \sum_{i=3}^{n} \frac{i(i-1)}{2} - \sum_{\substack{i=3, \\ i,\ not\ a\ prime}}^{n} \frac{3}{2}\left(\frac{i}{p(i)}-1\right)\left(\frac{i}{p(i)}-2\right) \\ &\quad + \sum_{\substack{i=3, \\ i,\ a\ prime}}^{n} \left((i-3) - \left[\frac{i+1}{4}\right] - \sum_{k=2}^{i-1} \frac{k}{p(k)}\right), \end{aligned}$$

which leads to the desired bound. □

3. Lower bound for $\Lambda(n)$

In this section, we give a lower bound for $\Lambda(n)$.

Theorem 4. *For any strongly 2-multiplicative graph of order n,*

$$\Lambda(n) \geq 10 + \sum_{\substack{i=6, \\ i,\ not\ a\ prime}}^{n} 3 + \sum_{\substack{i=6, \\ i,\ a\ prime}}^{n} \left((i-1)(i-3) - \sum_{k=1}^{i-4} \left[\frac{k(i-1)}{k+1}\right]\right).$$

Proof. Let $A = \{rst \mid 1 \leq r < s < t \leq n\}$. Then clearly $\Lambda(n) = |A|$. Consider the following $(n-2)$ triangular array of products:

$$\begin{array}{lllll}
3\cdot 2\cdot 1 & & & & \\
 & & & & \\
4\cdot 2\cdot 1 & & & & \\
4\cdot 3\cdot 1 & 4\cdot 3\cdot 2 & & & \\
 & & & & \\
5\cdot 2\cdot 1 & & & & \\
5\cdot 3\cdot 1 & 5\cdot 3\cdot 2 & & & \\
5\cdot 4\cdot 1 & 5\cdot 4\cdot 2 & 5\cdot 4\cdot 3 & & \\
\cdots & & & & \\
\cdots & & & & \\
n\cdot 2\cdot 1 & & & & \\
n\cdot 3\cdot 1 & n\cdot 3\cdot 2 & & & \\
n\cdot 4\cdot 1 & n\cdot 4\cdot 2 & n\cdot 4\cdot 3 & & \\
n\cdot 5\cdot 1 & n\cdot 5\cdot 2 & n\cdot 5\cdot 3 & n\cdot 5\cdot 4 & \\
... & & & & \\
n\cdot(n-1)\cdot 1 & n\cdot(n-1)\cdot 2 & n\cdot(n-1)\cdot 3 & \cdots & n\cdot(n-1)\cdot(n-2).
\end{array}$$

Now, let A_3 be set of products in the first triangular array of products, A_4 be set of products in the second triangular array of products, A_5 be set of products in the third triangular array of products. Next if k is not a prime and $6 \leq k \leq n$, let A_k be the set of products in the k^{th} triangular array of products which are greater than $(k-3)(k-2)(k-1)$. If k is a prime and $6 < k \leq n$, let A_k be the set of distinct products in the following triangular array of numbers.

$$\begin{array}{lllll}
2\cdot 1 & & & & \\
3\cdot 1 & 3\cdot 2 & & & \\
4\cdot 1 & 4\cdot 2 & 4\cdot 3 & & \\
5\cdot 1 & 5\cdot 2 & 5\cdot 3 & 5\cdot 4 & \\
\cdots & & & & \\
\cdots & & & & \\
(k-1)\cdot 1 & (k-1)\cdot 2 & (k-1)\cdot 3 & \cdots & (k-1)\cdot(k-2).
\end{array}$$

Then clearly $A_i \bigcap A_j = \emptyset$, for $i \neq j$, $3 \leq i, j \leq n$ and $A_i \subset A$ for $3 \leq i \leq n$. Hence $\Lambda(n) = |A| \geq \sum_{i=3}^{n} |A_i|$. Now, one can see that $|A_3| = 1$, $|A_4| = 3$, $|A_5| = 6$. If k is not a prime and $6 \leq k \leq n$, since the products in the k^{th} triangular array which are greater than $(k-3)(k-2)(k-1)$ are $k(k-3)(k-2)$, $k(k-3)(k-1)$, $k(k-2)(k-1)$, we have $|A_k| = 3$. If k is a prime then by the Theorem 1 we have

$$|A_k| \geq (k-1)(k-3) - \sum_{i=1}^{k-4} \left[\frac{i(k-1)}{i+1}\right].$$

Hence,

$$\begin{aligned}\Lambda(n) &= |A| \\ &\geq \sum_{i=3}^{n} |A_i| \\ &\geq 1+3+6+\sum^{n}_{\substack{i=6, \\ i \text{ is not a prime}}} 3 \; + \\ &\quad \sum^{n}_{\substack{i=6, \\ i \text{ is a prime}}} \left((i-1)(i-3) - \sum_{k=1}^{i-4} \left[\frac{k(i-1)}{k+1}\right]\right),\end{aligned}$$

and this leads to the desired bound. □

The following table gives the values of $\Lambda(n)$, upper bounds for $\Lambda(n)$ and lower bounds for $\Lambda(n)$.

Table 1

n	3	4	5	6	7	8	9	10
$\Lambda(n)$	1	4	10	16	29	43	62	77
Upper bounds for $\Lambda(n)$	1	7	14	26	41	60	93	120
lower bounds for $\Lambda(n)$	1	4	10	13	26	29	32	35

4. Some strongly 2-multiplicative graphs

Definition 2. A ladder L_n is a graph with vertex set $V(L_n) = \{v_i : 1 \leq i \leq 2n\}$ and edge set $E(L_n) = \{v_{2i}v_{2i+2}, v_{2i-1}v_{2i+1} : 1 \leq i \leq n-1\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq n\}$.

Definition 3. A triangular ladder is a graph T_n, whose vertex set is $V(T_n) = \{v_i : 1 \leq i \leq 2n\}$ and whose edge set is $E(T_n) = E(L_n) \bigcup \{v_{2i}v_{2i+1} : 1 \leq i \leq n-1\}$.

We first note that for a graph to be strongly 2-multiplicative, it has to have at least 3 vertices.

Theorem 5. *The triangular ladder graph T_n is strongly 2-multiplicative.*

Proof. Consider the triangular ladder graph T_n with vertex set $V(T_n) = \{v_1, v_2, v_3, \ldots, v_{2n}\}$, as shown below.

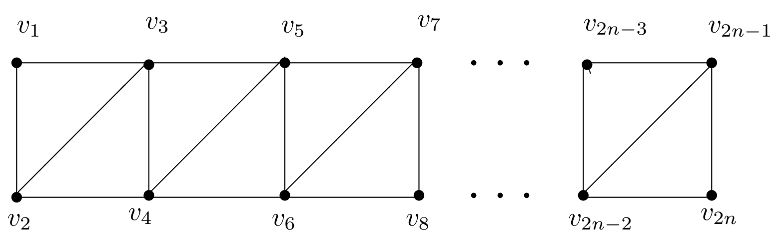

Figure 1

Then $\mathcal{A}$
consists of $8n-12$ distinct path homotopy classes $\mathcal{P}_{2i,2i-1,2i+1}$, $\mathcal{P}_{2i-1,2i,2i+2}$, $\mathcal{P}_{2i-1,2i+1,2i+2}$, $\mathcal{P}_{2i-1,2i+1,2i+3}$, $\mathcal{P}_{2i,2i+2,2i+4}$, $\mathcal{P}_{2i,2i+1,2i+3}$, $\mathcal{P}_{2i,2i+2,2i+1}$, $\mathcal{P}_{2i,2i+2,2i+3}$ corresponding to path homotopy classes of paths having vertex sets $\{v_{2i}, v_{2i-1}, v_{2i+1}\}$, $\{v_{2i-1}, v_{2i}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i}, v_{2i+2}, v_{2i+4}\}$, $\{v_{2i}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i}, v_{2i+2}, v_{2i+1}\}$ and $\{v_{2i}, v_{2i+2}, v_{2i+3}\}$ respectively, for $1 \le i \le n-2$ and path homotopy classes $\mathcal{P}_{2n-2,2n-3,2n-1}$, $\mathcal{P}_{2n-3,2n-2,2n}$, $\mathcal{P}_{2n-3,2n-1,2n}$, $\mathcal{P}_{2n-2,2n,2n-1}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{2n-2}, v_{2n-3}, v_{2n-1}\}$, $\{v_{2n-3}, v_{2n-2}, v_{2n}\}$, $\{v_{2n-3}, v_{2n-1}, v_{2n}\}$ and $\{v_{2n-2}, v_{2n}, v_{2n-1}\}$ respectively. We label the vertices as follows: v_i=i, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$. Since $(2i)\cdot(2i-1)\cdot(2i+1) < (2i-1)\cdot(2i)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+3) < (2i)\cdot(2i+2)\cdot(2i+1) < (2i)\cdot(2i+1)\cdot(2i+3) < (2i)\cdot(2i+2)\cdot(2i+3) < (2i)\cdot(2i+2)\cdot(2i+4)$, $(2i)\cdot(2i+2)\cdot(2i+4) < (2i+2)\cdot(2i+1)\cdot(2i+3)$, for $1 \le i \le n-2$ and $(2i)\cdot(2i-1)\cdot(2i+1) < (2i-1)\cdot(2i)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i)\cdot(2i+2)\cdot(2i+1)$ for $i = n-1$ it follows that $h(\mathcal{P}_{2,1,3}) < h(\mathcal{P}_{1,2,4}) < \cdots < h(\mathcal{P}_{2n-2,2n,2n-1})$. Therefore h is injective and the graph T_n is strongly 2-multiplicative. □

Definition 4. A crown product of G_1 and G_2, denoted by $G = G_1 \odot G_2$ and is defined as follows: Fix a vertex v in G_2. Take $|V(G_1)|$ copies of G_2 and attach the i-th copy of G_2 to the i-th vertex of G_1 by identifying the vertex v in the i-th copy of G_2 with the i-th vertex of G_1.

Theorem 6. *The graph $P_2 \odot C_n$ is strongly 2-multiplicative.*

Proof. Consider the cycles $C_n = (v_1, v_2, v_3, \ldots, v_n, v_1)$ and $C_n' = (v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}, v_{n+1})$ such that the path $P_2 = v_n v_{n+1}$ which joins C_n and C_n'. Let p_1 be the largest prime less than n and p_2 be the largest prime such that $n < p_2 < 2n$. We label the vertices as follows $v_i = i$, for $1 \le i \le p_1 - 1$ and $p_2 + 1 \le i \le 2n$, $v_i = i-1$, for $n+2 \le i \le p_2$, $v_i = i+1$, for $p_1 \le i < n$, $v_n = p_1$, $v_{n+1} = p_2$. Then for $n = 3$, $\mathcal{A}$ consists of 6 distinct path homotopy classes and for $n \ge 4$, $\mathcal{A}$ consists of $2n+4$ distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, ..., \mathcal{P}_{2n}$, $\mathcal{P}_{2n+1}$, $\mathcal{P}_{2n+2}$, $\mathcal{P}_{2n+3}$, $\mathcal{P}_{2n+4}$, where $\mathcal{P}_i$, for $1 \le i \le n-2, n+1 \le i \le 2n-2$ is the path homotopy class of paths having the vertex set $\{v_i, v_{i+1}, v_{i+2}\}$ and $\mathcal{P}_{n-1}$, $\mathcal{P}_n$, $\mathcal{P}_{2n-1}$, $\mathcal{P}_{2n}$, $\mathcal{P}_{2n+1}$, $\mathcal{P}_{2n+2}$, $\mathcal{P}_{2n+3}$ and $\mathcal{P}_{2n+4}$ are the path homotopy classes of paths having the vertex set $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$, $\{v_{2n-1}, v_{2n}, v_{n+1}\}$, $\{v_{2n}, v_{n+1}, v_{n+2}\}$, $\{v_{n-1}, v_n, v_{n+1}\}$, $\{v_1, v_n, v_{n+1}\}$, $\{v_n, v_{n+1}, v_{n+2}\}$, $\{v_n, v_{n+1}, v_{2n}\}$. Then $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $1 \le i \le p_1 - 3$, $h(\mathcal{P}_{p_1-2}) = (p_1-2)(p_1-1)(p_1+1)$, $h(\mathcal{P}_{p_1-1}) = (p_1-1)(p_1+1)(p_1+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p_1 \le i \le n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p_1)$ or $h(\mathcal{P}_{n-2}) = (n-2).n.p_1$, if p_1 is the prime which is immediate predecessor of n, $h(\mathcal{P}_{n-1}) = n\cdot p_1 \cdot 1$, $h(\mathcal{P}_n) = p_1 \cdot 1 \cdot 2$, $h(\mathcal{P}_{n+1})$=$p_2(n+1)(n+2)$ or $h(\mathcal{P}_{n+1}) = (n+1)\cdot(n+3)\cdot p_2$, if p_2 is the prime which is immediate successor of $(n+1)$, $h(\mathcal{P}_i) = (i-1)(i)(i+1)$, for $n+2 \le i \le p_2 - 2$, $h(\mathcal{P}_{p_2-1}) = (p_2-2)(p_2-1)(p_2+1)$, $h(\mathcal{P}_{p_2}) = (p_2-1)(p_2+1)(p_2+2)$, $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $p_2+1 \le i \le 2n-2$,

$h(\mathcal{P}_{2n-1}) = (2n-1)(2n)(p_2)$ or $h(\mathcal{P}_{2n-1}) = (2n-2)\cdot 2n \cdot p_2$, if p_2 is the prime which is immediate predecessor of 2n, $h(\mathcal{P}_{2n}) = 2n \cdot p_2 \cdot (n+1)$, $h(\mathcal{P}_{2n+1}) = n \cdot p_1 \cdot p_2$, $h(\mathcal{P}_{2n+2}) = 1 \cdot p_1 \cdot p_2$, $h(\mathcal{P}_{2n+3}) = p_1 \cdot p_2 \cdot (n+1)$, $h(\mathcal{P}_{2n+4}) = p_1 \cdot p_2 \cdot 2n$. Then it follows from the definition that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, $1 \le i \le n-4$, $n+2 \le i \le 2n-3$ and $h(\mathcal{P}_n) < h(\mathcal{P}_{n-1}) < h(\mathcal{P}_{n-2})$, $h(\mathcal{P}_{n+1}) < h(\mathcal{P}_{2n}) < h(\mathcal{P}_{2n-1})$ and $h(\mathcal{P}_{2n+2}) < h(\mathcal{P}_{2n+1}) < h(\mathcal{P}_{2n+3}) < h(\mathcal{P}_{2n+4})$. Also $h(\mathcal{P}_i) \neq h(\mathcal{P}_j)$, for $n-2 \le j \le n$ and $1 \le i \le n-3$, $n+1 \le i \le 2n$, since $h(\mathcal{P}_j)$ is divisible by p_1, whereas $h(\mathcal{P}_i)$ is not. Further $h(\mathcal{P}_i) \neq h(\mathcal{P}_j)$ for $j = 2n, 2n-1, n+1$ and $1 \le i \le n$, $n+2 \le i \le 2n-2$, since $h(\mathcal{P}_j)$ is divisible by p_2, whereas $h(\mathcal{P}_i)$ is not. Finally $h(\mathcal{P}_{2n+1})$, $h(\mathcal{P}_{2n+2})$, $h(\mathcal{P}_{2n+3})$, $h(\mathcal{P}_{2n+4})$ are not equal to any other, since these are divisible by $p_1 \cdot p_2$ and others are not. Therefore h is injective and the graph is strongly 2-multiplicative. □

Definition 5. Duplication of an edge $e = v_i v_{i+1}$ by a vertex w in a graph G produces a new graph $G^{'}$ such that $N(w) = \{v_i, v_{i+1}\}$, the set of vertices which are adjacent to w.

Theorem 7. *The graph obtained by duplication of an arbitrary edge by new vertex in path P_n is strongly 2-multiplicative.*

Proof. Consider the path P_n with vertex set $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$. We duplicate the edge e with end vertices v_{n-1} and v_n by an vertex v_{n+1}. Let the graph obtained by duplication of arbitrary edge by new vertex is G. Then $|V(G)| = n+1$, as shown below.

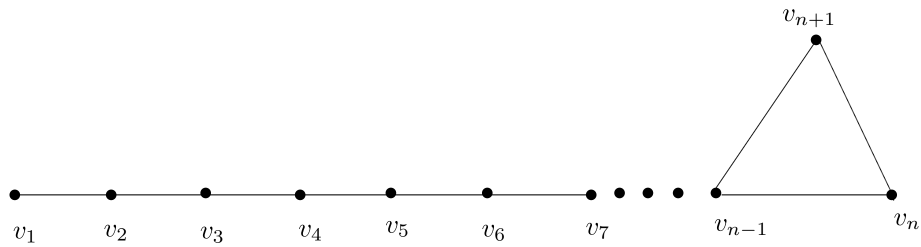

Figure 2

Let p be the largest prime less than n. We label the vertices as follows: $v_i = i$ for $1 \le i \le p-1$ and for $i = n+1$, $v_i = i+1$, for $p \le i \le n-1$ and $v_n = p$. Then $\mathcal{A}$ consists of n distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_n$, where $\mathcal{P}_i$, for $1 \le i \le n-2$ is the path homotopy class of paths having the vertex set $\{v_i, v_{i+1}, v_{i+2}\}$, $\mathcal{P}_{n-1}$, $\mathcal{P}_n$, are the path homotopy classes of paths having the vertex set $\{v_{n-2}, v_{n-1}, v_{n+1}\}$, $\{v_{n-1}, v_n, v_{n+1}\}$. Then $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $1 \le i \le p-3$, $h(\mathcal{P}_{p-2}) = (p-2)(p-1)(p+1)$, $h(\mathcal{P}_{p-1}) = (p-1)(p+1)(p+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p \le i \le n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p)$ or $h(\mathcal{P}_{n-2}) = (n-2)\cdot n \cdot p$, if p is the prime which is immediate predecessor of n, $h(\mathcal{P}_{n-1}) = (n-1)\cdot n \cdot (n+1)$ or $h(\mathcal{P}_{n-1}) = (n-2)\cdot n \cdot (n+1)$, if p is the prime which is immediate predecessor of n, $h(\mathcal{P}_n) = p \cdot n \cdot (n+1)$. Then from the definition it follows that $h(\mathcal{P}_1) < h(\mathcal{P}_2) < h(\mathcal{P}_3) < ... < h(\mathcal{P}_{n-3}) < h(\mathcal{P}_{n-1})$ and

$h(\mathcal{P}_{n-2}) < h(\mathcal{P}_n)$. $h(\mathcal{P}_i) \neq h(\mathcal{P}_j)$ for $1 \leq i \leq n-3$ and $i = (n-1)$, $n-2 \leq j \leq n$, $j \neq (n-1)$. Since $h(\mathcal{P}_j)$ is divisible by p, whereas $h(\mathcal{P}_i)$ is not, h is injective and the graph is strongly 2-multiplicative. □

Definition 6. Duplication of a vertex v by a new edge $e = uw$ in a graph G produces a new graph G' such that $N(u)=\{v, w\}$ and $N(w)= \{v, u\}$.

Theorem 8. *The graph obtained by duplicating all vertices by new edges in path P_n is strongly 2-multiplicative.*

Proof. Consider the graph G obtained by duplicating all vertices by new edges in path P_n with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_{3n}\}$ as shown below.

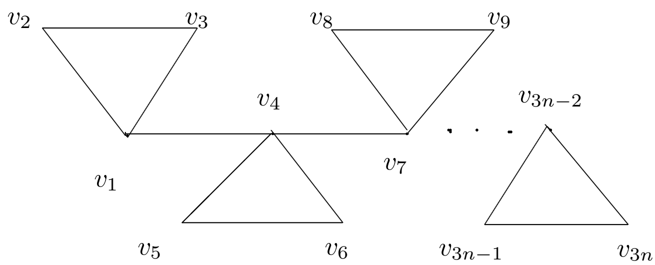

Figure 3

Then $\mathcal{A}$ consists of $6n-6$ distinct path homotopy classes $\mathcal{P}_{3i-2,3i-1,3i}$, $\mathcal{P}_{3i-2,3i-1,3i+1}$, $\mathcal{P}_{3i-2,3i,3i+1}$, $\mathcal{P}_{3i-2,3i+1,3i+2}$, $\mathcal{P}_{3i-2,3i+1,3i+3}$, $\mathcal{P}_{3i-2,3i+1,3i+4}$, corresponding to path homotopy classes of paths having vertex sets $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$, $\{v_{3i-2}, v_{3i-1}, v_{3i+1}\}$, $\{v_{3i-2}, v_{3i}, v_{3i+1}\}$, $\{v_{3i-2}, v_{3i+1}, v_{3i+2}\}$, $\{v_{3i-2}, v_{3i+1}, v_{3i+3}\}$ and $\{v_{3i-2}, v_{3i+1}, v_{3i+4}\}$ respectively, for $1 \leq i \leq n-2$ and path homotopy classes $\mathcal{P}_{3n-5,3n-4,3n-3}$, $\mathcal{P}_{3n-4,3n-5,3n-2}$, $\mathcal{P}_{3n-3,3n-5,3n-2}$, $\mathcal{P}_{3n-1,3n-2,3n-5}$, $\mathcal{P}_{3n,3n-2,3n-5}$, $\mathcal{P}_{3n-1,3n-2,3n}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{3n-5}, v_{3n-4}, v_{3n-3}\}$, $\{v_{3n-4}, v_{3n-5}, v_{3n-2}\}$, $\{v_{3n-3}, v_{3n-5}, v_{3n-2}\}$, $\{v_{3n-1}, v_{3n-2}, v_{3n-5}\}$, $\{v_{3n}, v_{3n-2}, v_{3n-5}\}$ and $\{v_{3n-1}, v_{3n-2}, v_{3n}\}$ respectively.
We label the vertices as follows: $v_i = i$, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$. Since $(3i-2)\,(3i-1)\,(3i) < (3i-2)\,(3i-1)\,(3i+1) < (3i-2)\,(3i)\,(3i+1) < (3i-2)(3i+1)(3i+2) < (3i-2)(3i+1)(3i+3) < (3i-2)(3i+1)(3i+4)$ and $(3i-2)(3i+1)(3i+4) < (3i+1)(3i+2)(3i+3)$ for $1 \leq i \leq n-2$ and $(3n-5)(3n-4)(3n-3) < (3n-4)(3n-5)(3n-2) < (3n-3)(3n-5)(3n-2) < (3n-1)(3n-2)(3n-5) < (3n)(3n-2)(3n-5) < (3n-1)(3n-2)(3n)$ it follows that $h(\mathcal{P}_{2,1,3}) < h(\mathcal{P}_{2,1,4}) < \cdots < h(\mathcal{P}_{3n-2,3n-1,3n})$. Therefore h is injective and the graph G is strongly 2-multiplicative. □

Theorem 9. *The graph $P_m \odot P_n$ is strongly 2-multiplicative*

Proof. Consider the graph $P_m \bigodot P_n$ of order mn with vertex set $V(P_m \bigodot P_n) = \{v_1, v_2, v_3, v_4, \ldots, v_{mn}\}$ as shown below.

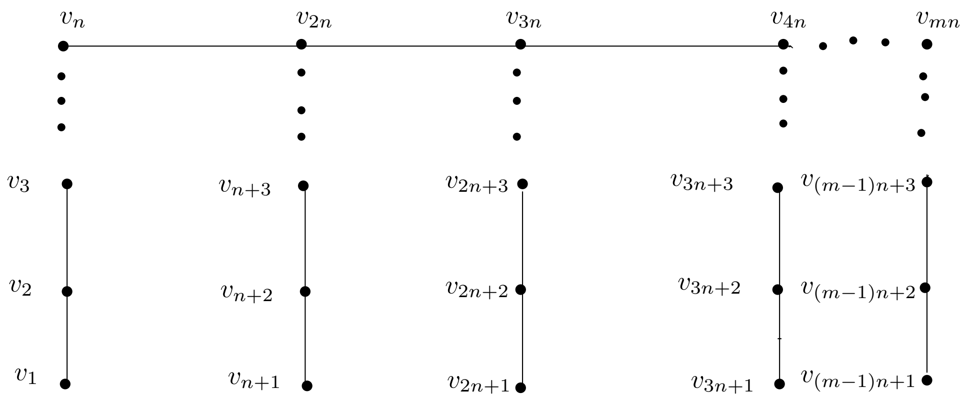

Figure 4

The graph $P_m \bigodot P_n$ is a path for $1 \leq m \leq 2$ which is strongly 2-multiplicative. If $m \geq 3$, then $\mathcal{A}$ consists of $mn + m - 4$ distinct path homotopy classes $\mathcal{P}_{i,j}$ corresponding to path homotopy classes of paths having vertex set $\{v_{i+j}, v_{i+j+1}, v_{i+j+2}\}$ for i=$0, n, 2n, 3n, ..., (m-1)n$ and $1 \leq j \leq n-2$, path homotopy classes $\mathcal{P}_{i,i-1,i+n}$, $\mathcal{P}_{i,i+n,(i+n)-1}$, $\mathcal{P}_{i,i+n,i+2n}$ corresponding to path homotopy class of paths having the vertex sets $\{v_i, v_{i-1}, v_{i+n}\}$ $\{v_i, v_{i+n}, v_{(i+n)-1}\}$, $\{v_i, v_{i+n}, v_{i+2n}\}$ for i=$n, 2n, 3n, ..., (m-2)n$ and $\mathcal{P}_{(m-1)n,(m-1)n-1,mn}$, $\mathcal{P}_{(m-1)n,mn,mn-1}$ corresponding to path homotopy class of paths having the vertex sets $\{v_{(m-1)n}, v_{(m-1)n-1}, v_{mn}\}$, $\{v_{(m-1)n}, v_{mn}, v_{(mn-1)}\}$. We label the vertices as follows: $v_i = i$, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$ and $h(\mathcal{P}_{i,j}) = (i+j) \cdot (i+j+1) \cdot (i+j+2)$. Since $(i) \cdot (i-1) \cdot (i+n) < (i) \cdot (i+n) \cdot ((i+n)-1) < (i) \cdot (i+n) \cdot (i+2n)$, $(i) \cdot (i+n) \cdot (i+2n) < (i+n) \cdot ((i+n)-1) \cdot (i+2n)$, for $i = n, 2n, 3n, ..., (m-2)n$ and $(m-1)n \cdot (m-1)n - 1 \cdot (mn) < (m-1)n \cdot (mn) \cdot (mn-1)$ it follows that $h(\mathcal{P}_{n-1,n,2n}) < h(\mathcal{P}_{n,2n,2n-1}) < \cdots < h(\mathcal{P}_{(m-1)n,mn,mn-1})$ and $h(\mathcal{P}_{0,1}) < h(\mathcal{P}_{0,2}) < \cdots < h(\mathcal{P}_{(m-1)n,mn-2})$. $h(\mathcal{P}_{i,j,k}) \neq h(\mathcal{P}_{i,j})$ because $h(\mathcal{P}_{i,j})$ have the factors (i) and $(i+n)$, whereas $h(\mathcal{P}_{i,j,k})$ do not. Therefore h is injective and graph $P_m \bigodot P_n$ is strongly 2-multiplicative. $\square$

Acknowledgement: The first author is thankful to University Grants Commission, India for the financial support under the grant No. F. 510/12/DRS-II/2018(SAP-I). The authors are thankful to the unknown referee for his valuable suggestions which has considerably improved the quality of the paper.

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