On strongly 2-multiplicative graphs

D.D. Somashekara¹, H.E. Ravi¹,∗ and C.R. Veena²

¹Department of Studies in Mathematics, University of Mysore
Manasagangotri, Mysore-570006, India
dsomashekara@yahoo.com ravihemath64@gmail.com

²Department of Mathematics, JSS College of Arts, Commerce and Science
Mysore-570025, India
veenacr.maths@gmail.com

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Abstract: A simple connected graph G of order n ≥ 3 is a strongly 2-multiplicative if there is an injection f : V(G) → {1, 2, . . . , n} such that the induced mapping h : A → ℤ⁺ defined by h(P) = \prod_{i=1}^{3} f(v_{j_i}), where j₁, j₂, j₃ ∈ {1, 2, . . . , n}, and P is the path homotopy class of paths having the vertex set \{v_{j₁}, v_{j₂}, v_{j₃}\}, is injective. Let Λ(n) be the number of distinct path homotopy classes in a strongly 2-multiplicative graph of order n. In this paper we obtain an upper bound and also a lower bound for Λ(n). Also we prove that triangular ladder, P₂ ⨀ Cₙ, Pₘ ⨀ Pₙ, the graph obtained by duplication of an arbitrary edge by a new vertex in path Pₙ and the graph obtained by duplicating all vertices by new edges in a path Pₙ are strongly 2-multiplicative.

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1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. During the past five decades thousands of research papers on graph labelings and their applications have been published. Many of the methods on graph labelings are motivated by certain practical problems. For more details one may refer the survey article by Gallian [7].

In 2001, Beineke and Hegde [6] have introduced the concept of strongly multiplicative graphs. A graph with n vertices is said to be strongly multiplicative if there is an injection f : V → {1, 2, . . . , n} such that the induced mapping f∗ : E(G) → ℤ defined

∗ Corresponding Author

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by \( f^*(e) = f(u)f(v) \), where \( e = uv \), is injective. They have proved that certain classes of graphs are strongly multiplicative. They have also obtained an upper bound for \( \lambda(n) \), the maximum number of edges for a given strongly multiplicative graph of order \( n \). In [2], Adiga, Ramaswamy and Somashekara have given a lower bound for \( \lambda(n) \) and proved that the complete bipartite graph \( K_{n,n} \) is strongly multiplicative if and only if \( n \leq 4 \). In [3], Adiga, Ramaswamy and Somashekara have given a formula for \( \lambda(n) \) and proved that every wheel is strongly multiplicative. In [4], Adiga, Ramaswamy and Somashekara have given an upper bound for \( \lambda(n) \), improving the upper bound obtained by Beineke and Hegde [6]. Seoud and Zid [9], Adiga and Smitha [5], Acharya, Germina and Ajitha [1], Vaidya and Kanani [11] and Muthusamy, Raajasekar and Basker Babujee [8] are among many others who contributed to the concept of strongly multiplicative graphs.

Motivated by this Somashekara, Veena and Ravi [10] have introduced the concept of strongly \( k \)-multiplicative graphs as follows: Consider a simple connected graph \( G \) of order \( n \). Let \( P_1 \) and \( P_2 \) be two paths in \( G \) with the same vertex set \( S \). Then we say that \( P_1 \) and \( P_2 \) are path homotopic with respect to \( S \). We denote this by \( P_1 \simeq_S P_2 \). One can easily prove that this relation is an equivalence relation. Let \( P \) be the path homotopy class consisting of those paths which are path homotopic to the path \( P \) with a given vertex set and let \( \mathcal{A} \) denote the set of all distinct path homotopy classes in \( G \).

**Definition 1.** A simple connected graph \( G \) of order \( n \) is said to be strongly \( k \)-multiplicative if there is an injective mapping \( f : V(G) \to \{1, 2, \ldots, n\} \) such that the induced mapping \( h : \mathcal{A} \to \mathbb{Z}^+ \) defined by \( h(P) = \prod_{i=1}^{k+1} f(v_{j_i}) \), where \( j_1, j_2, \ldots, j_{k+1} \in \{1, 2, \ldots, n\} \), \( k+1 \leq n \) and \( \mathcal{P} \) is the path homotopy class of paths having the vertex set \( \{v_{j_1}, v_{j_2}, \ldots, v_{j_{k+1}}\} \), is injective.

In particular if \( k=2 \) we call \( G \), strongly 2- multiplicative and if \( k = 1 \), then \( G \) is strongly 1-multiplicative or strongly multiplicative.

The following results of Adiga, Ramaswamy and Somashekara [2, 4] will be used in section 2.

**Theorem 1.** ([2]) If \( \lambda(n) \) denotes the maximum number of edges in a strongly multiplicative graph of order \( n \), then

\[
\lambda(n) \geq n(n-2) - \sum_{k=1}^{n-3} \left\lfloor \frac{kn}{k+1} \right\rfloor, n \geq 4
\]

where \([x]\) denotes the largest integer less than or equal to \( x \).
Theorem 2. ([4]) If \( \lambda(n) \) denotes the maximum number of edges in a strongly multiplicative graph of order \( n \), then

\[
\lambda(n) \leq \frac{n(n+1)}{2} - \sum_{i=2}^{n} \frac{i}{p(i)} + (n - 2) - \left\lfloor \frac{(n+2)}{4} \right\rfloor
\]

where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \).

In [10], Somashekara, Veena and Ravi have proved that certain classes of graphs like path, cycle, ladder etc., are strongly 2-multiplicative. In this paper we obtain an upper bound and also a lower bound for \( \Lambda(n) \), where \( \Lambda(n) \) is the number of distinct path homotopy classes in a strongly 2-multiplicative graph of order \( n \). It is easy to see that \( \Lambda(n) = |\{rst | 1 \leq r < s < t \leq n\}| \). Also we prove that triangular ladder, \( P_2 \odot C_n \), \( P_m \odot P_n \), the graph obtained by duplication of an arbitrary edge by a new vertex in path \( P_n \) and the graph obtained by duplicating all vertices by new edges in a path \( P_n \) are strongly 2-multiplicative.

2. Upper bound for \( \Lambda(n) \)

Let \( \delta(3) = \Lambda(3) \) and \( \delta(n) = \Lambda(n) - \Lambda(n - 1), n \geq 4 \), the number of new products one can get by going from \( n - 1 \) to \( n \) as the largest factor. We first give an upper bound for \( \delta(n) \) in the following Lemma.

Lemma 1. Let \( p(n) = p \) denote the least prime divisor of \( n \). Then

\[
\delta(n) \leq \left\{ \begin{array}{ll}
\frac{n(n-1)}{2} - \sum_{i=2}^{n-1} \frac{i}{p(i)} + (n - 3) - \left\lfloor \frac{(n+1)}{4} \right\rfloor, & \text{if } n \text{ is a prime} \\
\frac{n(n-1)}{2} - \frac{3}{2}(n - 1)(\frac{n}{p} - 2), & \text{if } n \text{ is not a prime,}
\end{array} \right.
\]

where \( \lfloor x \rfloor \) denotes the largest integer less than or equal to \( x \).

Proof. To find an upper bound for \( \delta(n) \), we need to consider, an array of new products having \( n \) as one of the factors, as shown below.

\[
\begin{align*}
n \cdot 2 \cdot 1 \\
n \cdot 3 \cdot 1 & \quad n \cdot 3 \cdot 2 \\
n \cdot 4 \cdot 1 & \quad n \cdot 4 \cdot 2 & \quad n \cdot 4 \cdot 3 \\
n \cdot 5 \cdot 1 & \quad n \cdot 5 \cdot 2 & \quad n \cdot 5 \cdot 3 & \quad n \cdot 5 \cdot 4 \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
n \cdot (n - 1) \cdot 1 & \quad n \cdot (n - 1) \cdot 2 & \quad n \cdot (n - 1) \cdot 3 & \quad \cdots & \quad n \cdot (n - 1) \cdot (n - 2).
\end{align*}
\]
Now, if $n$ is a prime, then the number of distinct products in (1) is same as the number of distinct products in the array

\[
\begin{array}{cccc}
2 & 1 & 3 & 1 \\
3 & 1 & 4 & 1 \\
5 & 1 & 5 & 2 \\
\vdots & & \vdots & \\
(n-1) & (n-1) & (n-1) & (n-2) \\
\end{array}
\]

(2)

But the number of distinct products in (2) is the cardinality of the set \(\{st \mid 1 \leq s < t \leq n-1\}\). Hence by the Theorem 2, it follows that

\[
\delta(n) \leq \frac{n(n-1)}{2} - \sum_{i=2}^{n-1} \frac{i}{p(i)} + (n-3) - \left\lfloor \frac{(n+1)}{4} \right\rfloor.
\]

Next, if $n$ is not a prime, let $p(n)$ denote the smallest prime factor of $n$. If $p(n) = p$, then $p < n$ and the product in the first column, namely $n \cdot 2 \cdot 1, n \cdot 3 \cdot 1, \ldots, n \cdot (\frac{n}{p} - 1) \cdot 1$ can be written respectively as $\frac{n}{p} \cdot 2p \cdot 1, \frac{n}{p} \cdot 3p \cdot 1, \ldots, \frac{n}{p} \cdot p(\frac{n}{p} - 1) \cdot 1$. Thus at most $(\frac{n}{p} - 1) - 1$ products possibly repeat. Also $n \cdot 2 \cdot 1, n \cdot 3 \cdot 1, \ldots, n \cdot (\frac{n}{p} - 1) \cdot 1$ can also be written respectively as $\frac{n}{p} \cdot 2 \cdot p, \frac{n}{p} \cdot 3 \cdot p, \ldots, \frac{n}{p} \cdot (\frac{n}{p} - 1) \cdot p$. Again $(\frac{n}{p} - 1) - 1$ products possibly repeat. Let $p_1(n)$ denote the smallest prime factor of $n$, $p_2(n)$ denote the second smallest prime factor of $n$. Let $p_1(n) = p_1$ and $p_2(n) = p_2$. Then the products in first column namely $n \cdot 2 \cdot 1, n \cdot 3 \cdot 1, \ldots, n \cdot (\frac{n}{p} - 1) \cdot 1$ can be written respectively as $\frac{n}{p_1p_2} \cdot 2p_1 \cdot p_2, \frac{n}{p_1p_2} \cdot 3p_1 \cdot p_2, \ldots, \frac{n}{p_1p_2} \cdot (\frac{n}{p_1} - 1) \cdot p_1 \cdot p_2$. Thus at most $(\frac{n}{p} - 1) - 1$ products possibly repeat. Thus in all at most $3[(\frac{n}{p} - 1) - 1]$ products possibly repeat. Applying similar argument to second column, we find that at most $3[(\frac{n}{p} - 1) - 2]$ products possibly repeat in the array. Proceeding like this, we find that in the $((\frac{n}{p} - 2))$th column, $3[(\frac{n}{p} - 1) - (\frac{n}{p} - 2)]$ products possibly repeat in the array. Since the array (1) contains $\frac{(n-1)(n-2)}{2} < \frac{n(n-1)}{2}$ products, we have

\[
\delta(n) \leq \frac{n(n-1)}{2} - 3 \sum_{j=1}^{\frac{n}{p}-2} \left( \frac{n}{p} - 1 - j \right)
\]

\[
= \frac{n(n-1)}{2} - 3 \left( \frac{n}{p} - 1 \right) \left( \frac{n}{p} - 2 \right) - \frac{1}{2} \left( \frac{n}{p} - 2 \right) \left( \frac{n}{p} - 1 \right)
\]

\[
= \frac{n(n-1)}{2} - 3 \left( \frac{1}{2} \left( \frac{n}{p} - 1 \right) \left( \frac{n}{p} - 2 \right) \right)
\]

\[
\leq \frac{n(n-1)}{2} - \frac{3}{2} \left( \frac{n}{p} - 1 \right) \left( \frac{n}{p} - 2 \right).
\]
Theorem 3. For any strongly 2-multiplicative graph of order $n$, 

$$\Lambda(n) \leq \frac{(n-1)n(n+1) - 6}{6} - \frac{3}{2} \sum_{i=3, \text{not a prime}}^{n} \left( \frac{i}{p(i)} - 1 \right) \left( \frac{i}{p(i)} - 2 \right)$$

$$+ \sum_{i=3, \text{a prime}}^{n} (i - 3) - \sum_{i=3, \text{a prime}}^{n} \left[ \frac{i+1}{4} \right] - \sum_{i=3, \text{a prime}}^{n} \sum_{k=2}^{i-1} \frac{k}{p(k)}.$$

Proof. Since 

$$\delta(3) = \Lambda(3)$$

and 

$$\delta(n) = \Lambda(n) - \Lambda(n-1), \text{ for } n \geq 4$$

we have 

$$\Lambda(n) = \sum_{i=3}^{n} \delta(i).$$

Using Lemma 1, we obtain 

$$\Lambda(n) = \sum_{i=3}^{n} \delta(i)$$

$$\leq \sum_{i=3}^{n} \frac{i(i-1)}{2} - \sum_{i=3, \text{not a prime}}^{n} \frac{3}{2} \left( \frac{i}{p(i)} - 1 \right) \left( \frac{i}{p(i)} - 2 \right)$$

$$+ \sum_{i=3, \text{a prime}}^{n} (i - 3) - \left[ \frac{i+1}{4} \right] - \sum_{k=2}^{i-1} \frac{k}{p(k)},$$

which leads to the desired bound. \qed

3. Lower bound for $\Lambda(n)$

In this section, we give a lower bound for $\Lambda(n)$.

Theorem 4. For any strongly 2-multiplicative graph of order $n$, 

$$\Lambda(n) \geq 10 + \sum_{i=6, \text{not a prime}}^{n} 3 + \sum_{i=6, \text{a prime}}^{n} \left( (i-1)(i-3) - \sum_{k=1}^{i-4} \left[ \frac{k(i-1)}{k+1} \right] \right).$$
Proof. Let \( A = \{rst \mid 1 \leq r < s < t \leq n\} \). Then clearly \( \Lambda(n) = |A| \). Consider the following \((n-2)\) triangular array of products:

\[
\begin{align*}
3 \cdot 2 \cdot 1 \\
4 \cdot 2 \cdot 1 & \quad 4 \cdot 3 \cdot 2 \\
5 \cdot 2 \cdot 1 & \quad 5 \cdot 3 \cdot 2 & \quad 5 \cdot 4 \cdot 3 \\
\ldots & \quad \ldots & \quad \ldots \\
n \cdot 2 \cdot 1 & \quad n \cdot 3 \cdot 2 & \quad n \cdot 4 \cdot 3 & \quad n \cdot 5 \cdot 4 \\
\ldots & \quad \ldots \\
n \cdot (n-1) \cdot 1 & \quad n \cdot (n-1) \cdot 2 & \quad n \cdot (n-1) \cdot 3 & \ldots & \quad n \cdot (n-1) \cdot (n-2).
\end{align*}
\]

Now, let \( A_3 \) be set of products in the first triangular array of products, \( A_4 \) be set of products in the second triangular array of products, \( A_5 \) be set of products in the third triangular array of products. Next if \( k \) is not a prime and \( 6 \leq k \leq n \), let \( A_k \) be the set of products in the \( k^{th} \) triangular array of products which are greater than \((k-3)(k-2)(k-1)\). If \( k \) is a prime and \( 6 < k \leq n \), let \( A_k \) be the set of distinct products in the following triangular array of numbers.

\[
\begin{align*}
2 \cdot 1 \\
3 \cdot 1 & \quad 3 \cdot 2 \\
4 \cdot 1 & \quad 4 \cdot 2 & \quad 4 \cdot 3 \\
5 \cdot 1 & \quad 5 \cdot 2 & \quad 5 \cdot 3 & \quad 5 \cdot 4 \\
\ldots & \quad \ldots \\
(k-1) \cdot 1 & \quad (k-1) \cdot 2 & \quad (k-1) \cdot 3 & \ldots & \quad (k-1) \cdot (k-2).
\end{align*}
\]

Then clearly \( A_i \cap A_j = \emptyset \), for \( i \neq j, 3 \leq i, j \leq n \) and \( A_i \subset A \) for \( 3 \leq i \leq n \). Hence \( \Lambda(n) = |A| \geq \sum_{i=3}^{n} |A_i| \). Now, one can see that \( |A_3| = 1, |A_4| = 3, |A_5| = 6 \). If \( k \) is not a prime and \( 6 \leq k \leq n \), since the products in the \( k^{th} \) triangular array which are greater than \((k-3)(k-2)(k-1)\) are \( k(k-3)(k-2), k(k-3)(k-1), k(k-2)(k-1) \), we have \( |A_k| = 3 \). If \( k \) is a prime then by the Theorem 1 we have

\[
|A_k| \geq (k-1)(k-3) - \sum_{i=1}^{k-4} \left\lfloor \frac{i(k-1)}{i+1} \right\rfloor.
\]
Hence,
\begin{align*}
\Lambda(n) &= |A| \\
& \geq \sum_{i=3}^{n} |A_i| \\
& \geq 1 + 3 + 6 + \sum_{i \text{ is not a prime}}^{n} 3 + \sum_{i \text{ is a prime}}^{n} (i - 1)(i - 3) - \sum_{k=1}^{i-4} \left[ \frac{k(i-1)}{k+1} \right],
\end{align*}
and this leads to the desired bound.

The following table gives the values of $\Lambda(n)$, upper bounds for $\Lambda(n)$ and lower bounds for $\Lambda(n)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda(n)$</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>16</td>
<td>29</td>
<td>43</td>
<td>62</td>
<td>77</td>
</tr>
<tr>
<td>Upper bounds for $\Lambda(n)$</td>
<td>1</td>
<td>7</td>
<td>14</td>
<td>26</td>
<td>41</td>
<td>60</td>
<td>93</td>
<td>120</td>
</tr>
<tr>
<td>Lower bounds for $\Lambda(n)$</td>
<td>1</td>
<td>4</td>
<td>10</td>
<td>13</td>
<td>26</td>
<td>29</td>
<td>32</td>
<td>35</td>
</tr>
</tbody>
</table>

4. Some strongly 2-multiplicative graphs

**Definition 2.** A ladder $L_n$ is a graph with vertex set $V(L_n) = \{v_i : 1 \leq i \leq 2n\}$ and edge set $E(L_n) = \{v_i v_{i+2}, v_{2i-1} v_{2i+1} : 1 \leq i \leq n - 1\} \cup \{v_{2i-1} v_{2i} : 1 \leq i \leq n\}$.

**Definition 3.** A triangular ladder is a graph $T_n$, whose vertex set is $V(T_n) = \{v_i : 1 \leq i \leq 2n\}$ and whose edge set is $E(T_n) = E(L_n) \cup \{v_{2i-1} v_{2i+1} : 1 \leq i \leq n - 1\}$.

We first note that for a graph to be strongly 2-multiplicative, it has to have at least 3 vertices.

**Theorem 5.** The triangular ladder graph $T_n$ is strongly 2-multiplicative.

**Proof.** Consider the triangular ladder graph $T_n$ with vertex set $V(T_n) = \{v_1, v_2, v_3, \ldots, v_{2n}\}$, as shown below.

![Diagram of the triangular ladder graph](Figure 1)
Then $\mathcal{A}$ consists of $8n - 12$ distinct path homotopy classes $P_{2i,2i-1,2i+1}$, $P_{2i-1,2i,2i+2}$, $P_{2i-1,2i+1,2i+2}$, $P_{2i,2i+1,2i+2}$, $P_{2i,2i+2,2i+1}$, $P_{2i,2i+3,2i+1}$, $P_{2i,2i+3,2i+2}$, $P_{2i,2i+3,2i+3}$ corresponding to path homotopy classes of paths having vertex sets $\{v_{2i}, v_{2i-1}, v_{2i+1}\}$, $\{v_{2i-1}, v_{2i}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i}, v_{2i+2}, v_{2i+4}\}$, $\{v_{2i}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i}, v_{2i+2}, v_{2i+1}\}$ and $\{v_{2i}, v_{2i+2}, v_{2i+3}\}$ respectively, for $1 \leq i \leq n - 2$ and path homotopy classes $P_{2n-2,2n-3,2n-1}$, $P_{2n-3,2n-2,2n}$, $P_{2n-3,2n-1,2n}$, $P_{2n-2,2n,2n-1}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{2n-2}, v_{2n-3}, v_{2n-1}\}$, $\{v_{2n-3}, v_{2n-2}, v_{2n}\}$, $\{v_{2n-3}, v_{2n-1}, v_{2n}\}$ and $\{v_{2n-2}, v_{2n}, v_{2n-1}\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i$. Then $h(P_{i,j,k}) = i \cdot j \cdot k$. Since $(2i) \cdot (2i - 1) < (2i) \cdot (2i + 1) < (2i - 1) \cdot (2i + 1)$, $P_{i,j,k}$ is strongly $2$-multiplicative.

**Definition 4.** A crown product of $G_1$ and $G_2$, denoted by $G = G_1 \circ G_2$ and is defined as follows: Fix a vertex $v$ in $G_2$. Take $|V(G_1)|$ copies of $G_2$ and attach the $i$-th copy of $G_2$ to the $i$-th vertex of $G_1$ by identifying the vertex $v$ in the $i$-th copy of $G_2$ with the $i$-th vertex of $G_1$.

**Theorem 6.** The graph $P_2 \circ C_n$ is strongly $2$-multiplicative.

**Proof.** Consider the cycles $C_n = (v_1, v_2, v_3, \ldots, v_n, v_1)$ and $C_n' = (v_{n+1}, v_{n+2}, v_{n+3}, \ldots, v_{2n}, v_{n+1})$ such that the path $P_2 = v_nv_{n+1}$ which joins $C_n$ and $C_n'$. Let $p_1$ be the largest prime less than $n$ and $p_2$ be the largest prime such that $n < p_2 < 2n$. We label the vertices as follows: $v_i = i$, for $1 \leq i \leq p_1 - 1$ and $p_2 + 1 \leq i \leq 2n$, $v_i = i - 1$, for $n + 2 \leq i \leq p_2$, $v_i = i + 1$, for $p_1 \leq i < n$, $v_n = p_1$, $v_{n+1} = p_2$. Then for $n = 3$, $\mathcal{A}$ consists of 6 distinct path homotopy classes and for $n \geq 4$, $\mathcal{A}$ consists of $2n + 4$ distinct path homotopy classes $P_{1,2}, P_{2,3}, \ldots, P_{2n, 2n-2}$, $P_{2n+1, 2n+2}, P_{2n+3, 2n+4}$, where $P_i$, for $1 \leq i \leq n - 2, n + 1 \leq i \leq 2n - 2$ is the path homotopy class of paths having the vertex set $\{v_1, v_{i+1}, v_{i+2}\}$ and $P_{n-1, n}, P_{n, n-1}, P_{2n, P_{2n-1}, P_{2n+1}, P_{2n+2, 2n+3}}$ and $P_{2n+4}$ are the path homotopy classes of paths having the vertex set $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$, $\{v_{2n-1}, v_{2n}, v_{2n+1}\}$, $\{v_{2n}, v_{n+1}, v_{n+2}\}$, $\{v_{n-1}, v_{n+1}, v_{n+2}\}$, $\{v_1, v_n, v_{n+1}\}$, $\{v_n, v_{n+1}, v_{n+2}\}$, $\{v_n, v_{n+1}, v_{2n}\}$. Then $h(P_i) = (i)(i+1)(i+2)$, for $1 \leq i \leq p_1 - 1$, $h(P_{p_1-2}) = (p_1 - 2)(p_1 - 1)(p_1 + 1)$, $h(P_{p_1-1}) = (p_1 - 1)(p_1 + 1)(p_1 + 2)$, $h(P_i) = (i)(i+1)(i+2)$, for $p_1 \leq i \leq n - 3$, $h(P_{n-2}) = (n-1)n(p_1)$, $h(P_{n-2}) = (n-2)n(p_1)$, if $p_1$ is the prime which is immediate predecessor of $n$. $h(P_{n-1}) = n \cdot p_1 \cdot 1$, $h(P_n) = p_1 \cdot 2$, $h(P_{n+1}) = p_2(n+1)(n+2)$ or $h(P_{n+1}) = (n+1)(n+3)p_2$, if $p_2$ is the prime which is immediate successor of $(n+1)$, $h(P_1) = (i)(i+1)(i+2)$, for $n + 2 \leq i \leq p_2 - 2$, $h(P_{p_2-1}) = (p_2 - 2)(p_2 - 1)(p_2 + 1)$, $h(P_{p_2}) = (p_2 - 1)(p_2 + 1)(p_2 + 2)$, $h(P_i) = (i)(i+1)(i+2)$, for $p_2 + 1 \leq i \leq 2n - 2$,
$h(P_{2n-1}) = (2n - 1)(2n)(p_2)$ or $h(P_{2n-1}) = (2n - 2) \cdot 2n \cdot p_2$, if $p_2$ is the prime which is immediate predecessor of $2n$, $h(P_{2n}) = 2n \cdot p_2 \cdot (n + 1)$, $h(P_{2n+1}) = n \cdot p_1 \cdot p_2$, $h(P_{2n+2}) = 1 \cdot p_1 \cdot p_2$, $h(P_{2n+3}) = p_1 \cdot p_2 \cdot (n + 1)$, $h(P_{2n+4}) = p_1 \cdot p_2 \cdot 2n$.

Then it follows from the definition that $h(P_i) < h(P_{i+1})$, $1 \leq i \leq n - 4$, $n + 2 \leq i \leq 2n - 3$ and $h(P_n) < h(P_{n-1}) < h(P_{n-2})$, $h(P_{n+1}) < h(P_{2n}) < h(P_{2n-1})$ and $h(P_{2n+2}) < h(P_{2n+1}) < h(P_{2n+3}) < h(P_{2n+4})$. Also $h(P_i) \neq h(P_j)$, for $n - 2 \leq j \leq n$ and $1 \leq i \leq n - 3$, $n + 1 \leq i \leq 2n$, since $h(P_j)$ is divisible by $p_1$, whereas $h(P_i)$ is not. Further $h(P_i) \neq h(P_j)$ for $j = 2n, 2n - 1, n + 1$ and $1 \leq i \leq n$, $n + 2 \leq i \leq 2n - 2$, since $h(P_j)$ is divisible by $p_2$, whereas $h(P_i)$ is not. Finally $h(P_{2n+1}), h(P_{2n+2}), h(P_{2n+3}), h(P_{2n+4})$ are not equal to any other, since these are divisible by $p_1 \cdot p_2$ and others are not. Therefore $h$ is injective and the graph is strongly 2-multiplicative.

**Definition 5.** Duplication of an edge $e = v_i, v_{i+1}$ by a vertex $w$ in a graph $G$ produces a new graph $G'$ such that $N(w) = \{v_i, v_{i+1}\}$, the set of vertices which are adjacent to $w$.

**Theorem 7.** The graph obtained by duplication of an arbitrary edge by new vertex in path $P_n$ is strongly 2-multiplicative.

**Proof.** Consider the path $P_n$ with vertex set $V(P_n) = \{v_1, v_2, v_3, ..., v_n\}$. We duplicate the edge $e$ with end vertices $v_{n-1}$ and $v_n$ by an vertex $v_{n+1}$. Let the graph obtained by duplication of arbitrary edge by new vertex is $G$. Then $|V(G)| = n + 1$, as shown below.

![Figure 2](image-url)

Let $p$ be the largest prime less than $n$. We label the vertices as follows: $v_i = i$ for $1 \leq i \leq p - 1$ and for $i = n + 1$, $v_i = i + 1$, for $p \leq i \leq n - 1$ and $v_n = p$. Then $A$ consists of $n$ distinct path homotopy classes $P_1, P_2, P_3, ..., P_n$, where $P_i$, for $1 \leq i \leq n - 2$ is the path homotopy class of paths having the vertex set $\{v_i, v_{i+1}, v_{i+2}\}$, $P_{n-1}$, $P_n$, are the path homotopy classes of paths having the vertex set $\{v_{n-2}, v_{n-1}, v_{n+1}\}$, $\{v_{n-1}, v_n, v_{n+1}\}$. Then $h(P_i) = (i)(i+1)(i+2)$, for $1 \leq i \leq p - 3$, $h(P_{p-2}) = (p-2)(p-1)(p+1)$, $h(P_{p-1}) = (p-1)(p+1)(p+2)$, $h(P_i) = (i+1)(i+2)(i+3)$, for $p \leq i \leq n - 3$, $h(P_{n-2}) = (n-1)(n)(p)$ or $h(P_{n-2}) = (n-2) \cdot n \cdot p$, if $p$ is the prime which is immediate predecessor of $n$, $h(P_{n-1}) = (n-1) \cdot n \cdot (n + 1)$ or $h(P_{n-1}) = (n-2) \cdot n \cdot (n + 1)$, if $p$ is the prime which is immediate predecessor of $n$, $h(P_n) = p \cdot n \cdot (n + 1)$. Then from the definition it follows that $h(P_1) < h(P_2) < h(P_3) < ... < h(P_{n-3}) < h(P_{n-1})$ and

```plaintext
\[
\begin{align*}
\frac{\ln(2n)}{\ln(2)} & \leq \int_{2}^{2n} \frac{1}{\ln(x)} \, dx \\
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\frac{\ln(2n)}{\ln(2)} & \leq \int_{2}^{2n} \frac{1}{\ln(x)} \, dx \\
\end{align*}
\]```
Since $h(P_{n-2}) < h(P_n)$, $h(P_i) \neq h(P_j)$ for $1 \leq i \leq n-3$ and $i = (n-1)$, $n-2 \leq j \leq n$, $j \neq (n-1)$. Since $h(P_j)$ is divisible by $p$, whereas $h(P_i)$ is not, $h$ is injective and the graph is strongly 2-multiplicative.

**Definition 6.** Duplication of a vertex $v$ by a new edge $e = uw$ in a graph $G$ produces a new graph $G'$ such that $N(u) = \{v, w\}$ and $N(w) = \{v, u\}$.

**Theorem 8.** The graph obtained by duplicating all vertices by new edges in path $P_n$ is strongly 2-multiplicative.

**Proof.** Consider the graph $G$ obtained by duplicating all vertices by new edges in path $P_n$ with vertex set $V(G) = \{v_1, v_2, v_3, \ldots, v_{3n}\}$ as shown below.

![Diagram of graph $G$ with vertices labeled $v_1$ to $v_{3n}$ and edges connecting them in a specific pattern.](attachment:image.png)

Then $A$ consists of $6n - 6$ distinct path homotopy classes $P_{3i-2,3i-1,3i}$, $P_{3i-2,3i-1,3i+1}$, $P_{3i-2,3i,3i+1}$, $P_{3i-2,3i+1,3i+2}$, $P_{3i-2,3i+1,3i+3}$, $P_{3i-2,3i+1,3i+4}$, corresponding to path homotopy classes of paths having vertex sets $\{v_{3i-2}, v_{3i-1}, v_{3i}\}$, $\{v_{3i-2}, v_{3i-1}, v_{3i+1}\}$, $\{v_{3i-2}, v_{3i}, v_{3i+1}\}$, $\{v_{3i-2}, v_{3i+1}, v_{3i+2}\}$, $\{v_{3i-2}, v_{3i+1}, v_{3i+3}\}$ and $\{v_{3i-2}, v_{3i+1}, v_{3i+4}\}$ respectively, for $1 \leq i \leq n - 2$ and path homotopy classes $P_{3n-5,3n-4,3n-3}$, $P_{3n-4,3n-5,3n-2}$, $P_{3n-3,3n-5,3n-2}$, $P_{3n-1,3n-2,3n-5}$, $P_{3n-3,3n-2,3n-5}$, $P_{3n-1,3n-2,3n}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{3n-5}, v_{3n-4}, v_{3n-3}\}$, $\{v_{3n-4}, v_{3n-5}, v_{3n-2}\}$, $\{v_{3n-3}, v_{3n-5}, v_{3n-2}\}$, $\{v_{3n-1}, v_{3n-2}, v_{3n-5}\}$, $\{v_{3n}, v_{3n-2}, v_{3n-5}\}$ and $\{v_{3n-1}, v_{3n-2}, v_{3n}\}$ respectively.

We label the vertices as follows: $v_i = i$, for all $i$. Then $h(P_{i,j,k}) = i \cdot j \cdot k$. Since $(3i - 2)$ $(3i - 1)$ $(3i) < (3i - 2)$(3i) $(3i + 1) < (3i - 2)$(3i) $(3i + 1)$ $(3i + 2) < (3i - 2)(3i + 1)(3i + 2) < (3i - 2)(3i + 1)(3i + 3) < (3i - 2)(3i + 1)(3i + 4)$ and $(3i - 2)(3i + 1)(3i + 4) < (3i + 1)(3i + 2)(3i + 3)$ for $1 \leq i \leq n - 2$ and $(3n - 5)(3n - 4)(3n - 3) < (3n - 4)(3n - 5)(3n - 2) < (3n - 3)(3n - 5)(3n - 2) < (3n - 1)(3n - 2)(3n - 5) < (3n)(3n - 2)(3n - 5) < (3n - 1)(3n - 2)(3n)$ it follows that $h(P_{2,1,3}) < h(P_{2,1,4}) < \cdots < h(P_{3n-2,3n-1,3n})$. Therefore $h$ is injective and the graph $G$ is strongly 2-multiplicative.

**Theorem 9.** The graph $P_m \odot P_n$ is strongly 2-multiplicative.
Proof. Consider the graph $P_m \odot P_n$ of order $mn$ with vertex set $V(P_m \odot P_n) = \{v_1, v_2, v_3, v_4, \ldots, v_{mn}\}$ as shown below.

![Graph](image)

Figure 4

The graph $P_m \odot P_n$ is a path for $1 \leq m \leq 2$ which is strongly 2-multiplicative. If $m \geq 3$, then $\mathcal{A}$ consists of $mn + m - 4$ distinct path homotopy classes $\mathcal{P}_{i,j}$ corresponding to path homotopy classes of paths having vertex set $\{v_i, v_{i+1}, v_{i+2}\}$ for $i=0, n, 2n, 3n, \ldots, (m-1)n$ and $1 \leq j \leq n-2$, path homotopy classes $\mathcal{P}_{i-1,i+n}$, $\mathcal{P}_{i,i+n,(i+n)-1}$, $\mathcal{P}_{i,i+n,i+2n}$ corresponding to path homotopy class of paths having the vertex sets $\{v_i, v_{i-1}, v_{i+n}\}$, $\{v_i, v_{i+n}, v_i\}$, $\{v_i, v_{i+n}, v_{i+2n}\}$ for $i=n, 2n, 3n, \ldots, (m-2)n$ and $\mathcal{P}_{(m-1)n,(m-1)n-1,mn}$, $\mathcal{P}_{(m-1)n,mn,mn-1}$ corresponding to path homotopy class of paths having the vertex sets $\{v_{(m-1)n}, v_{(m-1)n-1}, v_{mn}\}$, $\{v_{(m-1)n}, v_{mn}, v_{(mn-1)}\}$. We label the vertices as follows: $v_i = i$, for all $i$. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$ and $h(\mathcal{P}_{i,j}) = (i + j) \cdot (i + j + 1) \cdot (i + j + 2)$. Since $(i) \cdot (i-1) \cdot (i+n) < (i) \cdot (i+n) \cdot ((i+n)-1) < (i) \cdot (i+n) \cdot (i+2n)$, $(i) \cdot (i+n) \cdot (i+2n) < (i+n) \cdot ((i+n)-1) \cdot (i+2n)$, for $i = n, 2n, 3n, \ldots, (m-2)n$ and $(m-1)n \cdot (m-1)n-1 \cdot (mn) < (m-1)n \cdot (mn) \cdot (mn-1)$ it follows that $h(\mathcal{P}_{n-1,n,2n}) < h(\mathcal{P}_{n,2n,2n-1}) < \ldots < h(\mathcal{P}_{(m-1)n,mn,mn-1})$ and $h(\mathcal{P}_{0,1}) < h(\mathcal{P}_{0,2}) < \ldots < h(\mathcal{P}_{(m-1)n,mn-2})$. $h(\mathcal{P}_{i,j,k}) \neq h(\mathcal{P}_{i,j})$ because $h(\mathcal{P}_{i,j})$ have the factors $(i)$ and $(i+n)$, whereas $h(\mathcal{P}_{i,j,k})$ do not. Therefore $h$ is injective and graph $P_m \odot P_n$ is strongly 2-multiplicative.

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