Abstract: Let $k \geq 1$ be an integer, and let $G$ be a finite and simple graph with vertex set $V(G)$. A weak signed Roman $k$-dominating function (WSR$k$DF) on a graph $G$ is a function $f : V(G) \rightarrow \{-1, 1, 2\}$ satisfying the conditions that $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V(G)$, where $N[v]$ is the closed neighborhood of $v$. The weight of a WSR$k$DF $f$ is $w(f) = \sum_{v \in V(G)} f(v)$. The weak signed Roman $k$-domination number $\gamma_{wsR}^k(G)$ of $G$ is the minimum weight of a WSR$k$DF on $G$. In this paper we initiate the study of the weak signed Roman $k$-domination number of graphs, and we present different bounds on $\gamma_{wsR}^k(G)$. In addition, we determine the weak signed Roman $k$-domination number of some classes of graphs. Some of our results are extensions of well-known properties of the signed Roman $k$-domination number $\gamma_{sR}^k(G)$, introduced and investigated by Henning and Volkmann [5] as well as Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1] for the case $k = 1$.

Keywords: Weak signed Roman $k$-dominating function, weak signed Roman $k$-domination number, Signed Roman $k$-dominating function, Signed Roman $k$-domination number

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1. Terminology and introduction

For notation and graph theory terminology, we in general follow Haynes, Hedetniemi and Slater [4]. Specifically, let $G$ be a graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph $G$, respectively. The open neighborhood of a vertex $v$ is $N_G(v) = N(v) = \{u \in V(G) | uv \in E(G)\}$, and the closed neighborhood of $v$ is $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is $d_G(v) = d(v) = |N(v)|$. The minimum and maximum degree of a graph $G$ are denoted by $\delta(G) = \delta$ and $\Delta(G) = \Delta$, respectively. For a set $X \subseteq V(G)$, its open neighborhood is
the set \( N_G(X) = N(X) = \bigcup_{v \in X} N(v) \), and its closed neighborhood is the set \( N_G[X] = N[X] = N(X) \cup X \). The complement of a graph \( G \) is denoted by \( \overline{G} \). For sets \( A, B \subseteq V(G) \), we say that \( A \) dominates \( B \) if \( B \subseteq N[A] \). A leaf of a graph \( G \) is a vertex of degree 1, while a support vertex of \( G \) is a vertex adjacent to a leaf. An edge incident with a leaf is called a pendant edge. The star \( K_{1,t} \) has on vertex of degree \( t \) and \( t \) leaves. A spider is the graph formed by subdividing all edges of a star \( K_{1,t} \). Let \( P_n, C_n \) and \( K_n \) be the path, cycle and complete graph of order \( n \), and let \( K_{p,p} \) be the complete bipartite graph of order \( 2p \).

All along this paper we will assume that \( k \) is a positive integer. In 1985, Fink and Jacobson [3] introduced the concept of \( k \)-dominating sets. A subset \( D \subseteq V(G) \) is a \( k \) dominating set if every vertex in \( V(D) - D \) has at least \( k \) neighbors in \( D \). The minimum cardinality of a \( k \)-dominating set is the \( k \)-domination number, denoted by \( \gamma_k(G) \).

In this paper we continue the study of Roman dominating functions in graphs and digraphs. For a subset \( S \subseteq V(G) \) of vertices of a graph \( G \) and a function \( f : V(G) \to \mathbb{R} \), we define \( f(S) = \sum_{x \in S} f(x) \). For a vertex \( v \), we denote \( f(N[v]) \) by \( f[v] \) for notational convenience.

If \( k \geq 1 \) is an integer, then Henning and Volkmann [5] defined the signed Roman \( k \)-dominating function (SRkDF) on a graph \( G \) as a function \( f : V(G) \to \{-1, 1, 2\} \) such that \( f[v] \geq k \) for every \( v \in V(G) \), and every vertex \( u \) for which \( f(u) = -1 \) is adjacent to a vertex \( v \) for which \( f(v) = 2 \). The weight of an SRkDF \( f \) on a graph \( G \) is \( \omega(f) = \sum_{v \in V(G)} f(v) \). The signed Roman \( k \)-domination number \( \gamma^k_{sR}(G) \) of \( G \) is the minimum weight of an SRkDF on \( G \). The special case \( k = 1 \) was introduced and investigated by Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1]. Sheikholeslami and Volkmann [6] studied the signed Roman domination number in digraphs. A \( \gamma^k_{sR}(G) \)-function is a signed Roman \( k \)-dominating function on \( G \) of weight \( \gamma^k_{sR}(G) \).

A weak signed Roman \( k \)-dominating function (WSRkDF) on a graph \( G \) is defined as a function \( f : V(G) \to \{-1, 1, 2\} \) having the property \( f[v] \geq k \) for every \( v \in V(G) \). The weight of a WSRkDF \( f \) on a graph \( G \) is \( \omega(f) = \sum_{v \in V(G)} f(v) \). The weak signed Roman \( k \)-domination number \( \gamma^k_{wsR}(G) \) of \( G \) is the minimum weight of a WSRkDF on \( G \). The special case \( k = 1 \) was introduced and investigated by Volkmann [7]. A \( \gamma^k_{wsR}(G) \)-function is a weak signed Roman \( k \)-dominating function on \( G \) of weight \( \gamma^k_{wsR}(G) \). For a WSRkDF \( f \) on \( G \), let \( V_i(f) = \{ v \in V(G) : f(v) = i \} \) for \( i = -1, 1, 2 \). A weak signed Roman \( k \)-dominating function \( f : V(G) \to \{-1, 1, 2\} \) can be represented by the ordered partition \( (V_{-1}, V_1, V_2) \) of \( V(G) \).

The weak signed Roman \( k \)-domination number exists when \( \delta \geq \frac{k}{2} - 1 \). Therefore we assume in this paper that \( \delta \geq \frac{k}{2} - 1 \). The definitions lead to \( \gamma^k_{wsR}(G) \leq \gamma^k_{sR}(G) \). Therefore each lower bound of \( \gamma^k_{wsR}(G) \) is also a lower bound of \( \gamma^k_{sR}(G) \) and each upper bound of \( \gamma^k_{sR}(G) \) is also an upper bound of \( \gamma^k_{wsR}(G) \).

Our purpose in this work is to initiate the study of the weak signed Roman \( k \)-
domination number. We present basic properties and sharp bounds on \( \gamma_{wsR}^k(G) \). In particular, we show that many lower bounds on \( \gamma_{sR}^k(G) \) are also valid for \( \gamma_{wsR}^k(G) \). Some of our results are extensions of well-known properties of the signed Roman \( k \)-domination number and the weak signed Roman domination number \( \gamma_{wsR}^1(G) = \gamma_{wsR}^k(G) \), given by Ahangar, Henning, Zhao, Löwenstein and Samodivkin [1], Amjadi, Nazari-Moghaddam, Sheikholeslami and Volkmann [2], Henning and Volkmann [5] and Volkmann [7].

2. Preliminary results

In this section we present basic properties of the weak signed Roman \( k \)-dominating functions and the weak signed Roman \( k \)-domination numbers.

**Proposition 1.** If \( f = (V_{-1}, V_1, V_2) \) is a WSRkDF on a graph \( G \) of order \( n \), then

(a) \( |V_{-1}| + |V_1| + |V_2| = n \).

(b) \( \omega(f) = |V_1| + 2|V_2| - |V_{-1}| \).

(c) \( V_1 \cup V_2 \) is a \( \lceil \frac{k+1}{2} \rceil \)-dominating set.

**Proof.** Since (a) and (b) are immediate, we only prove (c). If \( |V_{-1}| = 0 \) then \( V_1 \cup V_2 = V(G) \) is a \( \lceil \frac{k+1}{2} \rceil \)-dominating set. Let now \( |V_{-1}| \geq 1 \) and let \( v \in V_{-1} \) an arbitrary vertex. Assume that \( v \) has \( j \) neighbors in \( V_1 \) and \( q \) neighbors in \( V_2 \). The condition \( f[v] \geq k \) leads to \( j + 2q - 1 \geq k \) and so \( q \geq \frac{k+1-j}{2} \). This implies

\[
j + q \geq j + \frac{k+1-j}{2} = \frac{k+1+j}{2} \geq \frac{k+1}{2}.
\]

Therefore \( v \) has at least \( j + q \geq \lceil \frac{k+1}{2} \rceil \) neighbors in \( V_1 \cup V_2 \). Since \( v \) was an arbitrary vertex in \( V_{-1} \), we deduce that \( V_1 \cup V_2 \) is a \( \lceil \frac{k+1}{2} \rceil \)-dominating set. \( \square \)

**Corollary 1.** If \( G \) is a graph of order \( n \), then \( \gamma_{wsR}^k(G) \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(G) - n \).

**Proof.** Let \( f = (V_{-1}, V_1, V_2) \) be a \( \gamma_{wsR}^k(G) \)-function. Then it follows from Proposition 1 that

\[
\gamma_{wsR}^k(G) = |V_1| + 2|V_2| - |V_{-1}| = 2|V_1| + 3|V_2| - n \geq 2|V_1 \cup V_2| - n \geq 2\gamma_{\lceil \frac{k+1}{2} \rceil}(G) - n.
\]

The graphs \( K_n \) and \( qK_2 \) show that Corollary 1 is sharp for \( k = 1 \) and \( k = 2 \). The proof of the next proposition is identically with the proof of Proposition 2 in [5] and is therefore omitted.
**Proposition 2.** Assume that \( f = (V_1, V_2, V_3) \) is a WSRkDF on a graph \( G \) of order \( n \), \( \Delta = \Delta(G) \) and \( \delta = \delta(G) \). Then

(i) \( (2\Delta + 2 - k)|V_2| + (\Delta + 1 - k)|V_1| \geq (\delta + k + 1)|V_1| \).

(ii) \( (2\Delta + \delta + 3)|V_2| + (\Delta + \delta + 2)|V_1| \geq (\delta + k + 1)n \).

(iii) \( (\Delta + \delta + 2)\omega(f) \geq (\delta - \Delta + 2k)n + (\delta - \Delta)|V_2| \).

(iv) \( \omega(f) \geq (\delta - 2\Delta + 2k - 1)n/(2\Delta + \delta + 3) + |V_2| \).

3. **Bounds on the weak signed Roman \( k \)-domination number**

We start with a general upper bound, and we characterize all extremal graphs.

**Theorem 1.** Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq \lceil \frac{k}{2} \rceil - 1 \). Then \( \gamma_{wsR}^{k}(G) \leq 2n \), with equality if and only if \( k \) is even, \( \delta(G) = \frac{k}{2} - 1 \), and each vertex of \( G \) is of minimum degree or adjacent to a vertex of minimum degree.

*Proof.* Define the function \( g : V(G) \to \{-1, 1, 2\} \) by \( g(x) = 2 \) for each vertex \( x \in V(G) \). Since \( \delta(G) \geq \lceil \frac{k}{2} \rceil - 1 \), the function \( g \) is a WSRkDF on \( G \) of weight \( 2n \) and thus \( \gamma_{wsR}^{k}(G) \leq 2n \).

Now let \( k \) be even, \( \delta(G) = \frac{k}{2} - 1 \), and assume that each vertex of \( G \) is of minimum degree or adjacent to a vertex of minimum degree. Let \( f \) be a WSRkDF on \( G \), and let \( x \in V(G) \) be an arbitrary vertex. If \( d(x) = \frac{k}{2} - 1 \), then \( f(x) = 2 \). If \( x \) is not of minimum degree, then \( x \) is adjacent to a vertex \( w \) of minimum degree. The condition \( f[w] \geq k \) implies \( f(x) = 2 \). Thus \( f \) is of weight \( 2n \), and we obtain \( \gamma_{wsR}^{k}(G) = 2n \) in this case.

Conversely, assume that \( \gamma_{wsR}^{k}(G) = 2n \). If \( k = 2p + 1 \) is odd, then \( \delta(G) \geq p \). Define the function \( h : V(G) \to \{-1, 1, 2\} \) by \( h(w) = 1 \) for an arbitrary vertex \( w \) and \( h(x) = 2 \) for each vertex \( x \in V(G) \setminus \{w\} \). Then

\[
h[v] = \sum_{x \in N[v]} f(x) \geq 1 + 2\delta(G) \geq 1 + 2p = k
\]

for each vertex \( v \in V(G) \). Thus the function \( h \) is a WSRkDF on \( G \) of weight \( 2n - 1 \), and we obtain the contradiction \( \gamma_{wsR}^{k}(G) \leq 2n - 1 \).

Let now \( k \) even, and assume that there exists a vertex \( w \) with \( d(w) \geq \frac{k}{2} \) and \( d(x) \geq \frac{k}{2} \) for each \( x \in N(w) \). Define the function \( h_{1} : V(G) \to \{-1, 1, 2\} \) by \( h_{1}(w) = 1 \) and \( h_{1}(x) = 2 \) for each vertex \( x \in V(G) \setminus \{w\} \). Then \( h_{1}[v] \geq k + 1 \) for each \( v \in N[w] \) and \( h_{1}[x] \geq k \) for each \( x \not\in N[w] \). Hence the function \( h_{1} \) is a WSRkDF on \( G \) of weight \( 2n - 1 \), a contradiction to the assumption \( \gamma_{wsR}^{k}(G) = 2n \). This completes the proof. \[\square\]

The next result is an immediate corollary of Theorem 1.
Corollary 2. Let $G$ be a graph of order $n$ with $\delta(G) \geq \lceil \frac{k}{2} \rceil - 1$. Then $\gamma^k_{sR}(G) \leq 2n$, with equality if and only if $k$ is even, $\delta(G) = \frac{k}{2} - 1$, and each vertex of $G$ is of minimum degree or adjacent to a vertex of minimum degree.

The next known corollary follows immediately from Corollary 2.

Corollary 3. ([2]) Let $T$ be a tree of order $n$. Then $\gamma^4_{sR}(T) \leq 2n$, with equality if and only if every vertex of $T$ is either a leaf or a support vertex.

Observation 2. If $G$ is a graph of order $n$ with $\delta(G) \geq k-1$, then $\gamma^k_{wsR}(G) \leq n$.

Proof. Define the function $f : V(G) \rightarrow \{-1, 1, 2\}$ by $f(x) = 1$ for each vertex $x \in V(G)$. Since $\delta(G) \geq k-1$, the function $f$ is an SR$k$DF on $G$ of weight $n$ and thus $\gamma^k_{wsR}(G) \leq \frac{kn}{r+1}$.

As an application of Proposition 2 (iii), we obtain a lower bound on the weak signed Roman $k$-domination number for $r$-regular graphs.

Corollary 4. If $G$ is an $r$-regular graph of order $n$ with $r \geq \frac{k}{2} - 1$, then

$$\gamma^k_{wsR}(G) \geq \frac{kn}{r+1}.$$ 

Example 1. If $H$ is a $(k-1)$-regular graph of order $n$, then it follows from Corollary 4 that $\gamma^k_{wsR}(H) \geq n$ and thus $\gamma^k_{wsR}(H) = n$, according to Observation 2.

Example 1 shows that Observation 2 and Corollary 4 are both sharp. The proof of the next observation is analogously to the proof of Proposition 3 in [7] and is therefore omitted.

Observation 3. If $G$ is a graph of order $n$ with $\delta(G) \geq \frac{k}{2} - 1$, then

$$\gamma^k_{wsR}(G) \geq k + 1 + \Delta(G) - n.$$ 

Let $n \geq k \geq 2$ be integers. Then it was shown in [5] that $\gamma^k_{sR}(K_n) = k$. This implies $\gamma^k_{wsR}(K_n) \leq \gamma^k_{sR}(K_n) = k$. According to Corollary 4, we deduce that $\gamma^k_{wsR}(K_n) \geq k$. Therefore we obtain $\gamma^k_{wsR}(K_n) = k$ for $n \geq k \geq 2$. This example shows that Observation 3 is sharp.

Corollary 5. Let $G$ be a graph of order $n$, minimum degree $\delta \geq \frac{k}{2} - 1$ and maximum degree $\Delta$. If $\delta < \Delta$, then

$$\gamma^k_{wsR}(G) \geq \frac{-2\Delta + 2\delta + 3k}{2\Delta + \delta + 3}n.$$
Proof. Multiplying both sides of the inequality in Proposition 2 (iv) by $\Delta - \delta$ and adding the resulting inequality to the inequality in Proposition 2 (iii), we obtain the desired lower bound.

Since $\gamma_{sR}^k(G) \geq \gamma_{wsR}^k(G)$, Corollary 5 leads immediately to the next lower bound, given by Henning and Volkmann [5].

**Corollary 6.** ([5]) Let $G$ be a graph of order $n$, minimum degree $\delta \geq \frac{k}{2} - 1$ and maximum degree $\Delta$. If $\delta < \Delta$, then

$$\gamma_{sR}^k(G) \geq \frac{-2\Delta + 2\delta + 3k}{2\Delta + \delta + 3}n.$$ 

Examples 9 and 10 in [5] demonstrate that Corollary 6 is sharp and therefore Corollary 5 too. The special case $k = 1$ of Corollary 6 can be found in [1].

A set $S \subseteq V(G)$ is a 2-packing of the graph $G$ if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices $u, v \in S$. The 2-packing number $\rho(G)$ of $G$ is defined by

$$\rho(G) = \max\{|S| : S \text{ is a 2-packing of } G\}.$$ 

**Theorem 4.** If $G$ is a graph of order $n$ with $\delta(G) \geq \frac{k}{2} - 1$, then

$$\gamma_{wsR}^k(G) \geq \rho(G)(k + \delta(G) + 1) - n.$$ 

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a 2-packing of $G$, and let $f$ be a $\gamma_{wsR}^k(G)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(G)} N[v_i]$, then $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ is a 2-packing, we have that

$$|A| = \sum_{i=1}^{\rho(G)} (d(v_i) + 1) \geq \rho(G)(\delta(G) + 1).$$

It follows that

$$\gamma_{wsR}^k(G) = \sum_{x \in V(G)} f(x) = \sum_{i=1}^{\rho(G)} f(v_i) + \sum_{x \in V(G) - A} f(x)$$

$$\geq k\rho(G) + \sum_{x \in V(G) - A} f(x) \geq k\rho(G) - n + |A|$$

$$\geq k\rho(G) - n + \rho(G)(\delta(G) + 1)$$

$$= \rho(G)(k + \delta(G) + 1) - n.$$ 

Theorem 4 yields the next result immediately.
Corollary 7. ([5]) If $G$ is a graph of order $n$ with $\delta(G) \geq \frac{k}{2} - 1$, then

$$\gamma^k_{sR}(G) \geq \rho(G)(k + \delta(G) + 1) - n.$$ 

In [5], the authors presented an infinite family of graphs achieving equality in Corollary 7. Thus Corollary 7 and Theorem 4 are sharp. Using Corollary 4, one can prove the following Nordhaus-Gaddum type inequality analogously to Theorem 13 in [5].

Theorem 5. If $G$ is an $r$-regular graph of order $n$ such that $r \geq \frac{k}{2} - 1$ and $n - r - 1 \geq \frac{k}{2} - 1$, then

$$\gamma^k_{wsR}(G) + \gamma^k_{wsR}(\overline{G}) \geq 4k(n)/(n + 1).$$

If $n$ is even, then $\gamma^k_{wsR}(G) + \gamma^k_{wsR}(\overline{G}) \geq 4k(n + 1)/(n + 2)$.

Let $k \geq 1$ be an odd integer, and let $H$ and $\overline{H}$ be $(k - 1)$-regular graphs of order $n = 2k - 1$. By Example 1, we have $\gamma^k_{wsR}(H) = \gamma^k_{wsR}(\overline{H}) = n$. Consequently,

$$\gamma^k_{wsR}(H) + \gamma^k_{wsR}(\overline{H}) = 2n = \frac{4kn}{n + 1}.$$ 

Thus the Nordhaus-Gaddum bound of Theorem 5 is sharp for odd $k$.

4. The weak signed Roman $k$-domination number of $K_{p,p}$

Example 2. Let $k \geq 1$ and $p \geq k + 1$ be integers.

1. If $p \geq k + 3$, then $\gamma^k_{wsR}(K_{p,p}) = 2k + 2$.

2. If $k + 1 \leq p \leq k + 2$, then $\gamma^k_{wsR}(K_{p,p}) = p + k - 1$.

3. If $k \geq 2$, then $\gamma^k_{wsR}(K_{k,k}) = 2k$ and $\gamma^1_{wsR}(K_{1,1}) = 1$.

Proof. Let $X = \{x_1, x_2, \ldots, x_p\}$ and $Y$ be a bipartition of the complete bipartite graph $K_{p,p}$.

1. First we show that $\gamma^k_{wsR}(K_{p,p}) \geq 2k + 2$. Let $f : V(K_{p,p}) \rightarrow \{-1, 1, 2\}$ be a WSRkDF. If $f(u) \geq 1$ for each $u \in V(K_{p,p})$, then $\omega(f) \geq 2p \geq 2k + 2$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. If $w \in Y$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k + p - 1 \geq 2k + 2.$$ 

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. We deduce that

$$\omega(f) = f[x] + f[y] - f(x) - f(y) \geq 2k + 2.$$
Since we have discussed all possible cases, we obtain $\gamma_{wsR}^k(K_{p,p}) \geq 2k + 2$.

As $\gamma_{sR}^k(K_{p,p}) = 2k + 2$ for $p \geq k + 2$ (see Example 14 in [5]), we have the converse inequality $\gamma_{wsR}^k(K_{p,p}) \leq \gamma_{sR}^k(K_{p,p}) = 2k + 2$ and so the desired result.

(2) First we show that $\gamma_{wsR}^k(K_{p,p}) \geq p + k - 1$. Let $f : V(K_{p,p}) \to \{-1, 1, 2\}$ be a WSRkDF. If $f(u) \geq 1$ for each $u \in V(K_{p,p})$, then $\omega(f) \geq 2p \geq p + k - 1$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. If $w \in Y$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k + p - 1.$$  

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. We deduce that

$$\omega(f) = f[x] + f[y] - f(x) - f(y) \geq 2k + 2 \geq k + p \geq p + k - 1.$$  

If $p = k + 2$, then define the function $g : V(K_{p,p}) \to \{-1, 1, 2\}$ by $g(x_1) = g(x_2) = -1$, $g(x_3) = 2$ and $g(x) = 1$ otherwise. Then $g$ is a WSRkDF function on $K_{p,p}$ of weight $2k + 1 = p + k - 1$ and thus $\gamma_{wsR}^k(K_{p,p}) \leq p + k - 1$ and so $\gamma_{wsR}^k(K_{p,p}) = p + k - 1$ in this case.

If $p = k + 1$, then define the function $h : V(K_{p,p}) \to \{-1, 1, 2\}$ by $h(x_1) = -1$ and $h(x) = 1$ otherwise. Then $h$ is a WSRkDF function on $K_{p,p}$ of weight $2k = p + k - 1$ and thus $\gamma_{wsR}^k(K_{p,p}) \leq p + k - 1$ and so $\gamma_{wsR}^k(K_{p,p}) = p + k - 1$ also in this case.

(3) Clearly, $\gamma_{wsR}^k(K_{1,1}) = 1$. Let now $k \geq 2$.

First we show that $\gamma_{wsR}^k(K_{k,k}) \geq 2k$. Let $f : V(K_{k,k}) \to \{-1, 1, 2\}$ be a WSRkDF. If $f(u) \geq 1$ for each $u \in V(K_{k,k})$, then $\omega(f) \geq 2k$. Assume next, without loss of generality, that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) \geq 1$ for each $y \in Y$. Then $f(u) = 2$ for at least one vertex $u \in Y$. If $w \in Y$ with $w \neq u$, then it follows that

$$\omega(f) = f[w] + \sum_{y \in Y - \{w\}} f(y) \geq k + k = 2k.$$  

Finally, assume that $f(x) = -1$ for at least one vertex $x \in X$ and $f(y) = -1$ for at least one vertex $y \in Y$. We deduce that

$$\omega(f) = f[x] + f[y] - f(x) - f(y) \geq 2k + 2.$$  

Applying Observation 2, we obtain $\gamma_{wsR}^k(K_{k,k}) = 2k$. \hfill $\Box$

Example 1 implies $\gamma_{wsR}^k(K_{k-1,k-1}) = 2k - 2$ for $k \geq 2$.  


5. Cycles

Let $C_n$ be a cycle of length $n \geq 3$. In [1], the authors have shown that $\gamma_{sR}(C_n) = \lceil 2n/3 \rceil$. In addition, in [7] it is proved that $\gamma_{wsR}(C_n) = \lceil n/3 \rceil$ when $n \equiv 0, 1 \pmod{3}$ and $\gamma_{wsR}(C_n) = \lceil n/3 \rceil + 1$ when $n \equiv 2 \pmod{3}$. Now we determine $\gamma_{sR}^k(C_n)$ as well as $\gamma_{sR}(C_n)$ for $2 \leq k \leq 6$.

Theorems 1 and 2 immediately lead to $\gamma_{wsR}^6(C_n) = \gamma_{sR}^6(C_n) = 2n$. In addition, according to Corollary 4 and Observation 2, we have $\gamma_{wsR}^3(C_n) = \gamma_{sR}^3(C_n) = n$.

**Example 3.** For $n \geq 3$, we have $\gamma_{wsR}^5(C_n) = \gamma_{sR}^5(C_n) = \lceil \frac{5n}{3} \rceil$.

**Proof.** Corollary 4 implies $\gamma_{wsR}^5(C_n) \geq \gamma_{sR}^5(C_n) \geq \lceil \frac{5n}{3} \rceil$. For the converse inequality $\gamma_{wsR}^5(C_n) \leq \gamma_{sR}^5(C_n) \leq \lceil \frac{5n}{3} \rceil$, we distinguish three cases.

Case 1. Assume that $n = 3t$ with an integer $t \geq 1$. Let $C_{3t} = v_0v_1 \ldots v_{3t-1}v_0$. Define the function $f : V(C_{3t}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$ and $f(v_{3i+1}) = f(v_{3i+2}) = 2$ for $0 \leq i \leq t - 1$. Then $f[v_j] = 5$ for each $0 \leq j \leq 3t - 1$ and therefore $f$ is an SR5DF on $C_{3t}$ of weight $\omega(f) = 5t$. Thus $\gamma_{wsR}^5(C_{3t}) \leq \gamma_{sR}^5(C_{3t}) \leq 5t$. Consequently, $\gamma_{wsR}^5(C_n) = \gamma_{sR}^5(C_n) = 5t = \lceil \frac{5n}{3} \rceil$ in this case.

Case 2. Assume that $n = 3t + 1$ with an integer $t \geq 1$. Let $C_{3t+1} = v_0v_1 \ldots v_{3t+1}v_0$. Define the function $f : V(C_{3t+1}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$, $f(v_{3i+1}) = f(v_{3i+2}) = 2$ for $0 \leq i \leq t - 1$ and $f(v_{3t}) = 2$. Then $f[v_j] \geq 5$ for each $0 \leq j \leq 3t$ and therefore $f$ is an SR5DF on $C_{3t+1}$ of weight $\omega(f) = 5t + 2$. Thus $\gamma_{wsR}^5(C_{3t+1}) \leq \gamma_{sR}^5(C_{3t+1}) \leq 5t + 2$. Consequently, $\gamma_{wsR}^5(C_n) = \gamma_{sR}^5(C_n) = 5t + 2 = \lceil \frac{5n}{3} \rceil$ also in this case.

Case 3. Assume that $n = 3t + 2$ with an integer $t \geq 1$. Let $C_{3t+2} = v_0v_1 \ldots v_{3t+2}v_0$. Define the function $f : V(C_{3t+2}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = 1$, $f(v_{3i+1}) = f(v_{3i+2}) = 2$ for $0 \leq i \leq t - 1$ and $f(v_{3t}) = f(v_{3t+1}) = 2$. Then $f[v_j] \geq 5$ for each $0 \leq j \leq 3t + 1$ and therefore $f$ is an SR5DF on $C_{3t+2}$ of weight $\omega(f) = 5t + 4$. Thus $\gamma_{wsR}^5(C_{3t+2}) \leq \gamma_{sR}^5(C_{3t+2}) \leq 5t + 4$. Consequently, $\gamma_{wsR}^5(C_n) = \gamma_{sR}^5(C_n) = 5t + 4 = \lceil \frac{5n}{3} \rceil$ also in the last case.

**Example 4.** For $n \geq 3$, we have $\gamma_{wsR}^4(C_n) = \gamma_{sR}^4(C_n) = \lceil \frac{4n}{3} \rceil$.

**Proof.** Corollary 4 implies $\gamma_{wsR}^4(C_n) \geq \gamma_{sR}^4(C_n) \geq \lceil \frac{4n}{3} \rceil$. For the converse inequality $\gamma_{wsR}^4(C_n) \leq \gamma_{sR}^4(C_n) \leq \lceil \frac{4n}{3} \rceil$, we distinguish three cases.

Case 1. Assume that $n = 3t$ with an integer $t \geq 1$. Let $C_{3t} = v_0v_1 \ldots v_{3t-1}v_0$. Define the function $f : V(C_{3t}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = f(v_{3i+1}) = 1$ and $f(v_{3i+2}) = 2$ for $0 \leq i \leq t - 1$. Then $f[v_j] = 4$ for each $0 \leq j \leq 3t - 1$ and therefore $f$ is an SR4DF on $C_{3t}$ of weight $\omega(f) = 4t$. Thus $\gamma_{wsR}^4(C_{3t}) \leq \gamma_{sR}^4(C_{3t}) \leq 4t$. Consequently, $\gamma_{wsR}^4(C_n) = \gamma_{sR}^4(C_n) = 4t = \lceil \frac{4n}{3} \rceil$.

Case 2. Assume that $n = 3t + 1$ with an integer $t \geq 1$. Let $C_{3t+1} = v_0v_1 \ldots v_{3t+1}v_0$. Define the function $f : V(C_{3t+1}) \rightarrow \{-1, 1, 2\}$ by $f(v_{3i}) = f(v_{3i+1}) = 1$, $f(v_{3i+2}) = 2$ for $0 \leq i \leq t - 1$ and $f(v_{3t}) = 2$. Then $f[v_j] \geq 4$ for each $0 \leq j \leq 3t$ and therefore $f$ is
In [2], the authors have shown that \( \gamma_{wsR}(P_{3t+1}) \leq 4t+2 \). Thus \( \gamma_{wsR}(C_{3t+1}) \leq 4t+2 \). Consequently, \( \gamma_{wsR}(C_n) = \gamma_{wsR}(P_n) = 4t + 2 = \left \lfloor \frac{4n}{3} \right \rfloor \).

Case 3. Assume that \( n = 3t + 2 \) with an integer \( t \geq 1 \). Let \( C_{3t+2} = v_0v_1 \ldots v_{3t+1}v_0 \). Define the function \( f : V(C_{3t+2}) \rightarrow \{ -1, 1, 2 \} \) by \( f(v_{3i}) = f(v_{3i+1}) = 1 \), \( f(v_{3i+2}) = 2 \) for \( 0 \leq i \leq t-1 \), \( f(v_{3t}) = 1 \) and \( f(v_{3t+1}) = 2 \). Then \( f(v_i) \geq 4 \) for each \( 0 \leq j \leq 3t+1 \) and therefore \( f \) is an SR4DF on \( C_{3t+2} \) of weight \( \omega(f) = 4t + 3 \). Thus \( \gamma_{wsR}(C_{3t+2}) \leq 4t + 3 \). Consequently, \( \gamma_{wsR}(C_n) = \gamma_{wsR}(C_n) = 4t + 3 = \left \lfloor \frac{4n}{3} \right \rfloor \).

Examples 3 and 4 also show the sharpness of Corollary 4. In [5], we have determined \( \gamma_{sR}^2(C_n) \).

**Example 5.** ([5]) For \( n \geq 3 \), we have \( \gamma_{sR}^2(C_n) = \left \lfloor \frac{2n}{3} \right \rfloor + \left \lceil \frac{n}{3} \right \rceil - \left \lfloor \frac{n}{3} \right \rfloor \).

Analogously to Example 5, one can determine the weak signed Roman 2-domination number of a cycle.

**Example 6.** For \( n \geq 3 \), we have \( \gamma_{wsR}^2(C_n) = \left \lfloor \frac{2n}{3} \right \rfloor + \left \lceil \frac{n}{3} \right \rceil - \left \lfloor \frac{n}{3} \right \rfloor \).

6. Trees

Let \( P_n \) be a path of order \( n \). Our aim in the section is to determine \( \gamma_{wsR}(P_n) \), and to establish lower bounds on the weak signed Roman \( k \)-domination number of a tree for \( 2 \leq k \leq 4 \).

We start with the path \( P_n \). In [1] it is proved that \( \gamma_{sR}(P_n) = \left \lfloor \frac{2n}{3} \right \rfloor \) and in [7], the author has shown that \( \gamma_{wsR}(P_2) = 1 \), \( \gamma_{wsR}(P_n) = \lceil n/3 \rceil \) when \( n \equiv 1 \pmod{3} \) and \( \gamma_{wsR}(P_n) = \lceil n/3 \rceil + 1 \) when \( n \equiv 0, 2 \pmod{3} \) and \( n \geq 3 \). Now we determine \( \gamma_{wsR}^k(P_n) \) for \( 2 \leq k \leq 4 \). In [5], one can find the following result.

**Example 7.** If \( 2 \leq n \leq 7 \) then \( \gamma_{sR}^2(P_n) = n \), and if \( n \geq 8 \), then \( \gamma_{sR}^2(P_n) = \left \lfloor \frac{2n+5}{3} \right \rfloor \).

If \( f \) is an SR2DF on \( P_n \), then \( f \) is also a WSR2DF on \( P_n \). Now assume that \( g \) is a WSR2DF on \( P_n \). If \( g(v) = -1 \), then \( g[v] \geq 2 \) implies that \( v \) has a neighbor \( w \) with \( g(w) = 2 \). Therefore \( g \) is also an SR2DF on \( P_n \). This observation and Example 7 lead to the next result immediately.

**Example 8.** If \( 2 \leq n \leq 7 \) then \( \gamma_{wsR}^2(P_n) = n \), and if \( n \geq 8 \), then \( \gamma_{wsR}^2(P_n) = \left \lfloor \frac{2n+5}{3} \right \rfloor \).

In [2], the authors have shown that \( \gamma_{sR}^3(P_n) = n + 2 \) when \( n \geq 4 \) and \( \gamma_{sR}^4(P_n) = \left \lfloor \frac{4n}{3} \right \rfloor + 2 \) when \( n \geq 3 \). The same arguments as above lead to \( \gamma_{wsR}^3(P_2) = n + 2 \) when \( n \geq 4 \) and \( \gamma_{wsR}^4(P_n) = \left \lfloor \frac{4n}{3} \right \rfloor + 2 \) when \( n \geq 3 \).
Observation 6. Let $T$ be a tree of order $n$ and let $f$ be a WSR2DF on $T$. Then the following holds.
(a) If $v$ is a leaf or a support vertex in $T$, then $f(v) \geq 1$.
(b) If $2 \leq n \leq 5$, then $\gamma_{isoR}^2(T) = n$.

The next result provided a lower bound on the weak signed Roman 2-domination number of a tree in terms of its order.

Theorem 7. If $T$ is a tree of order $n \geq 4$, then

$$\gamma_{isoR}^2(T) \geq \frac{n + 4}{2}.$$  

Proof. We proceed by induction on the order $n \geq 4$. If $n = 4$, then by Observation 6(b), $\gamma_{isoR}^2(T) = n = (n + 4)/2$. This establishes the base case when $n = 4$. Let $n \geq 5$ and suppose that if $T'$ is a tree of order $n'$ where $4 \leq n' < n$, then $\gamma_{isoR}^2(T') \geq (n' + 4)/2$. Let $T$ be a tree of order $n$. Choose an optimal WSR2DF $f$ on $T$, and so $\gamma_{isoR}^2(T) = \omega(f)$. If $f(x) \geq 1$ for each vertex $x \in V(T)$, then $\omega(f) \geq n > (n + 4)/2$. Now suppose that there is a vertex $v \in V(T)$ with $f(v) = -1$. Suppose that $T - v$ is the disjoint union of $r$ trees $T_1, T_2, \ldots, T_r$. Let $f_i$ be the restriction of $f$ on $T_i$ for $1 \leq i \leq r$. Clearly, $f_i$ is a WSR2DF on $T_i$ for $1 \leq i \leq r$. Since by Observation 6(a) a leaf and its only neighbor has a positive label, $r \geq 2$ and each $T_i$ has $n_i \geq 2$ vertices. If $n_i = 2$, then in fact $\omega(f_i) \geq 3 = (n_i + 4)/2$, and if $n_i = 3$, then $\omega(f_i) = 3$ or $\omega(f_i) \geq 4 > (n_i + 4)/2$. If $n_i \geq 4$, then by the induction hypothesis $\omega(f_i) \geq (n_i + 4)/2$. If $\omega(f_i) \geq (n_i + 4)/2$ for all $i$, then since $r \geq 2$,

$$\omega(f) = -1 + \sum_{i=1}^{r} \omega(f_i) \geq -1 + \sum_{i=1}^{r} \frac{n_i + 4}{2} = \frac{n + 4r - 3}{2} \geq \frac{n + 5}{2}.$$  

Hence we may assume that for some $i$, $n_i = 3$ and $\omega(f_i) = 3$, for otherwise the desired result follows. Assume that $T_1, T_2, \ldots, T_q$, $q \geq 1$, are exactly the trees with three vertices and with $\omega(f_i) = 3$, $1 \leq i \leq q$. We note that $f(w) = 1$ for each vertex $w$ that belongs to such a tree $T_i$ with $\omega(f_i) = 3$. If $r > q$, then

$$\omega(f) = -1 + \sum_{i=1}^{q} \omega(f_i) + \sum_{i=q+1}^{r} \omega(f_i) \geq -1 + 3q + \sum_{i=q+1}^{r} \frac{n_i + 4}{2} = \frac{n + 4(r - q) + 3(q - 1)}{2} \geq \frac{n + 4}{2},$$

as desired. If $r = q$, then $q \geq 3$, and we obtain

$$\omega(f) = -1 + \sum_{i=1}^{q} \omega(f_i) = -1 + 3q = n - 2 \geq \frac{n + 4}{2},$$
since $n \geq 10$ in this case. This completes the proof. \(\square\)

**Corollary 8.** ([5]) If $T$ is a tree of order $n \geq 4$, then

$$\gamma_{sR}^2(T) \geq \frac{n+4}{2}.$$  

In [2], one can find the following statement. If $T$ is a tree of order $n \geq 2$, then $\gamma_{sR}^3(T) \geq \frac{4n+7}{5}$, with equality if and only if $T = P_2$. The next examples demonstrates that this statement is not valid.

**Example 9.** Let $P = v_1v_2 \ldots v_{2p+1}$ be a path of order $2p+1$ with $p \geq 1$. Now attach two pendant edges to $v_1$ and $v_{2p+1}$ and three pendant edges to $v_{2i+1}$ for $1 \leq i \leq p-1$. The resulting tree $T_{5p+2}$ is of order $5p+2$. Define the function $f : V(T_{5p+2}) \rightarrow \{-1,1,2\}$ by $f(v_{2i+1}) = 2$ for $0 \leq i \leq p$, $f(v_{2i}) = -1$ for $1 \leq i \leq p$ and $f(x) = 1$ otherwise. Then $f$ is an SR3DF on $T_{5p+2}$ of weight

$$\omega(f) = 2(p+1) - p + 3p + 1 = 4p + 3 = \frac{4n(T_{5p+2}) + 7}{5}.$$  

Therefore $\gamma_{sR}^3(T_{5p+2}) \leq \frac{4n(T_{5p+2}) + 7}{5}$. Since $f(u) + f(v) \geq 3$ if $v$ is a leaf and $u$ its support vertex, it easy to verify that $\gamma_{sR}^3(T_{5p+2}) = \frac{4n(T_{5p+2}) + 7}{5}$.

**Example 10.** Let $S_\Delta$ be a spider and $w$ be a vertex of maximum degree $\Delta \geq 1$. In addition, let $v_1, v_2, \ldots, v_\Delta$ be the neighbors of $w$ and $u_i \neq w$ be the neighbor of $v_i$ for $1 \leq i \leq \Delta$. Now attach $\Delta+1$ pendant edges to $w$ and two pendant edges to $u_i$ for $1 \leq i \leq \Delta$. The resulting tree $H$ is of order $5\Delta+2$. Define the function $f : V(H) \rightarrow \{-1,1,2\}$ by $f(w) = 2$, $f(u_i) = 2$ for $1 \leq i \leq \Delta$, $f(v_i) = -1$ for $1 \leq i \leq \Delta$ and $f(x) = 1$ otherwise. Then $f$ is an SR3DF on $H$ of weight

$$\omega(f) = 4\Delta + 3 = \frac{4n(H) + 7}{5}.$$  

Therefore $\gamma_{sR}^3(H) = \frac{4n(H) + 7}{5}$.

I conjecture that the bound $\gamma_{sR}^3(T) \geq \frac{4n+7}{5}$ is really valid for each tree $T$ of order $n \geq 2$, however, I only can prove the following weaker bound.

**Theorem 8.** If $T$ is a tree of order $n \geq 2$, then

$$\gamma_{sR}^3(T) \geq \gamma_{wsR}^3(T) \geq \frac{3n+6}{4}.$$  

**Proof.** Clearly, it is enough to prove the right inequality. We proceed by induction on the order $n \geq 2$. If $n = 2$, then $\gamma_{wsR}^3(T) = 3 = (3n+6)/4$. If $n = 3$, then $\gamma_{wsR}^3(T) = 4 \geq (3n+6)/4$. Let now $n \geq 4$ and suppose that if $T'$ is a tree of order $n'$ where $2 \leq n' < n$, then $\gamma_{wsR}^3(T') \geq (3n'+6)/4$. Let $T$ be a tree of order $n$. Choose
an optimal WSR3DF $f = (V_{-1}, V_1, V_2)$ on $T$, and so $\gamma_{wsR}^3(T) = \omega(f)$. If $f(x) \geq 1$ for each vertex $x \in V(T)$, then $\omega(f) \geq n + 1 > (3n + 6)/4$.

Now suppose that there is a vertex $v \in V(T)$ with $f(v) = -1$. If $|V_{-1}| \geq 2$, then choose $u, v \in V_{-1}$ such that $d(u, v)$ is as large as possible. Suppose that $T - v$ is the disjoint union of $r$ trees $T_1, T_2, \ldots, T_r$. Let $f_i$ be the restriction of $f$ on $T_i$ for $1 \leq i \leq r$. Clearly, $f_i$ is a SWR3DF on $T_i$ for $1 \leq i \leq r$. Since a leaf and its only neighbor has a positive label, $r \geq 2$ and each $T_i$ has $n_i \geq 2$ vertices.

If $r \geq 3$, then we deduce from the induction hypothesis that

$$\omega(f) = -1 + \sum_{i=1}^{r} \omega(f_i) \geq -1 + \sum_{i=1}^{r} \frac{3n_i + 6}{4}$$

$$= -1 + \frac{3(n - 1) + 6r}{4}$$

$$\geq \frac{3n + 6}{4}.$$

Let now $r = 2$. By the choice of $u$ and $v$, we observe that $V_{-1} \cap V(T_1) = \emptyset$ or $V_{-1} \cap V(T_2) = \emptyset$, say $V_{-1} \cap V(T_2) = \emptyset$. Note that $\omega(f_i) = 4$ if $n_i = 2$. If $n_2 = 2$ then we deduce from the induction hypothesis that

$$\omega(f) = \omega(f_1) + \omega(f_2) - 1 \geq \frac{3n_1 + 6}{4} + 4 - 1$$

$$\geq \frac{3(n - 3) + 6 + 12}{4} \geq \frac{3n + 6}{4}.$$ 

If $n_2 \geq 3$ then, $\omega(f_2) \geq n_2 + 1$, and we deduce from the induction hypothesis that

$$\omega(f) = \omega(f_1) + \omega(f_2) - 1 \geq \frac{3n_1 + 6}{4} + n_2 + 1 - 1$$

$$\geq \frac{3(n_1 + n_2 + 1) + 6}{4} \geq \frac{3n + 6}{4},$$

and the proof is complete.

**Observation 9.** Let $T$ be a tree, and let $f$ be a WSR4DF on $T$. If $v$ is a leaf or a support vertex in $T$, then $f(v) = 2$.

**Theorem 10.** If $T$ is a tree of order $n \geq 4$, then $\gamma_{wsR}^4(T) \geq n + 4$.

**Proof.** We proceed by induction on the order $n \geq 4$. If $n = 4$, then by Observation 9, $\gamma_{wsR}^4(T) = 8 = n + 4$. This establishes the base case when $n = 4$. Let $n \geq 5$ and
suppose that if $T'$ is a tree of order $n'$ where $4 \leq n' < n$, then $\gamma_{sR}^4(T') \geq n' + 4$. Let $T$ be a tree of order $n$. Choose an optimal WSR4DF $f$ on $T$, and so $\gamma_{wsR}^4(T) = \omega(f)$. Assume first that $f(x) \geq 1$ for each vertex $x \in V(T)$. Since $n \geq 5$, the tree has at least 4 leaves or support vertices $v_1, v_2, v_3, v_4$. According to Observation 9, we note that $f(v_1) = f(v_2) = v(v_3) = f(v_4) = 2$ and hence $\omega(f) \geq 8 + n - 4 = n + 4$ in this case.

Now assume that there is a vertex $v \in V(T)$ with $f(v) = -1$. Suppose that $T - v$ is the disjoint union of $r$ trees $T_1, T_2, \ldots, T_r$. Let $f_i$ be the restriction of $f$ on $T_i$ for $1 \leq i \leq r$. Clearly, $f_i$ is a WSR4DF on $T_i$ for $1 \leq i \leq r$. Since $f[v] \geq 4$, we deduce that $r \geq 3$ and each $T_i$ has $n_i \geq 3$ vertices. If $n_i = 3$, then in fact $\omega(f_i) = 6$, and if $n_i \geq 4$, then by the induction hypothesis $\omega(f_i) \geq n_i + 4$. Now assume that $n_1 = n_2 = \ldots = n_q = 3$ for $0 \leq q \leq r$ and $n_i \geq 4$ for $q + 1 \leq i \leq r$. We deduce from the induction hypothesis that

$$\omega(f) = -1 + \sum_{i=1}^{q} \omega(f_i) + \sum_{i=q+1}^{r} \omega(f_i)$$

$$\geq -1 + 6q + \sum_{i=q+1}^{r} (n_i + 4)$$

$$= -1 + 6q + (n - 3q - 1) + 4(r - q)$$

$$= n + 3q - 2 + 4(r - q) \geq n + 4.$$

\square

**Corollary 9.** If $T$ is a tree of order $n \geq 4$, then $\gamma_{sR}^4(T) \geq n + 4$.

Note that if $H \in \{K_{1,3}, P_4, P_5, P_6\}$, then $\gamma_{wsR}^4(H) = \gamma_{sR}^4(H) = n(H) + 4$. Corollary 9 is an improvement of the bound $\gamma_{sR}^4(T) \geq n + 2$, given in [2].

**References**


