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Twin signed total Roman domatic numbers in digraphs

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Abstract: Let D be a finite simple digraph with vertex set $V(D)$ and arc set $A(D)$. A twin signed total Roman dominating function (TSTRDF) on the digraph D is a function $f : V(D) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N^-(v)} f(x) \geq 1$ and $\sum_{x\in N^+(v)} f(x) \geq 1$ for each $v \in V(D)$, where $N^-(v)$ (resp. $N^+(v)$) consists of all in-neighbors (resp. out-neighbors) of v, and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v and an out-neighbor w with $f(v) = f(w) = 2$. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct twin signed total Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a twin signed total Roman dominating family (of functions) on D. The maximum number of functions in a twin signed total Roman dominating family on D is the twin signed total Roman domatic number of D, denoted by $d^*_{stR}(D)$. In this paper, we initiate the study of the twin signed total Roman domatic number in digraphs and present some sharp bounds on $d^*_{stR}(D)$. In addition, we determine the twin signed total Roman domatic number of some classes of digraphs.

Keywords: twin signed total Roman dominating function, twin signed total Roman domination number, twin signed total Roman domatic number, directed graph

AMS Subject classification: 05C69

1. Introduction

In this paper we continue the study of signed total Roman dominating functions in graphs and digraphs. Let G be a finite and simple graph with vertex set $V(G)$, and let $N_G(v) = N(v)$ be the open neighborhood of the vertex v. A *signed total Roman dominating function* (STRDF) on a graph G is defined in [12] as a function $f : V(G) \longrightarrow \{-1, 1, 2\}$ such that $\sum_{x\in N(v)} f(x) \geq 1$ for each $v \in V(G)$, and every vertex $u \in V(G)$ for which $f(u) = -1$ is adjacent to a vertex v with $f(v) = 2$. The *weight* of an STRDF f is the value $\omega(f) = \sum_{v\in V(G)} f(v)$. The *signed total Roman domination number* $\gamma_{stR}(G)$ of G is the minimum weight of an STRDF on G. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed total Roman dominating functions on G

with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed total Roman dominating family* (of functions) on G. The maximum number of functions in a signed total Roman dominating family (STRD family) on G is the *signed total Roman domatic number* of G, denoted by $d_{stR}(G)$. This parameter was introduced and investigated in [11]. Following this idea, we initiate the study of twin signed total Roman domatic numbers on digraphs.

Let D be a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). The integers $n = n(D) = |V(D)|$ and $m = m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. A digraph without directed cycles of length 2 is an *oriented graph*. An oriented graph D is called a tournament when either $(u, v) \in A(D)$ or $(v, u) \in A(D)$ for each pair of distinct vertices $u, v \in V(D)$. By D^{-1} we denote the digraph obtained by reversing all arcs of D. If (u, v) is an arc of D, then we also write $u \to v$, and we say that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex v, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. Let $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$. We write $d^+(v) = d_D^+(v)$ for the out-degree of a vertex v and $d^-(v) = d_D^-(v)$ for its in-degree. The *minimum* and *maximum in-degree* and *minimum* and *maximum out-degree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is *r-out-regular* (*r-in-regular*) if $\delta^+(D) = \Delta^+(D) = r$ ($\delta^-(D) = \Delta^-(D) = r$). In addition, let $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ and $\Delta = \Delta(D) = \max\{\Delta^+(D), \Delta^-(D)\}$ be the *minimum* and *maximum degree* of D, respectively. A digraph D is called *regular* or *r-regular* if $\delta(D) = \Delta(D) = r$. For a real-valued function $f : V \longrightarrow \mathbb{R}$ the weight of f is $\omega(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $\omega(f) = f(V)$. Consult [7] for the notation and terminology which are not defined here.

A *signed total Roman dominating function* (abbreviated STRDF) on D is defined in [13] as a function $f : V \longrightarrow \{-1, 1, 2\}$ such that (i) $f(N^-(v)) = \sum_{x \in N^-(v)} f(x) \geq 1$ for each vertex $v \in V$ and (ii) every vertex u for which $f(u) = -1$ has an in-neighbor v for which $f(v) = 2$. The *signed total Roman domination number* $\gamma_{stR}(D)$ of D is the minimum weight of an STRDF on D. A $\gamma_{stR}(D)$-function is an STRDF on D of weight $\gamma_{stR}(D)$.

In [2], an STRDF of D is called a *twin signed total Roman dominating function* (briefly TSTRDF) if it also is a signed total Roman dominating function of D^{-1}, i.e., $f(N^+(v)) \geq 1$ for every $v \in V$ and every vertex u for which $f(u) = -1$ has an out-neighbor v for which $f(v) = 2$. The *twin signed total Roman domination number* for a digraph D is $\gamma^*_{stR}(D) = \min\{\omega(f) \mid f \text{ is a TSTRDF of } D\}$. A $\gamma^*_{stR}(D)$-function is a twin signed total Roman dominating function on D of weight $\gamma^*_{stR}(D)$. As the assumption $\delta(D) \geq 1$ is necessary, we always assume that when we discuss $\gamma^*_{stR}(D)$, all digraphs involved satisfy $\delta(D) \geq 1$. Since every TSTRDF of D is an STRDF on both D and D^{-1} and since the constant function 1 is a TSTRDF of D, we have

$$\max\{\gamma_{stR}(D), \gamma_{stR}(D^{-1})\} \leq \gamma^*_{stR}(D) \leq n. \tag{1}$$

The corresponding concepts have been defined and studied for twin domination number [3, 9], twin signed domination number [6], twin signed total domination number [4], twin minus domination number [5], twin minus total domination number [10], and twin signed Roman domination number [8].
A set $\{f_1, f_2, \ldots, f_d\}$ of distinct twin signed total Roman dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a *twin signed total Roman dominating family* (of functions) on D. The maximum number of functions in a twin signed total Roman dominating family (TSTRD family) on D is the *twin signed total Roman domatic number* of D, denoted by $d^*_{stR}(D)$. Since the set consisting of the TSTRDF with constant value 1 forms an TSTRD family on D, the twin signed total Roman domatic number is well-defined and

$$d^*_{stR}(D) \geq 1 \tag{2}$$

for all digraphs D. Since every TSTRD family of D is an STRD family on both D and D^{-1}, we have

$$d^*_{stR}(D) \leq \min\{d_{stR}(D), d_{stR}(D^{-1})\}. \tag{3}$$

In this paper, we initiate the study of the twin signed total Roman domatic number in digraphs and present some sharp bounds on $d^*_{stR}(D)$. In addition, we determine the twin signed total Roman domatic number of some classes of digraphs.

An *orientation* of a graph G is an assignment of orientations to its edges. The *associated digraph* G^* of a graph G is obtained by replacing each edge of G by a pair of two mutually opposite oriented edges. Since $N^-_{G^*}(v) = N^+_{G^*}(v) = N_G(v)$ for each $v \in V(G) = V(G^*)$, the following useful observation is valid.

Observation 1. For any graph G, $\gamma_{stR}(G) = \gamma^*_{stR}(G^*)$ and $d_{stR}(G) = d^*_{stR}(G^*)$.

We make use of the following results in this paper.

Propsotion A. ([12]) *If K_n is the complete graph of order $n \geq 3$, then $\gamma_{stR}(K_n) = 3$.*

Propsotion B. ([11]) If $k \geq 0$ is an integer, then $d_{stR}(K_{9k+6}) = 3k + 2$.

Observations 1, Propositions A and B lead to the next results immediately.

Corollary 1. If K^*_n is the complete digraph of order $n \geq 3$, then $\gamma^*_{stR}(K^*_n) = 3$.

Corollary 2. If $k \geq 0$ is an integer, then $d^*_{stR}(K^*_{9k+6}) = 3k + 2$.

Propsotion C. ([11]) *If D is an r-out-regular digraph of order n with $r \geq 1$, then $\gamma_{stR}(D) \geq \lceil n/r \rceil$.*

Inequality (1) and Proposition C imply the next result immediately.

Corollary 3. If D is an r-out-regular or r-in-regular digraph of order n with $r \geq 1$, then $\gamma^*_{stR}(D) \geq \lceil n/r \rceil$.

Propsotion D. ([12]) *Let C_n be a cycle of order $n \geq 3$. Then $\gamma_{stR}(C_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma_{stR}(C_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma_{stR}(C_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.*

Propsotion E. ([11]) *If C_n is a cycle of length $n \geq 3$. Then $d_{stR}(C_n) = 2$, when $n \equiv 0 \pmod 4$ and $d_{stR}(C_n) = 1$ when $n \not\equiv 0 \pmod 4$.*

Corollary 4. Let C^*_n be the associated digraph of cycle C_n of order $n \geq 3$. Then

1. $\gamma^*_{stR}(C^*_n) = n/2$ when $n \equiv 0 \pmod 4$, $\gamma^*_{stR}(C^*_n) = (n+3)/2$ when $n \equiv 1, 3 \pmod 4$ and $\gamma^*_{stR}(C^*_n) = (n+6)/2$ when $n \equiv 2 \pmod 4$.
2. $d^*_{stR}(C^*_n) = 2$ when $n \equiv 0 \pmod 4$ and $d^*_{stR}(C^*_n) = 1$ when $n \not\equiv 0 \pmod 4$.

Propsotion F. ([12]) *If P_n is a path of order $n \geq 3$, then $\gamma_{stR}(P_n) = n/2$ when $n \equiv 0 \pmod 4$, and $\gamma_{stR}(P_n) = \lceil (n+3)/2 \rceil$ otherwise.*

Corollary 5. If P^*_n is the associated digraph of path P_n of order $n \geq 3$, then $\gamma^*_{stR}(P^*_n) = n/2$ when $n \equiv 0 \pmod 4$, and $\gamma^*_{stR}(P^*_n) = \lceil (n+3)/2 \rceil$ otherwise.

Propsotion G. ([13]) *Let $K_{p,p}$ be the complete bipartite graph where $p \geq 1$. Then $\gamma_{stR}(K_{p,p}) = 2$, unless $p = 3$ in which case $\gamma_{stR}(K_{p,p}) = 4$.*

Propsotion H. ([11]) *Let $K_{p,p}$ be the complete bipartite graph where $p \geq 1$. Then $d_{stR}(K_{p,p}) = p$, unless $p = 3$ in which case $d_{stR}(K_{p,p}) = 1$.*

The next result follows immediately from Observation 1 and Propositions G and H.

Corollary 6. Let $K^*_{p,p}$ be the associated digraph of the complete bipartite graph $K_{p,p}$ where $p \geq 1$. Then

1. $\gamma^*_{stR}(K^*_{p,p}) = 2$, unless $p = 3$ in which case $\gamma^*_{stR}(K^*_{p,p}) = 4$.
2. $d^*_{stR}(K^*_{p,p}) = p$, unless $p = 3$ in which case $d^*_{stR}(K^*_{3,3}) = 1$.

Propsotion I. ([1]) If D is a digraph with $\delta^-(D) \geq 1$, then $d_{stR}(D) \leq \delta^-(D)$.

Propsotion J. ([13]) *If D is a digraph of order $n \geq 3$ with $\delta^-(D) \geq 1$, then*

$$\gamma_{stR}(D) \geq 4 + \delta^-(D) - n.$$

Propsotion K. ([2]) *If D is a digraph of order $n \geq 3$ with $\delta(D) \geq 1$, then*

$$\gamma^*_{stR}(D) \geq \frac{3}{2}(1+\sqrt{2n+1})-n.$$

2. Properties of the twin signed total Roman domatic number

In this section we present basic properties of $d^*_{stR}(D)$ and sharp bounds on this parameter. Using Proposition I and inequality (3), we obtain our first bound on $d^*_{stR}(D)$.

Theorem 2. If D is a digraph with $\delta(D) \geq 1$, then

$$d^*_{stR}(D) \leq \delta(D).$$

Corollary 6 (Item 2) shows that Theorem 2 is sharp. Theorem 2 and inequality (2) yield the next result immediately.

Corollary 7. For a digraph D with $\delta(D) = 1$, $d^*_{stR}(D) = 1$.

As we observed in (3), $d^*_{stR}(D) \leq d_{stR}(D)$. Next we show that the difference $d_{stR}(D) - d^*_{stR}(D)$ can be arbitrarily large.

Theorem 3. For every positive integer $k \geq 3$, there exists a digraph D such that

$$d_{stR}(D) - d^*_{stR}(D) \geq 3k+1.$$

Proof. Let $k \geq 3$ be an integer, and let D be the digraph obtained from two copies of K^*_{9k+6}, say G_1, G_2, by adding two new vertices x and y, arcs going from every vertex in $V(G_1) \cup V(G_2)$ to both x and y, and the opposite arcs (x, y) and (y, x). Since $d^+(x) = d^+(y) = 1$, we deduce from Corollary 7 that $d^*_{stR}(D) = 1$.
Let $V(G_j) = \{v^j_1, \ldots, v^j_{9k+6}\}$ for $j \in \{1, 2\}$. For $1 \leq p \leq 3k + 2$ and $j = 1, 2$, define the functions $f^j_1 : V(G_j) \to \{-1, 1, 2\}$ by $f^j_1(v^j_i) = -1$ for $i \in \{1, \ldots, 6k + 3\}$ and $f^j_1(v^j_i) = 2$ for $i \in \{6k + 4, \ldots, 9k + 6\}$, and $f^j_p(v^j_i) = f^j_{p-1}(v^j_{i+3})$ for $2 \leq p \leq 3k + 2$ where the indices are taken modulo $9k + 6$. Clearly $\{f^j_1, f^j_2, \ldots, f^j_{3k+2}\}$ is an STRD family on the digraph G_j for $j = 1, 2$ (see Example 10 [11]). For $1 \leq p \leq 3k + 2$, define $h_p : V(D) \to \{-1, 1, 2\}$ by $h_p(x) = h_p(y) = -1$, $h_p(u) = f^j_p(u)$ if $u \in V(G_j)$ for $j = 1, 2$. Clearly, $\{h_1, h_2, \ldots, h_{3k+2}\}$ is an STRD family of D and hence $d_{stR}(D) \geq 3k + 2$. Thus $d_{stR}(D) - d^*_{stR}(D) \geq 3k + 1$, and the proof is complete. □

Theorem 4. If D is a digraph of order n, then

$$\gamma^*_{stR}(D) \cdot d^*_{stR}(D) \leq n.$$

Moreover, the equality holds if and only if for each each TSTRD family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d^*_{stR}(D)$, each function f_i is a $\gamma^*_{stR}(D)$-function and $\sum_{i=1}^{d} f_i(v) = 1$ for each $v \in V(D)$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a TSTRD family on D with $d = d^*_{stR}(D)$ and let $v \in V(D)$. Then

$$d \cdot \gamma^*_{stR}(D) = \sum_{i=1}^{d} \gamma^*_{stR}(D) \le \sum_{i=1}^{d} \sum_{v \in V(D)} f_i(v) = \sum_{v \in V(D)} \sum_{i=1}^{d} f_i(v) \le \sum_{v \in V(D)} 1 = n.$$

□

Corollaries 1 and 2 demonstrate that Theorem 4 is sharp. Using Proposition G and Corollary 6 for $p \ge 4$, we have a further example which shows the sharpness of Theorem 4. As an application of Theorem 4 for some out-regular or in-regular digraphs we obtain the next result.

Corollary 8. Let D be an r-out-regular or r-in-regular digraph of order n such that n is not a multiple of r, then $d^*_{stR}(D) \le r - 1$.

Proof. Let $n = dr + s$ with integers $d \ge 1$ and $1 \le s \le r - 1$. According to Corollary 3, we have

$$\gamma^*_{stR}(D) \ge \left\lceil \frac{n}{r} \right\rceil = \left\lceil \frac{dr + s}{r} \right\rceil = d + 1.$$

Now Theorem 4 yields

$$d^*_{stR}(D) \le \frac{n}{d+1} < r$$

and therefore $d^*_{stR}(D) \le r - 1$. □

Corollary 6 demonstrates that Corollary 8 is not valid in general.

Corollary 9. If D is an oriented graph of order n such that $\delta(D) \ge 1$, then

$$d^*_{stR}(D) \le \frac{n-2}{2}.$$

Proof. If D is not a tournament or D is non-regular tournament, then $\delta^-(D) + \delta^+(D) \le n - 2$, and hence we deduce from Theorem 2 that

$$d^*_{stR}(D) \le \frac{\delta^-(D) + \delta^+(D)}{2} \le \frac{n-2}{2}.$$

Let now D be a δ-regular tournament. Then D^{-1} is a δ-regular tournament such that $n = 2\delta + 1$. Hence $n \not\equiv 0 \pmod{\delta}$ and it follows from Corollary 8 that

$$d^*_{stR}(D) \le \delta - 1 = \frac{n-3}{2} < \frac{n-2}{2}.$$

□

The next result is an immediate consequence of Corollary 3 and Theorem 4.

Corollary 10. *If D is an r-out-regular or r- in-regular digraph of order n with $r \ge 1$, then $d^*_{stR}(D) \le \frac{n}{2}$*

Corollary 6 is an example which shows that Corollary 10 is sharp for $p \ge 4$. The upper bound on the product $\gamma^*_{stR}(D).d^*_{stR}(D)$ leads to an upper bound on the sum of these two parameters.

Theorem 5. If D is a digraph of order n, then

$$\gamma^*_{stR}(D) + d^*_{stR}(D) \le n + 1.$$

Proof. It follows from Theorem 4 that

$$\gamma^*_{stR}(D) + d^*_{stR}(D) \le \frac{n}{d^*_{stR}(G)} + d^*_{stR}(D). \tag{4}$$

According to (2) and Theorem 2, we have $1 \le d^*_{stR}(G) \le n-1$. Using these bounds, and the fact that the function $g(x) = x + n/x$ is decreasing for $1 \le x \le \sqrt{n}$ and increasing for $\sqrt{n} \le x \le n$, we observe that

$$\gamma^*_{stR}(D) + d^*_{stR}(D) \le \frac{n}{d^*_{stR}(G)} + d^*_{stR}(D) \le \max\{n+1, \frac{n}{n-1} + n - 1\} = n + 1.$$

□

The *complement* $\overline{D}$ of a digraph D is the digraph with vertex set $V(D)$ such that for any two distinct vertices u and v, the arc (u, v) belongs to $\overline{D}$ if and only if (u, v) does not belong to D. Next, we present a so-called Nordhaus-Gaddum type inequality for the twin signed total Roman domination and twin signed total Roman domatic numbers of regular digraphs.

Theorem 6. *If D is an r-regular digraph of order n with $r \ge 1$, then*

$$\gamma^*_{stR}(D) + \gamma^*_{stR}(\overline{D}) \ge \frac{4n}{n-1}.$$

If n is even, then $\gamma^*_{stR}(D) + \gamma^*_{stR}(\overline{D}) \ge \frac{4(n-1)}{n-2}$.

Proof. Since D is r-regular, the complement $\overline{D}$ is $(n-r-1)$-regular. Therefore, if follows from Corollary 3 that

$$\gamma^*_{stR}(D)+\gamma^*_{stR}(\overline{D})\geq n\Big(\frac{1}{r}+\frac{1}{n-r-1}\Big).$$

The conditions $r\geq 1$ and $n-r-1\geq 1$ imply that $1\leq r\leq n-2$. As the function $g(x)=1/x+1/(n-x-1)$ has its minimum for $x=(n-1)/2$ when $1\leq x\leq n-2$, we obtain

$$\gamma^*_{stR}(D)+\gamma^*_{stR}(\overline{D})\geq n\Big(\frac{1}{r}+\frac{1}{n-r-1}\Big)\geq n\Big(\frac{2}{n-1}+\frac{2}{n-1}\Big)=\frac{4n}{n-1},$$

and this is the desired bound. If n is even, then the function g has its minimum for $r=x=(n-2)/2$ or $r=x=n/2$, since r is an integer. Hence this leads to

$$\gamma^*_{stR}(D)+\gamma^*_{stR}(\overline{D})\geq n\Big(\frac{1}{r}+\frac{1}{n-r-1}\Big)\geq n\Big(\frac{2}{n}+\frac{2}{n-2}\Big)=\frac{4(n-1)}{n-2},$$

and the proof is complete. □

Theorem 7. *Let D be a digraph of order n such that $min\{\delta(D),\delta(\overline{D})\}\geq 1$. Then*

$$d^*_{stR}(D)+d^*_{stR}(\overline{D})\leq n-1.$$

Furthermore, if $d^*_{stR}(D)+d^*_{stR}(\overline{D})=n-1$, then D is both in-regular and out-regular.

Proof. It follows from Theorem 2 that

$$\begin{aligned}
d^*_{stR}(D)+d^*_{stR}(\overline{D}) &\leq \min\{\delta^-(D),\delta^+(D)\}+\min\{\delta^-(\overline{D}),\delta^+(\overline{D})\}\\
&\leq \min\{\delta^-(D)+\delta^-(\overline{D}),\delta^+(D)+\delta^+(\overline{D})\}\\
&= \min\{\delta^-(D)+n-1-\Delta^-(D),\delta^+(D)+n-1-\Delta^+(D)\}\\
&= n-1+\min\{\delta^-(D)-\Delta^-(D),\delta^+(D)-\Delta^+(D)\}\\
&\leq n-1
\end{aligned}$$

and the proof of the Nordhaus-Gaddum bound is complete. If D is not in-regular or out-regular, then $\Delta^-(D)-\delta^-(D)\geq 1$ or $\Delta^+(D)-\delta^+(D)\geq 1$, respectively, and hence the above inequality chain implies the better bound $d^*_{stR}(D)+d^*_{stR}(\overline{D})\leq n-2$. □

The next result improves the bound of Theorem 7 for r-regular digraphs of order $n\geq 7$.

Theorem 8. *Let D be an r-regular digraph of order $n\geq 7$ such that $\delta^-(D),\delta^-(\overline{D})\geq 1$. Then $d^*_{stR}(D)+d^*_{stR}(\overline{D})\leq n-2$.*

Proof. Since D is r-regular, $\overline{D}$ is $\overline{r}$-regular such that $r+\overline{r}+1=n$. Assume, without loss of generality, that $\overline{r}\le r$. If $r=1$, then $n-1-r=\overline{r}=1$ that leads to the contradiction $n=3$. Let now $r\ge 2$. If $n=tr+s$ with integers $t\ge 1$ and $1\le s\le r-1$, then Theorem 2 and Corollary 8 lead to $d^*_{stR}(D)+d^*_{stR}(\overline{D})\le (r-1)+\overline{r}=n-2$ as desired. Thus assume that $n=tr$ with an integer $t\ge 2$. As $\overline{r}\le r$, we observe that $tr=n=r+\overline{r}+1\le 2r+1$ and so $t=2$. Therefore $n=2r$ and hence $\overline{r}=n-r-1=2r-r-1=r-1$.
If $\overline{r}=1$, then $r=2$ and $n=4$ which is a contradiction. Therefore, $\overline{r}\ge 2$ and thus $r=\overline{r}+1$. If $n=k\overline{r}+s$ with integers $k\ge 1$ and $1\le s\le \overline{r}-1$, then it follows from Theorem 2 and Corollary 8 that $d^*_{stR}(D)+d^*_{stR}(\overline{D})\le r+\overline{r}-1=n-2$ as desired. Now assume that $n=k\overline{r}$ with an integer $k\ge 2$. Altogether, we have

$$n=2r=k(r-1)$$

with $r\ge 3$. It is straightforward to verify that this identity is only possible for $k=3$ and $r=3$ and thus $\overline{r}=2$ and $n=6$ which is a contradiction. This completes the proof. □

As an application of Corollary 8, we improve Theorem 7 for r-regular digraphs.

Theorem 9. *Let D be an r-regular digraph of order n such that $\delta^-(D),\delta^-(\overline{D})\ge 1$ and $n\not\equiv 0\,(\mathrm{mod}\ (n-1-r)), n\not\equiv 0\,(\mathrm{mod}\ r)$. Then $d_{stR}(D)+d_{stR}(\overline{D})\le n-3$.*

Proof. Since D is an r-regular, the complement $\overline{D}$ is $(n-1-r)$-regular According to Corollary 8 and the hypothesis $n\not\equiv 0\,(\mathrm{mod}\ (n-1-r))$ and $n\not\equiv 0\,(\mathrm{mod}\ r)$, we deduce that $d^*_{stR}(D)+d^*_{stR}(\overline{D})\le r-1+(n-1-r-1)=n-3$ and this is the desired bound. □

Corollary 11. If T is a tournament of odd order $n\ge 3$, then $d^*_{stR}(T)+d^*_{stR}(\overline{T})\le n-3$.

Proof. If T is an r-regular tournament, then $\overline{T}$ is also an r-regular tournament such that $n=2r+1$. It is easy to see that $n\not\equiv 0\,(\mathrm{mod}\ r)$ and $n\not\equiv 0\,(\mathrm{mod}\ (n-1-r))$. According to Theorem 9, $d^*_{stR}(T)+d^*_{stR}(\overline{T})\le n-3$.
Assume now that T is not regular. Then $\delta^-(T)\le (n-3)/2$ and $\delta^-(\overline{T})\le (n-3)/2$, and we deduce from Theorem 2 that

$$d^*_{stR}(T)+d^*_{stR}(\overline{T})\le \delta^-(T)+\delta^-(\overline{T})\le \left(\frac{n-3}{2}\right)+\left(\frac{n-3}{2}\right)=n-3.$$

□

Using Observation 1, Theorems 2, 4, 5 and 7, we obtain the next known results.

Corollary 12. *([11]) Let G be a graph of order n. Then $d_{stR}(G)\le \delta(G)$, $\gamma_{stR}(G)\cdot d_{stR}(G)\le n$, $\gamma_{stR}(G)+d_{stR}(G)\le n+1$ and $d_{stR}(G)+d_{stR}(\overline{G})\le n-1$.*

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