Communications in Combinatorics and Optimization
Vol. 5 No. 2, 2020 pp.125-137
DOI: 10.22049/CCO.2019.26719.1136

CCO
Commun. Comb. Optim.

New results on upper domatic number of graphs

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Received: 10 November 2019; Accepted: 12 December 2019
Published Online: 15 December 2019

Abstract: For a graph $G = (V, E)$, a partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of the vertex set V is an *upper domatic partition* if V_i dominates V_j or V_j dominates V_i or both for every $V_i, V_j \in \pi$, whenever $i \neq j$. The *upper domatic number* $D(G)$ is the maximum order of an upper domatic partition of G. We study the properties of upper domatic number and propose an upper bound in terms of clique number. Further, we discuss the upper domatic number of certain graph classes including unicyclic graphs and power graphs of paths and cycles.

Keywords: Domination, upper domatic partition, upper domatic number, transitivity

AMS Subject classification: 05C69, 05C70

1. Introduction

Let $G = (V, E)$ be a graph of order $n = |V|$ and size $m = |E|$. Throughout this paper, we consider only finite, simple and undirected graphs. The *degree* of a vertex $v \in V$, denoted by $\deg(v)$, is the number of vertices adjacent to v. The maximum (minimum) degree of a vertex in a graph G is denoted by $\Delta(G)$ ($\delta(G)$). Complete graphs, cycles and paths of order n are represented as K_n, C_n, and P_n respectively. A maximal complete subgraph of a graph is called *clique*. The maximum cardinality of a clique of G is called the *clique number* $\omega(G)$.
A vertex $v \in V$ is a *leaf vertex* if $\deg(v)= 1$. Given two disjoint subsets A and B of the vertex set of a graph, the set A dominates B, denoted by $A \rightarrow B$, if every vertex of B is adjacent to at least one vertex in A, else we write $A \not\rightarrow B$. The dominating set of a graph G is a set $S \subset V$ such that $S \rightarrow V - S$. The domination number of a graph G is the minimum cardinality of a dominating set of G. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be

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a vertex partition of a graph G. A set $V_i \in \pi$ is a source set if V_i is a dominating set of G and a sink set if $V_j \to V_i$, for all $1 \leq j \leq k$.
Partitioning the vertex set of a graph into dominating sets is a problem posed by Cockayne and Hedetniemi in 1977 [1]. This gave rise to the concept of *domatic number* $d(G)$ of a graph G which is the maximum order k of a vertex partition $\{V_1, V_2, \ldots, V_k\}$ such that each V_i, $1 \leq i \leq k$, is a dominating set. This concept was further generalised by S. T. Hedetniemi and J. T. Hedetniemi taking into consideration the domination between the subsets of the vertex set [5]. Using this idea, they considered a vertex partition $\{V_1, V_2, \ldots, V_k\}$ of a graph G such that $V_i \to V_j$ for all i, j, $1 \leq i < j \leq k$ and defined *transitivity* $Tr(G)$ of a graph as the maximum order of such a partition. Haynes *et al.* obtained another generalisation of domatic number by defining the *upper domatic number* $D(G)$ which is the maximum order of a vertex partition $\{V_1, V_2, \ldots, V_k\}$ of a graph G such that for each pair i and j, $1 \leq i < j \leq k$, either $V_i \to V_j$ or $V_j \to V_i$ or both [4]. Any upper domatic partition of order $D(G)$ is referred to as a D-partition of G. It follows from these definitions that for any graph G,

$$1 \leq d(G) \leq Tr(G) \leq D(G) \leq n.$$

In this paper, we continue the study of upper domatic number. The results include upper bounds of $D(G)$ in terms of size and clique number of the graph G, and a characterisation of graphs having transitivity at least four. It has been proved that for unicyclic graphs the upper domatic number is the same as its transitivity. We have also examined the upper domatic number of powers of graphs and have determined the upper domatic number of powers of paths and powers of cycles.

2. Preliminary Results

Theorem 1. *[1] For any graph G, $d(G) \leq \delta(G) + 1$.*

Analogous to the result in Theorem 1, Haynes *et al.* obtained an upper bound for the upper domatic number in terms of maximum degree for any graph G [4] .

Theorem 2. *[4] For any graph G, $D(G) \leq \Delta(G) + 1$.*

The following results giving upper domatic number and transitivity of certain families of graphs are required for further discussion.

Theorem 3. *[4]*

1. *For the path P_n with $n \geq 4$, $Tr(P_n) = D(P_n) = 3$.*
2. *For the cycle C_n with $n \geq 3$, $Tr(C_n) = D(C_n) = 3$.*
3. *For the complete graph K_n, $Tr(K_n) = D(K_n) = n$.*

4. For any acyclic graph T, $D(T) = Tr(T)$.

Given any D-partition of a graph G, we make the following observations.

Observation 4. If $\pi = \{V_1, V_2, \ldots, V_k\}$ is a D-partition of a graph G such that the subset V_i contains a leaf vertex, then V_i is dominated by at most one set V_j, where $i \neq j$.

Observation 5. Every sink set in a D-partition is an independent set.

Observation 6. Let π be a D-partition of a graph G. Then for any $V_i \in \pi$, $D(H) \geq D(G) - 1$, where $H = G[\pi - V_i]$.

The following theorem gives a property of graphs with $D(G) < 5$.

Theorem 7. *[4] If $D(G) \leq 4$, then there exists a D-partition of graph G that contains a sink set.*

S. T. Hedetniemi and J. T. Hedetniemi noted that the transitivity of a graph is at least as large as the transitivity of any of its subgraphs [5].

Proposition 1. *[5] If H is a subgraph of a graph G, then $Tr(H) \leq Tr(G)$.*

Proposition 2. *[3] For any graph G and a subgraph H, if $V(G) - V(H)$ dominates H, then $Tr(G) \geq Tr(H) + 1$.*

Proposition 1 holds for the upper domatic number of a graph, provided the subgraph has a D-partition with a source set [4].

Proposition 3. *[4] If H is a subgraph of a graph G and H has a D-partition with source set, then $D(H) \leq D(G)$.*

This proposition can be extended to disconnected graphs as shown in the following theorem.

Theorem 8. *For a disconnected graph G with finite number of components, $D(G) \leq max\{D(G_i)$, where G_i is a component of $G\}$.*

Proof. Let G be a disconnected graph with r components such that $D(G) = k$ and π be a D-partition of G. If possible, assume that $D(G) > \max\{D(G_i)$, where G_i is a component of $G\}$. Then corresponding to each component G_i of G, there exists an element say $V_{G_i} \in \pi$ such that none of the vertices of G_i appears in V_{G_i}. Otherwise, the collection π_i of order k formed by taking the intersection of $V(G_i)$ with the k elements of π will constitute a D-partition of G_i, contradicting the assumption. Arbitrarily choose one component G_1 of G and let $V_{G_1} \in \pi$ be the corresponding set that contains no vertex of G_1. As the set V_{G_1} is not empty, choose a component G_2 having at least one vertex in V_{G_1} and let $V_{G_2} \in \pi$ be the set containing no vertex of G_2. Since $V_{G_2} \not\to V_{G_1}$, the set V_{G_1} dominates V_{G_2} and V_{G_2} does not contain any of the vertices of G_1 and G_2. Proceeding in this manner, choose the r^{th} component G_r and the set V_{G_r} disjoint from the vertex set of G_r. Then, the set V_{G_r} has empty intersection with the vertex set of components G_i, for $1 \leq i \leq r$ making V_{G_r} an empty set, a contradiction. Hence, $D(G) \leq \max\{D(G_i)$, where G_i is a component of $G\}$. □

Corollary 1. *For a disconnected graph G with finite number of components $G_1, G_2, \ldots, G_r$, if the component with maximum upper domatic number has a D-partition with source set, then $D(G) = max\{D(G_i)$, where G_i is a component of $G\}$.*

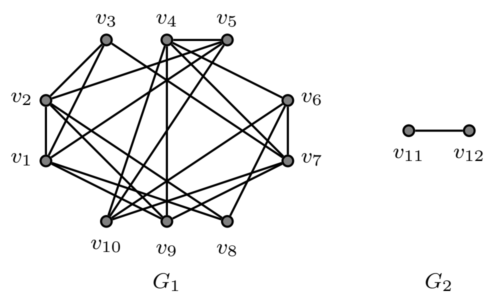

Figure 1. **Disconnected graph G with $D(G) < max\{D(G_1), D(G_2)\}$.**

Remark 1. The inequality in Theorem 8 is sharp. Figure 1 depicts a graph G with components G_1 and G_2. The fact that $\pi_1 = \{\{v_1\}, \{v_2\}, \{v_3, v_4\}, \{v_5, v_6\}, \{v_7, v_8\}, \{v_9, v_{10}\}\}$ is an upper domatic partition of G_1 and $\Delta(G_1) = 5$ implies that $D(G_1) = 6$, while it can be verified that $D(G) = 5 < D(G_1)$.

An analogous result for the transitivity of a disconnected graph can be found in [5].

Theorem 9. *[5] For a disconnected graph G with finite number of components $G_1, G_2, \ldots, G_r$, $Tr(G) = max\{Tr(G_i) \mid 1 \leq i \leq r\}$.*

From Proposition 3, it is straightforward to see that the clique number of a graph is a lower bound for its upper domatic number [4]. The following theorem shows that the mean of order and clique number of a graph serves as an upper bound for upper domatic number of the graph.

Theorem 10. *For any graph G with clique number $\omega(G)$, $D(G) \leq \dfrac{n+\omega(G)}{2}$.*

Proof. Assuming the contrary, let G be a graph with $D(G) > \dfrac{n+\omega(G)}{2}$. As $\omega(G) > 0$, $\dfrac{n+\omega(G)}{2} > \dfrac{n}{2}$ and there are at least $\omega(G)+1$ singleton sets in the D-partition of G. But the elements of the singleton sets in a D-partition induces a clique implying that G contains a clique of order $\omega(G)+1$ which contradicts the maximality of $\omega(G)$. Therefore, $D(G) \leq \dfrac{n+\omega(G)}{2}$. □

Remark 2. There exist graphs for which the bound in Theorem 10 is attained. For example, consider the complete bipartite graph $K_{s,s}$ where $s \geq 1$. Then $D(K_{s,s}) = s+1$ [4]. However, the difference between upper domatic number and the bound obtained in Theorem 10 can be arbitrarily large as one can see in the case of $K_{1,s}$, $s \geq 1$.

Further we observe that $D(G)$ has an obvious bound in terms of the size of G.

Observation 11. For any graph G of size m, $D(G) \leq \dfrac{1+\sqrt{1+8m}}{2}$. Moreover, $D(G) = \dfrac{1+\sqrt{1+8m}}{2}$ if and only if G is a complete graph.

The graph families admitting small values of upper domatic number was characterized in [4]. A *star* on n vertices is the complete bipartite graph $K_{1,n-1}$ and a *galaxy* is a disjoint union of stars.

Theorem 12. *[4] For any graph G of order n,*

1. *$D(G) = 1$ if and only if $G = \overline{K_n}$,*
2. *$D(G) = 2$ if and only if G is a galaxy with at least one edge,*
3. *$D(G) \geq 3$ if and only if G contains a K_3 or a P_3.*

Since transitivity serves as a lower bound for the upper domatic number of a graph, we provide a sufficient condition for $D(G) \geq 4$ by characterising the graphs with $Tr(G) \geq 4$. Let G_i, $1 \leq i \leq 12$ be the graphs shown in Figure 2 and $\mathcal{A} = \{G_i, 1 \leq i \leq 12\}$.

Theorem 13. *For any graph G, $Tr(G) \geq 4$ if and only if G contains at least one of the graphs in class $\mathcal{A}$.*

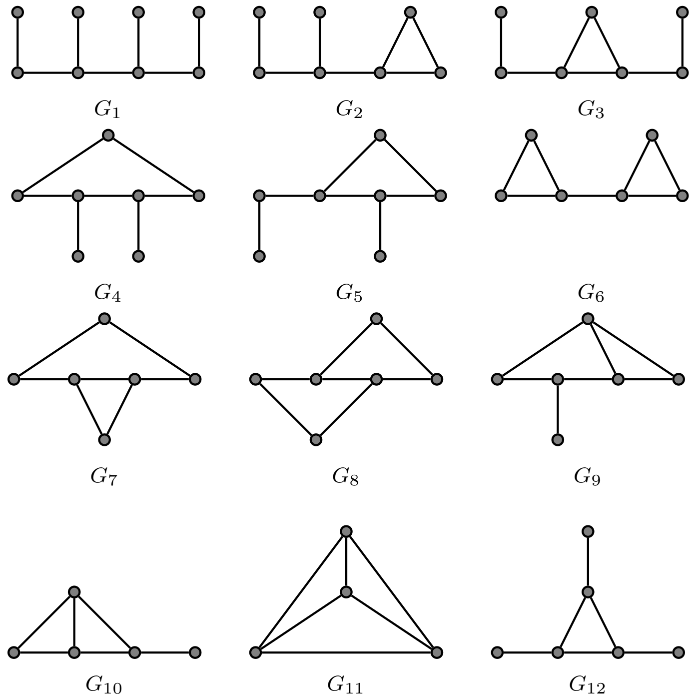

Figure 2. **Class $\mathcal{A}$.**

Proof. Let G be a graph containing one of the graphs G_i; $1 \leq i \leq 12$ in class $\mathcal{A}$ as its subgraph. Then by Proposition 1, $Tr(G) \leq Tr(G_i)$. Since $\Delta(G_i) = 3$ for each $G_i \in \mathcal{A}$, $Tr(G_i) \leq 4$ by Theorem 2. On the other hand every $G_i \in \mathcal{A}$ contains a subgraph $H \in \{P_4, C_3\}$, such that $[V(G_i) - V(H)] \to H$, hence by Proposition 2 and Theorem 3, $Tr(G) \geq 4$, thus proving the sufficiency part.
Conversely assume that G is a graph with $Tr(G) \geq 4$. Then any Tr-partition of G will have at least four elements. Consider a transitive partition $\pi = \{V_1, V_2, V_3, V_4\}$ where $V_i \subset V(G)$ of G. Then $V_i \to V_j$ for $i > j$, $i = 1, 2, 3$; $j = 2, 3, 4$. Each V_i; $1 \leq i \leq 4$ being non-empty, we can find a vertex, say $v_4 \in V_4$. Then there exist vertices $v_1 \in V_1$; $v_2 \in V_2$ and $v_3 \in V_3$ that dominate v_4, as π is a transitive partition. Again corresponding to $v_3 \in V_3$, there exist vertices say $v_2' \in V_2$ and $v_1' \in V_1$ that dominate v_3. It is to be noted that v_1' and v_2' need not be necessarily different from $v_1 \in V_1$ and $v_2 \in V_2$. In the same way let v_1'' and v_1''' be the vertices in V_1 that dominates v_2' and v_2 respectively. Then the following cases arise.
Case 1: All the vertices $v_1, v_2, v_3, v_4, v_1', v_1'', v_1'''$ and v_2' are distinct.
In this case, the graph induced by these vertices is isomorphic to G_1 as depicted in Figure 3.
Case 2: The vertices v_2 and v_2' are distinct.
In this case there arise eight possibilities depending upon the nature of vertices v_1,

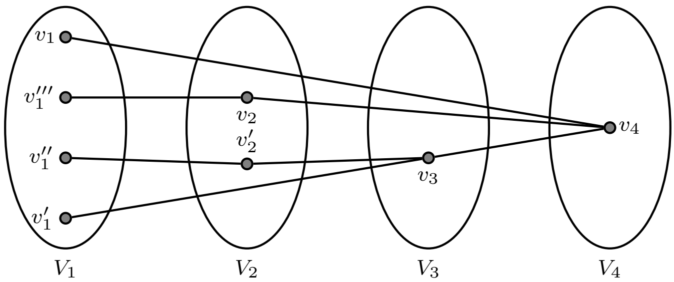

Figure 3. **The graph G_1.**

v_1', v_1'', v_1''' belonging to V_1.
(i) $v_1 = v_1'''$; $v_1' \neq v_1''$ or $v_1 \neq v_1'''$; $v_1' = v_1''$, in which case the resulting graph induced by these vertices is G_2.
(ii) $v_1 = v_1'$ and $v_1'' \neq v_1'''$, then the graph induced by these vertices is G_3.
(iii) $v_1'' = v_1'''$ and $v_1 \neq v_1'$, where these vertices induces the graph G_4.
(iv) $v_1 = v_1''$ and $v_1' \neq v_1'''$ or $v_1 \neq v_1''$ and $v_1' = v_1'''$, then the resulting graph induced by these vertices is G_5.
(v) $v_1 = v_1'''$ and $v_1' = v_1''$, in which case these vertices induces the graph G_6.
(vi) $v_1 = v_1'$ and $v_1'' = v_1'''$, then these vertices induces the graph G_7.
(vii) $v_1 = v_1''$ and $v_1' = v_1'''$, while these vertices induces the graph G_8.
(viii) $v_1 = v_1'' = v_1''' \neq v_1'$ or $v_1' = v_1'' = v_1''' \neq v_1$, where the graph induced by these vertices is G_9.
Case 3: The vertices v_2 and v_2' are the same.
In this case, it suffices to consider only three possibilities for the vertices in V_1.
(i) $v_1 = v_1' \neq v_1''$ or $v_1 = v_1'' \neq v_1'$ or $v_1' = v_1'' \neq v_1$, in which case these vertices induces the graph G_{10}.
(ii) $v_1 = v_1' = v_1''$, then the resulting graph induced is G_{11}.
(iii) v_1, v_1', v_1'' are all distinct, where these vertices induces the graph G_{12}.
Hence we have proved that every graph G, with $Tr(G) \geq 4$ will have one of the members of class $\mathcal{A}$ as its subgraph. □

Corollary 2. *For any graph G, if G contains at least one of the graphs in class $\mathcal{A}$, then $D(G) \geq 4$.*

Corollary 3. *For any graph G, $Tr(G) = 3$, if and only if G contains P_4 or C_3 and G is $H-$free, for all $H \in \mathcal{A}$.*

3. Unicyclic Graphs

We now explore the family of *unicyclic graphs*, obtained from trees by adding a single edge between two non-adjacent vertices of the tree. A unicyclic graph can also be understood as a connected graph containing exactly one cycle [2]. Primarily, in this section, we show that a unicyclic graph is another class of graph having equal upper domatic number and transitivity. By the definition of a unicyclic graph, it is immediate from Theorem 3 and Proposition 3 that the upper domatic number of a unicyclic graph is at least three.

Proposition 4. *For any unicyclic graph G, $D(G) \geq 3$.*

The bound in Proposition 4 is obviously sharp as seen in the case of C_n. Tadpole graph form another family of unicyclic graphs with upper domatic number three. The *tadpole graph* $T_{s,t}$ is obtained by joining any vertex of a cycle C_s and an end vertex of a path P_t with an edge. We establish a result necessary to determine the upper domatic number of a tadpole graph.

Lemma 1. *For any path P_n, $n \geq 4$, there exists no D-partition without a sink set.*

Proof. For $n \geq 4$, the upper domatic number of a path P_n being three, a D-partition $\pi' = \{V_1', V_2', V_3'\}$ can be either a transitive triple, such that $V_1' \to V_2'$, $V_1' \to V_3'$ and $V_2' \to V_3'$ or a cyclic triple, where $V_1' \to V_2'$, $V_2' \to V_3'$ and $V_3' \to V_1'$. We consider the D-partition to be a cyclic triple otherwise, π' has a sink set. To construct such a partition from a path with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, we assign v_1 to V_1', v_2 to V_3' as $V_3' \to V_1'$, and v_3 to V_2'. Assigning the vertices $v_4, v_5, \ldots v_n$ to the sets V_1', V_3' and V_2' sequentially, let v_n be assigned to the set V_i', for some i, $1 \leq i \leq 3$, there is no vertex in $V'_{(i-1)\text{mod } 3}$ to dominate v_n as v_{n-1} is in $V'_{(i+1)\text{mod } 3}$. □

Remark 3. Let G be a disjoint union of path, then there exists no D- partition of G without a sink set.

Theorem 14. *For a tadpole graph $T_{s,t}$, where $s \geq 3$ and $t \geq 1$, $D(T_{s,t}) = 3$.*

Proof. Let $T_{s,t}$, where $s \geq 3$ and $t \geq 1$, be a tadpole graph. By Theorem 2 and Proposition 4, it is evident that $3 \leq D(T_{s,t}) \leq 4$.
If possible, let $D(T_{s,t}) = 4$. Then by Theorem 7, $T_{s,t}$ has a D-Partition $\pi = \{V_1, V_2, V_3, V_4\}$ with a sink set. Without loss of generality, let V_4 be the sink set in π, then vertices belonging to V_4 has degree at least three. Since there exists only one vertex of degree three in $T_{s,t}$, V_4 is a singleton set. It can be observed that the graph induced by $T_{s,t} - V_4$ is a disconnected graph having two components which are paths

P_{s-1} and P_t and $D(T_{s,t} - V_4) = 3$.
The two end vertices of the path P_{s-1} and one end vertex of the path P_t are the only vertices adjacent to the vertex in the sink set, hence these vertices are contained in different V_i's, for $1 \leq i \leq 3$. Thus, by Observation 4, V_i cannot be a sink set, where $1 \leq i \leq 3$. But, by Remark 3 there exists no D-partition of a path without sink set proving that $D(T_{s,t}) = 3$. □

Theorem 15. *For any unicyclic graph G, $D(G) = Tr(G)$.*

Proof. For any graph G, we know that $Tr(G) \leq D(G)$. It remains to show that for any unicyclic graph G, $D(G) \leq Tr(G)$. Assuming that G is a unicyclic graph for which $Tr(G) < D(G)$, let G_0 be the unicyclic graph with least number of vertices such that $Tr(G_0) < D(G_0) = k$. The graph G_0 is not a cycle since the upper domatic number of a cycle coincides with its transitivity. Therefore, G_0 has at least one leaf vertex. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ be a D-partition of G_0. Without loss of generality, assume that V_1 contains a leaf vertex.
Claim 1: V_1 is a source set.
If V_1 is not a source set, then by Observation 4 there exists a set $V_j \in \pi$ that dominates V_1. Let $v \in V_1$ be the leaf vertex. Since V_j dominates V_1, the graph obtained from G_0 by the removal of v is still a unicyclic graph with the same upper domatic number as G_0, which contradicts the assumption that G_0 is a unicyclic graph with the least number of vertices such that $Tr(G_0) < D(G_0)$.
Now consider the graph G_0' obtained from G_0 by removing the source set V_1. Then G_0' is either a connected graph that is acyclic or unicyclic, or disconnected graph where the components are trees or trees and a unicyclic graph.
Claim 2: The subgraph G_0' has equal transitivity and upper domatic number.
The inequality $Tr(G_0') \leq D(G_0')$ follows immediately. If G_0' is acyclic, then by Theorem 3, $D(G_0') = Tr(G_0')$. On the other hand, if G_0' is unicyclic, then it follows by the assumption that $D(G_0') = Tr(G_0')$. If G_0' is a disconnected graph with unicyclic and acyclic components say $G_1, G_2, \ldots G_r$, by Theorem 8, $D(G_0') \leq max\{D(G_i) \mid 1 \leq i \leq r\}$. But for $1 \leq i \leq r$, $D(G_i) = Tr(G_i)$ and by Theorem 9, $Tr(G_0') = max\{Tr(G_i) \mid 1 \leq i \leq r\}$.
Therefore,

$$D(G_0') \leq max\{D(G_i) \mid 1 \leq i \leq r\} = max\{Tr(G_i) \mid 1 \leq i \leq r\} = Tr(G_0')\}.$$

Thus proving the claim.
By Observation 6, $D(G_0') \geq D(G_0) - 1$ and V_1 being a source set in the upper domatic partition π, we can conclude that,

$$Tr(G_0) \geq Tr(G_0') + 1 = D(G_0') + 1 \geq D(G_0).$$

Thus, $D(G_0) = Tr(G_0)$, which is a contradiction to the assumption that $Tr(G_0) < D(G_0)$. □

A graph with at most one cycle in each of its components is called a pseudoforest. Theorem 15 can be extended for pseudoforests as well.

Theorem 16. *If G is a pseudoforest, then $D(G) = Tr(G)$.*

Proof. Consider a pseudoforest G having finite number of components, say r. Since each component of G is either acyclic or unicyclic, by Theorem 3 and Theorem 15, $D(G_i) = Tr(G_i)$ for each component G_i of G. Hence, $D(G) \leq max\{D(G_i)|1 \leq i \leq r\} = max\{Tr(G_i)|1 \leq i \leq r\} = Tr(G)$, but $D(G) \geq Tr(G)$. Therefore, $D(G) = Tr(G)$. □

We close this section by characterising the unicyclic graphs with equal upper domatic number and domatic number.

Theorem 17. *For any unicyclic graph G, $D(G) = d(G)$ if and only if G is C_{3k}, where k is a positive integer.*

Proof. If $G = C_{3k}$, then $D(G) = d(G) = 3$. On the other hand, let G be a unicyclic graph with $D(G) = d(G)$. Note that by Proposition 4, $D(G) \geq 3$ for any unicyclic graph G. But, by Theorem 1, $d(G) \leq 2$ for any graph G with a leaf vertex. A unicyclic graph G with $d(G) \geq 3$, does not have a leaf vertex, which implies that G is a cycle. It is well known that $d(C_{3k+1}) = d(C_{3k+2}) = 2$ while $d(C_{3k}) = 3$ [1], where k is a positive integer. Hence, G is C_{3k}, if $D(G) = d(G)$. □

4. Powers of graphs

The k^{th} power of a graph G is the graph G^k, with vertex set $V(G^k) = V(G)$ and two vertices $u, v \in V(G^k)$ are adjacent if and only if the distance between u and v is at most k in G. It is to be noted that for the graph G^k, the value of k is at most the diameter of G. Moreover, for any graph G of order n, $G^k = K_n$, when k coincides with the diameter of G. In this section, we discuss the upper domatic number of powers of paths and cycles. The k^{th} powers of a path on n vertices and a cycle on n vertices are denoted as P_n^k and C_n^k respectively. The following properties which follow from the definition are required for further discussion.

Proposition 5. *For the graphs P_n^k and C_n^k,*

1. *P_n^k is an induced subgraph of P_{n+1}^k,*

2. P_n^k *is a subgraph (not induced) of* C_n^k*,*

3. $\Delta(P_n^k) = \Delta(C_n^k) = 2k$*,*

4. $\omega(P_n^k) = k+1$*,*

5. $\omega(C_n^k) = k+1$*,* $k < \lfloor \frac{n}{2} \rfloor$*.*

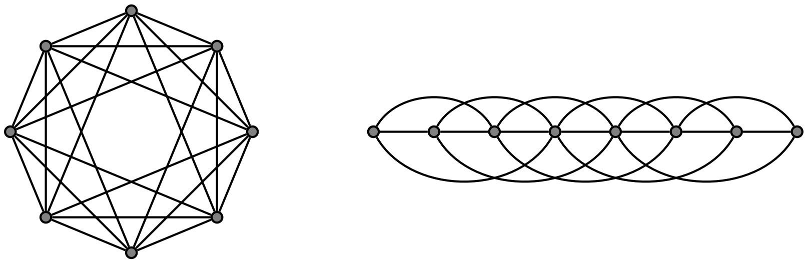

Figure 4. **Graphs C_8^3 and P_8^3.**

Theorem 18. *The upper domatic number of the* k^{th} *power of path* P_n *is*

$$D(P_n^k) = \begin{cases} k+1, & \textit{if } n = k+1, \\ l+k+1, & \textit{if } k+2 \le n \le 3k \ \textit{ where } l = \left\lfloor \frac{n-k-1}{2} \right\rfloor, \\ 2k+1, & \textit{if } n \ge 3k+1. \end{cases}$$

Proof. Consider the k^{th} power of a path P_n, $k \le n-1$ with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where $v_i v_{i+1} \in E(P_n)$, for $i = 1, 2, \ldots, n-1$. For different values of n, we consider the following cases.

Case 1: $n = k+1$

The k^{th} power of path on $k+1$ vertices is a complete graph of order $k+1$. Thus, Theorem 3 implies $D(P_{k+1}^k) = k+1$.

Case 2: $k+2 \le n \le 3k$

When $k+2 \le n \le 3k$, the vertices $v_{l+1}, v_{l+2}, \ldots, v_{l+k+1}$ where $l = \left\lfloor \frac{n-k-1}{2} \right\rfloor$ of P_n^k induces a clique of order $k+1$. Now partition the vertex set of P_n^k into $l+k+1$ sets in the following manner.

$$V_i = \begin{cases} \{v_{l+i}\}, & \text{if } 1 \le i \le k+1, \\ \{v_{i-k-1}, v_{l+i}\}, & \text{if } k+2 \le i \le l+k, \\ \{v_l, v_{2l+k+1}\}, & \text{if } i = l+k+1 \text{ and } n-k-1 \text{ is even}, \\ \{v_l, v_{2l+k+1}, v_n\}, & \text{if } i = l+k+1 \text{ and } n-k-1 \text{ is odd}. \end{cases}$$

We claim that the partition $\pi = \{V_1, V_2, \ldots, V_{l+k+1}\}$ is an upper domatic partition. Note that the vertices $v_1, v_2, \ldots v_l$ induces K_l, the vertices $v_{l+1}, v_{l+2}, \ldots, v_{l+k+1}$ induces K_{k+1} and the vertices $v_{l+k+2}, v_{l+k+3}, \ldots, v_n$ induces K_l or K_{l+1} depending on the parity of $n-k-1$. Since the sets $V_1, V_2, \ldots, V_{k+1}$ contains exactly one vertex from $\{v_{l+1}, v_{l+2}, \ldots, v_{l+k+1}\}$, the sets $V_1, V_2, \ldots, V_{k+1}$ dominate mutually. Further the sets $V_{k+2}, V_{k+3}, \ldots, V_{l+k+1}$ also dominate each other and each vertex of $V_1 \cup V_2 \cup \ldots \cup V_{k+1}$ is adjacent to at least one vertex from each of the sets $V_{k+2}, V_{k+3}, \ldots, V_{l+k+1}$. Therefore, the sets $V_{k+2}, V_{k+3}, \ldots, V_{l+k+1}$ are dominating sets and the partition π is an upper domatic partition. Thus, $D(P_n^k) \geq l+k+1$.

If $D(P_n^k) > l+k+1$, there exist an upper domatic partition of $l+k+2$ sets. By pigeonhole principle, in such a partition there should be at least $k+2$ singleton sets, which contradicts the fact that clique number of P_n^k is $k+1$. Thus, $D(P_n^k) = l+k+1$.

Case 3: $n = 3k+1$

For P_{3k+1}^k, the partition considered in Case 2 provides an upper domatic partition of order $l+k+1$. Hence, $D(P_{3k+1}^k) \geq l+k+1$ and by substituting for $l = \left\lfloor \frac{n-k-1}{2} \right\rfloor$ and $n = 3k+1$, we get $D(P_{3k+1}^k) \geq 2k+1$. Since $\Delta(P_n^k) = 2k$, by Theorem 2 $D(P_n^k) \leq 2k+1$, thus proving $D(P_n^k) = 2k+1$.

Case 4: $n > 3k+1$

For $n > 3k+1$, P_{3k+1}^k is a subgraph of P_n^k having a D-partition of order 2k+1 with a source set. Therefore, by Proposition 3 and Theorem 2, $D(P_n^k) = 2k+1$. □

Theorem 19. *For the graphs C_n^k,*

$$D(C_n^k) = \begin{cases} n, & \text{if } 2k \leq n \leq 2k+1, \\ l{+}k{+}1, & \text{if } 2k+1 < n \leq 3k \quad \text{where } l = \left\lfloor \frac{n-k-1}{2} \right\rfloor, \\ 2k+1, & \text{if } n \geq 3k+1. \end{cases}$$

Proof. We consider the following cases for different values of n.

Case 1: $2k \leq n \leq 2k+1$

Since $C_{2k}^k = K_{2k}$ and $C_{2k+1}^k = K_{2k+1}$, for $2k \leq n \leq 2k+1$, by Theorem 3, $D(C_n^k) = n$.

Case 2: $2k+1 < n \leq 3k$

Since P_n^k is a subgraph of C_n^k, for $2k+1 < n \leq 3k$, by Proposition 3 and Theorem 18, $D(C_n^k) \geq l+k+1$. But by pigeonhole principle, if $D(C_n^k) > l+k+1$, such an upper domatic partition will contain at least $k+2$ singleton sets which contradicts the fact that the clique number of C_n^k is $k+1$.

Case 3: $n \geq 3k+1$

By Theorem 18, $D(P_n^k) = 2k+1$, for $n \geq 3k+1$ and P_n^k is a subgraph of C_n^k, therefore $D(C_n^k) \geq 2k+1$. However, Theorem 2 implies that $D(C_n^k) \leq 2k+1$. Thus, $D(C_n^k) = 2k+1$. □

A graph is said to be *upper domatically full* if $D(G) = \Delta(G)+1$.

Corollary 4. *For $n \geq 3k+1$, the power graphs P_n^k and C_n^k are upper domatically full.*

Proof. Since $\Delta(P_n^k) = \Delta(C_n^k) = 2k$, it follows from Theorems 18 and 19 that both P_n^k and C_n^k are upper domatically full whenever $n \geq 3k + 1$. □

Acknowledgements

We would like to thank Prof. Teresa W. Haynes for introducing us to the concept of upper domatic number and for the constructive feedback that helped improve the presentation of the article. We gratefully acknowledge the precise and sharp reviews of the referee.

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