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Nonnegative signed total Roman domination in graphs

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Abstract: Let G be a finite and simple graph with vertex set $V(G)$. A nonnegative signed total Roman dominating function (NNSTRDF) on a graph G is a function $f : V(G) \to \{-1, 1, 2\}$ satisfying the conditions that (i) $\sum_{x\in N(v)} f(x) \geq 0$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, and (ii) every vertex u for which $f(u) = -1$ has a neighbor v for which $f(v) = 2$. The weight of an NNSTRDF f is $\omega(f) = \sum_{v\in V(G)} f(v)$. The nonnegative signed total Roman domination number $\gamma^{NN}_{stR}(G)$ of G is the minimum weight of an NNSTRDF on G. In this paper we initiate the study of the nonnegative signed total Roman domination number of graphs, and we present different bounds on $\gamma^{NN}_{stR}(G)$. We determine the nonnegative signed total Roman domination number of some classes of graphs. If n is the order and m is the size of the graph G, then we show that $\gamma^{NN}_{stR}(G) \geq \frac{3}{4}(\sqrt{8n+1}+1) - n$ and $\gamma^{NN}_{stR}(G) \geq (10n - 12m)/5$. In addition, if G is a bipartite graph of order n, then we prove that $\gamma^{NN}_{stR}(G) \geq \frac{3}{2}(\sqrt{4n+1}-1) - n$.

Keywords: Nonnegative signed total Roman dominating function, nonnegative signed total Roman domination number

AMS Subject classification: 05C69

1. Introduction

In this paper we continue the study of Roman dominating functions in graphs. Let G be a finite and simple graph with vertex set $V = V(G)$ and edge set $E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size

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of the graph G, respectively. We write $d_G(v) = d(v)$ for the degree of a vertex v. The minimum and maximum degree are $\delta(G) = \delta$ and $\Delta(G) = \Delta$. The sets $N_G(v) = N(v) = \{u \mid uv \in E(G)\}$ and $N_G[v] = N[v] = N(u) \cup \{v\}$ are called the open neighborhood and closed neighborhood of the vertex v, respectively. A graph G is regular or r-regular if $\Delta(G) = \delta(G) = r$. For disjoint subsets U and V of vertices, we denote by $[U, V]$ the set of edges between U and V. For a set $S \subseteq V(G)$, its open neighborhood is the set $N(S) = \bigcup_{v\in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. Also if $S \subseteq V(G)$, then $G[S]$ is the subgraph induced by S.
A cycle on n vertices is denoted by C_n, while a path on n vertices is denoted by P_n. We denote by K_n the complete graph on n vertices and by $K_{m,n}$ the complete bipartite graph with one partite set of cardinality m and the other of cardinality n. A star is a complete bipartite graph of the form $K_{1,n}$. A vertex of degree one is called a leaf. The complement of a graph G is denoted by $\overline{G}$.
For a real-valued function $f : V(G) \to R$, the weight of f is $\omega(f) = \sum_{v\in V(G)} f(v)$, and for $S \subseteq V(G)$, we define $f(S) = \sum_{v\in S} f(v)$, so $\omega(f) = f(V(G))$. Consult [4] and [5] for notation and terminology which are not defined here.
For an integer $k \geq 1$, a signed total Roman k-dominating function (STRkDF) on a graph G is defined in [8] as a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{u\in N_G(v)} f(u) \geq k$ for every $v \in V(G)$, and every vertex u for which $f(u) = -1$ is adjacent to a vertex v for which $f(v) = 2$. The weight of an STRkDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The signed total Roman k-domination number $\gamma^k_{stR}(G)$ of G is the minimum weight of an STRkDF on G. The special case $k = 1$ was introduced in [6]. Signed total Roman domination in graphs and digraphs is well studied in the literature, see for example [1–3, 7]. Following [8], we initiate the study of nonnegative signed total Roman dominating functions on graphs G.
Let G be a graph with $\delta(G) \geq 1$. A nonnegative signed total Roman dominating function (NNSTRDF) on G is defined as a function $f : V(G) \to \{-1, 1, 2\}$ such that $\sum_{u\in N(v)} f(u) \geq 0$ for every $v \in V(G)$ and every vertex u for which $f(u) = -1$ has a neighbor v for which $f(v) = 2$. The weight of an NNSTRDF f on a graph G is $\omega(f) = \sum_{v\in V(G)} f(v)$. The nonnegative signed total Roman domination number $\gamma^{NN}_{stR}(G)$ of G is the minimum weight of an NNSTRDF on G. A $\gamma^{NN}_{stR}(G)$-function is a nonnegative signed total Roman dominating function on G of weight $\gamma^{NN}_{stR}(G)$. For an NNSTRDF f on G, let $V_i = V_i^f = \{v \in V(G) : f(v) = i\}$ for $i = -1, 1, 2$. An NNSTRDF $f : V(G) \to \{-1, 1, 2\}$ can be represented by the ordered partition (V_{-1}, V_1, V_2) of $V(G)$. Further, we let $n_{-1} = |V_{-1}|$, $n_1 = |V_1|$, $n_2 = |V_2|$, and so $n = n_2 + n_1 + n_{-1}$. Therefore $\gamma^{NN}_{stR}(G) = 2n_2 + n_1 - n_{-1}$.
We present different sharp lower and upper bounds on $\gamma^{NN}_{stR}(G)$. We determine the nonnegative signed total Roman domination number of some classes of graphs. We show that $\gamma^{NN}_{stR}(G) \geq \frac{3}{4}(\sqrt{8n+1}+1)-n$ and $\gamma^{NN}_{stR}(G) \geq (10n-12m)/5$. In addition, if G is a bipartite graph of order n, then we prove that $\gamma^{NN}_{stR}(G) \geq \frac{3}{2}(\sqrt{4n+1}-1)-n$.

2. Special classes of graphs

In this section, we determine the nonnegative signed total Roman domination number of special classes of graphs.

Proposition 1. For $n \geq 1$, $\gamma_{stR}^{NN}(K_{1,n}) = 2$.

Proof. Let u be the central vertex, and let $\{u_1, u_2, \ldots, u_n\}$ be the leaves of the star $K_{1,n}$. If $n = 1, 2$, then it is easy to see that $\gamma_{stR}^{NN}(K_{1,n}) = 2$. Thus let $n \geq 3$. First we show that $\gamma_{stR}^{NN}(K_{1,n}) \geq 2$. Let f be a $\gamma_{stR}^{NN}(K_{1,n})$-function. Since $N(u_i) = \{u\}$ for every $1 \leq i \leq n$, we deduce that $f(u) \neq -1$. If $f(u) = 1$, then $f(u_i) \neq -1$ for every $1 \leq i \leq n$ and so $\gamma_{stR}^{NN}(K_{1,n}) = n + 1 > 2$. Now let $f(u) = 2$. Thus

$$\gamma_{stR}^{NN}(K_{1,n}) = \sum_{1 \leq i \leq n} f(u_i) + f(u) = f(N(u)) + f(u) \geq 0 + 2 = 2.$$

Now we show that $\gamma_{stR}^{NN}(K_{1,n}) \leq 2$. First let n be even. Define the function $f : V(K_{1,n}) \to \{-1, 1, 2\}$ by $f(u) = 2$ and $f(u_i) = (-1)^i$ for every $1 \leq i \leq n$. Then the function f is an NNSTRDF on $K_{1,n}$ of weight 2 and thus $\gamma_{stR}^{NN}(K_{1,n}) \leq 2$. This implies that $\gamma_{stR}^{NN}(K_{1,n}) = 2$ when n is even.
Now let n be odd. Define the function $f : V(K_{1,n}) \to \{-1, 1, 2\}$ by $f(u) = 2$, $f(u_1) = 2$, $f(u_2) = f(u_3) = -1$ and $f(u_i) = (-1)^i$ for every $4 \leq i \leq n$. Then the function f is an NNSTRDF on $K_{1,n}$ of weight 2 and so $\gamma_{stR}^{NN}(K_{1,n}) \leq 2$. This implies that $\gamma_{stR}^{NN}(K_{1,n}) = 2$ when n is odd and the proof is complete. □

Proposition 2. For $n \geq 2$, $\gamma_{stR}^{NN}(K_n) = 2$.

Proof. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$. First we show that $\gamma_{stR}^{NN}(K_n) \geq 2$. Let f be a $\gamma_{stR}^{NN}(K_n)$-function. If $f(u_i) \neq -1$ for every $1 \leq i \leq n$, then $\gamma_{stR}^{NN}(K_n) = n \geq 2$. Now we may assume that $f(u_1) = -1$. Thus there is an index $i \neq 1$, we may assume that $i = 2$, such that $f(u_2) = 2$. This leads to

$$\gamma_{stR}^{NN}(K_n) = \sum_{i \neq 2} f(u_i) + f(u_2) = f(N(u_2)) + f(u_2) \geq 0 + 2 = 2.$$

Now we show that $\gamma_{stR}^{NN}(K_n) \leq 2$. First let n be even. If $n = 2$, then Proposition 1 implies that $\gamma_{stR}^{NN}(K_2) = 2$. Now let $n \geq 4$. Define the function $f : V(K_n) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_2) = 2$, $f(u_3) = f(u_4) = -1$ and $f(u_i) = (-1)^i$ for each vertex $u_i \in V - \{u_1, u_2, u_3, u_4\}$. Then the function f is an NNSTRDF on K_n of weight 2 and thus $\gamma_{stR}^{NN}(K_n) \leq 2$. Hence $\gamma_{stR}^{NN}(K_n) = 2$ when n is even.
Now let n be odd and $n \geq 3$. Define the function $f : V(K_n) \to \{-1, 1, 2\}$ by $f(u_1) = 2$ and $f(u_i) = (-1)^i$ for each $2 \leq i \leq n$. Then f is an NNSTRDF on K_n of weight 2 and thus $\gamma_{stR}^{NN}(K_n) \leq 2$. Hence $\gamma_{stR}^{NN}(K_n) = 2$ when n is odd and $n \neq 1$. □

Proposition 3. For $n \geq 3$, $\gamma_{stR}^{NN}(P_n) = \lceil \frac{n}{2} \rceil$ when $n \equiv 0, 1, 3 \pmod 4$ and $\gamma_{stR}^{NN}(P_n) = \frac{n}{2} + 1$ when $n \equiv 2 \pmod 4$.

Proof. Let $P_n := u_1 u_2 \ldots u_n$ and let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stR}^{NN}(P_n)$-function. Then $n_{-1} \leq n_2$ and therefore

$$\gamma_{stR}^{NN}(P_n) = 2n_2 + n_1 - n_{-1} \geq n_2 + n_1 \geq \frac{n_2 + n_1 + n_{-1}}{2} = \frac{n}{2}.$$

This implies $\gamma_{stR}^{NN}(P_n) \geq \lceil \frac{n}{2} \rceil$. If $n = 3$, then Proposition 1 leads to the desired result. For $n \geq 4$ we distinguish four cases.

Case 1. Let $n = 4p$ for an integer $p \geq 1$. Define the function $f : V(P_n) \to \{-1, 1, 2\}$ by $f(u_{4i+1}) = f(u_{4i+4}) = -1$ and $f(u_{4i+2}) = f(u_{4i+3}) = 2$ for $0 \leq i \leq p - 1$. Then the function f is an NNSTRDF on P_n of weight $\omega(f) = \frac{n}{2}$ and thus $\gamma_{stR}^{NN}(P_n) = \frac{n}{2}$ in this case.

Case 2. Let $n = 4p + 1$ for an integer $p \geq 1$. Define the function $f : V(P_n) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_{4p+1}) = -1$, $f(u_2) = f(u_{4p}) = 2$, $f(u_3) = 1$, $f(u_{4i}) = f(u_{4i+3}) = 2$ and $f(u_{4i+1}) = f(u_{4i+2}) = -1$ for $1 \leq i \leq p - 1$. Then the function f is an NNSTRDF on P_n of weight $\omega(f) = 2p + 1 = \lceil \frac{n}{2} \rceil$ and thus $\gamma_{stR}^{NN}(P_n) = \lceil \frac{n}{2} \rceil$.

Case 3. Let $n = 4p + 3$ for an integer $p \geq 1$. Define the function $f : V(P_n) \to \{-1, 1, 2\}$ by $f(u_{4p+1}) = -1$, $f(u_{4p+2}) = 2$, $f(u_{4p+3}) = 1$, $f(u_{4i+1}) = f(u_{4i+4}) = -1$ and $f(u_{4i+2}) = f(u_{4i+3}) = 2$ for $0 \leq i \leq p - 1$. Then the function f is an NNSTRDF on P_n of weight $\omega(f) = 2p + 2 = \lceil \frac{n}{2} \rceil$ and thus $\gamma_{stR}^{NN}(P_n) = \lceil \frac{n}{2} \rceil$.

Case 4. Let $n = 4p + 2$ for an integer $p \geq 1$. If $n_1 \geq 1$, then it follows that

$$\gamma_{stR}^{NN}(P_n) = 2n_2 + n_1 - n_{-1} \geq n_2 + n_1 > \frac{n_2 + n_1 + n_{-1}}{2} = \frac{n}{2}.$$

This implies $\gamma_{stR}^{NN}(P_n) = \frac{n}{2} + 1$ when $n_1 \geq 1$. Now let $n_1 = 0$, and let g be a $\gamma_{stR}^{NN}(P_n)$-function. Since $g(u_2) = g(u_{4p+1}) = 2$, we observe that $g(u_1) + g(u_2) + g(u_3) \geq 3$ and $g(u_{4p}) + g(u_{4p+1}) + g(u_{4p+2}) \geq 3$. In addition, we note that $g(u_{4i}) + g(u_{4i+1}) + g(u_{4i+2}) + g(u_{4i+3}) \geq 2$ for $1 \leq i \leq p - 1$. Therefore we obtain

$$\begin{aligned}\gamma_{stR}^{NN}(P_n) &= g(u_1) + g(u_2) + g(u_3) + \sum_{i=1}^{p-1}(g(u_{4i}) + g(u_{4i+1}) + g(u_{4i+2}) + g(u_{4i+3})) \\ &+ g(u_{4p}) + g(u_{4p+1}) + g(u_{4p+2}) \geq 3 + 2(p-1) + 3 = 2p + 4 > \frac{n}{2} + 1.\end{aligned}$$

Thus $\gamma_{stR}^{NN}(P_n) \geq \frac{n}{2} + 1$. For the converse define the function $f : V(P_n) \to \{-1, 1, 2\}$ by $f(u_1) = f(u_{4p+2}) = -1$, $f(u_2) = f(u_{4p+1}) = 2$, $f(u_3) = f(u_{4p}) = 1$, $f(u_{4i}) = f(u_{4i+3}) = 2$ and $f(u_{4i+1}) = f(u_{4i+2}) = -1$ for $1 \leq i \leq p - 1$. Then the function f is an NNSTRDF on P_n of weight $\omega(f) = 2p + 2 = \frac{n}{2} + 1$ and hence $\gamma_{stR}^{NN}(P_n) = \frac{n}{2} + 1$ in this case. □

By using an argument similar to that described in the proof of Proposition 3, we obtain the next proposition.

Proposition 4. For $n \geq 3$, $\gamma_{stR}^{NN}(C_n) = \lceil \frac{n}{2} \rceil$ when $n \equiv 0, 1, 3 \pmod 4$ and $\gamma_{stR}^{NN}(C_n) = \frac{n}{2} + 1$ when $n \equiv 2 \pmod 4$.

In Proposition 1, we determined exact values of the nonnegative signed total Roman domination number of $K_{1,n}$. In the following, we determine exact values of the nonnegative signed total Roman domination number of $K_{m,n}$ for $n, m \geq 2$.

Proposition 5. For $n \geq 2$,

$$\gamma_{stR}^{NN}(K_{2,n}) = \begin{cases} 2 & n = 2 \text{ or } n = 4 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Let $K_{2,n}$ be a complete bipartite graph with partite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. If $n = 2$, then by Proposition 4, $\gamma_{stR}^{NN}(K_{2,n}) = 2$. Now let $n = 4$. Define the function $f : V(K_{2,4}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(y_2) = 1$ and $f(x_2) = f(y_3) = f(y_4) = -1$. Then the function f is an NNSTRDF on $K_{2,4}$ of weight 2 and thus $\gamma_{stR}^{NN}(K_{2,4}) \leq 2$. Now let g be a $\gamma_{stR}^{NN}(K_{2,4})$-function. If $g(x_1), g(x_2) \neq 2$, then for each i, $g(y_i) \neq -1$. Thus $\gamma_{stR}^{NN}(K_{2,4}) = \omega(g) = \sum_{u \in X \cup Y} g(u) \geq 4$, a contradiction. Now let $g(x_1) = 2$. If for each i, $g(y_i) \neq 2$, then $g(x_2) \neq -1$. Thus $\gamma_{stR}^{NN}(K_{2,4}) = \omega(g) = g(x_1) + g(x_2) + g(N(x_2)) \geq 2 + 1 + 0 = 3$, a contradiction. Next let, without loss of generality, $g(y_1) = 2$. It is easy to see that $\sum_{1 \leq i \leq 4} g(y_i) \geq 1$ and thus

$$\gamma_{stR}^{NN}(K_{2,4}) = \omega(g) = g(x_1) + g(x_2) + \sum_{1 \leq i \leq 4} g(y_i) \geq 2 - 1 + 1 = 2.$$

Now let $n \neq 2, 4$. If n is even, then $n \geq 6$ and define the function $f : V(K_{2,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = f(y_2) = 2$, $f(x_2) = f(y_3) = f(y_4) = f(y_5) = f(y_6) = -1$ and $f(y_i) = (-1)^i$ for $7 \leq i \leq n$. Thus the function f is an NNSTRDF on $K_{2,n}$ of weight 1 and so $\gamma_{stR}^{NN}(K_{2,n}) \leq 1$. If n is odd, then define the function $f : V(K_{2,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = f(y_2) = f(y_3) = -1$ and $f(y_i) = (-1)^i$ for $4 \leq i \leq n$. Thus f is an NNSTRDF on $K_{2,n}$ of weight 1 and hence $\gamma_{stR}^{NN}(K_{2,n}) \leq 1$. Now let g be a $\gamma_{stR}^{NN}(K_{2,n})$-function. If $g(x_1), g(x_2) \neq 2$, then for each i, $g(y_i) \neq -1$. It follows that $\gamma_{stR}^{NN}(K_{2,n}) = \omega(g) = \sum_{u \in X \cup Y} g(u) \geq 4$, a contradiction. Assume next, without loss of generality, that $g(x_1) = 2$. Then

$$\gamma_{stR}^{NN}(K_{2,n}) = \omega(g) = g(x_1) + g(x_2) + g(N(x_2)) \geq 2 - 1 + 0 = 1,$$

and this completes the proof. □

Proposition 6. For $n \geq m \geq 3$,

$$\gamma_{stR}^{NN}(K_{m,n}) = \begin{cases} 2 & m = n = 4 \\ 1 & m = 3 \text{ and } n = 4 \text{ or } m = 4 \text{ and } n \geq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $K_{m,n}$ be a complete bipartite graph with partite sets $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. First let $m = n = 4$. Define the function $f : V(K_{4,4}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = f(y_2) = 1$ and $f(x_3) = f(x_4) = f(y_3) = f(y_4) = -1$. Then the function f is an NNSTRDF on $K_{4,4}$ of weight 2 and thus $\gamma_{stR}^{NN}(K_{4,4}) \leq 2$. Now let g be a $\gamma_{stR}^{NN}(K_{4,4})$-function. If $g(x_i) \neq 2$ for every i ($g(y_j) \neq 2$ for every j is similar), then for each j, $g(y_j) \neq -1$. Thus $\gamma_{stR}^{NN}(K_{4,4}) = \omega(g) = \sum_{u \in X \cup Y} g(u) \geq 5$, a contradiction. Next let, without loss of generality, $g(x_1) = g(y_1) = 2$. It is easy to see that $\sum_{1 \leq i \leq 4} g(x_i) \geq 1$ and $\sum_{1 \leq j \leq 4} g(y_j) \geq 1$. Thus

$$\gamma_{stR}^{NN}(K_{4,4}) = \omega(g) = \sum_{1 \leq i \leq 4} g(x_i) + \sum_{1 \leq j \leq 4} g(y_j) \geq 1 + 1 = 2.$$

Assume now that $m = 4$ or $n = 4$. If $m = 3$ and $n = 4$, then define the function $f : V(K_{3,4}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(y_2) = 1$ and $f(x_2) = f(x_3) = f(y_3) = f(y_4) = -1$. Thus f is an NNSTRDF on $K_{3,4}$ of weight 1 and so $\gamma_{stR}^{NN}(K_{3,4}) \leq 1$. Now let $m = 4$ and $n \geq 5$. If n is even, then $n \geq 6$. Define the function $f : V(K_{4,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = f(y_2) = 2$, $f(x_2) = 1$, $f(x_3) = f(x_4) = f(y_3) = f(y_4) = f(y_5) = f(y_6) = -1$ and $f(y_i) = (-1)^i$ for $7 \leq i \leq n$. Thus the function f is an NNSTRDF on $K_{4,n}$ of weight 1 and then $\gamma_{stR}^{NN}(K_{4,n}) \leq 1$. If n is odd, then define the function $f : V(K_{4,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = 1$, $f(x_3) = f(x_4) = f(y_2) = f(y_3) = -1$ and $f(y_i) = (-1)^i$ for $4 \leq i \leq n$. Thus f is an NNSTRDF on $K_{4,n}$ of weight 1 and hence $\gamma_{stR}^{NN}(K_{4,n}) \leq 1$. Now let g be a $\gamma_{stR}^{NN}(K_{m,n})$-function. If $g(x_i) \neq 2$ for every i ($g(y_j) \neq 2$ for every j is similar), then for each j, $g(y_j) \neq -1$. Then $\gamma_{stR}^{NN}(K_{m,n}) = \omega(g) = \sum_{u \in X \cup Y} g(u) \geq 4$, a contradiction. Next assume, without loss of generality, that $g(x_1) = g(y_1) = 2$. If $m = 3$ and $n = 4$, then it is easy to see that $\sum_{1 \leq j \leq 4} g(y_j) \geq 1$. Thus

$$\gamma_{stR}^{NN}(K_{3,4}) = \omega(g) = \sum_{1 \leq i \leq 3} g(x_i) + \sum_{1 \leq j \leq 4} g(y_j) = f(N(y_1)) + \sum_{1 \leq j \leq 4} g(y_j) \geq 0 + 1 = 1.$$

If $m = 4$ and $n \geq 5$, then $\sum_{1 \leq j \leq 4} g(x_i) \geq 1$. Thus

$$\gamma_{stR}^{NN}(K_{4,n}) = \omega(g) = \sum_{1 \leq i \leq 4} g(x_i) + \sum_{1 \leq j \leq n} g(y_j) = \sum_{1 \leq i \leq 4} g(x_i) + f(N(x_1)) \geq 1 + 0 = 1.$$

Now let $m, n \neq 4$. If $m = n = 3$, then define the function $f : V(K_{3,3}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$ and $f(x_2) = f(x_3) = f(y_2) = f(y_3) = -1$. Then f is an

NNSTRDF on $K_{3,3}$ of weight 0 and thus $\gamma_{stR}^{NN}(K_{3,3}) \leq 0$. Next let $m = 3$ and $n \geq 5$. If n is even, then define $f : V(K_{3,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = f(y_2) = 2$, $f(x_2) = f(x_3) = f(y_3) = f(y_4) = f(y_5) = f(y_6) = -1$, $f(y_i) = (-1)^i$ for $7 \leq i \leq n$. Then f is an NNSTRDF on $K_{3,n}$ of weight 0 and thus $\gamma_{stR}^{NN}(K_{3,n}) \leq 0$. If n is odd, then define the function $f : V(K_{3,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = f(x_3) = f(y_2) = f(y_3) = -1$ and $f(y_i) = (-1)^i$ for $4 \leq i \leq n$. Then f is an NNSTRDF on $K_{3,n}$ of weight 0 and thus $\gamma_{stR}^{NN}(K_{3,n}) \leq 0$. Now assume that $m \geq 5$. First let $m + n$ is even. If m and n are even, then define the function $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(x_2) = f(y_1) = f(y_2) = 2$, $f(x_3) = f(x_4) = f(x_5) = f(x_6) = f(y_3) = f(y_4) = f(y_5) = f(y_6) = -1$, $f(x_i) = (-1)^i$ for $7 \leq i \leq m$ and $f(y_j) = (-1)^j$ for $7 \leq j \leq n$. Then f is an NNSTRDF on $K_{m,n}$ of weight 0 and thus $\gamma_{stR}^{NN}(K_{m,n}) \leq 0$. If m and n are odd, then define the function $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = 2$, $f(x_2) = f(x_3) = f(y_2) = f(y_3) = -1$, $f(x_i) = (-1)^i$ for $4 \leq i \leq m$ and $f(y_j) = (-1)^j$ for $4 \leq j \leq n$. Then f is an NNSTRDF on $K_{m,n}$ of weight 0 and thus $\gamma_{stR}^{NN}(K_{m,n}) \leq 0$. Now let $m + n$ be odd. We may assume that m is odd and n is even (the case m is even and n is odd is similar). Then define the function $f : V(K_{m,n}) \to \{-1, 1, 2\}$ by $f(x_1) = f(y_1) = f(y_2) = 2$, $f(x_2) = f(x_3) = f(y_3) = f(y_4) = f(y_5) = f(y_6) = -1$, $f(x_i) = (-1)^i$ for $4 \leq i \leq m$ and $f(y_j) = (-1)^j$ for $7 \leq j \leq n$. Then f is an NNSTRDF on $K_{m,n}$ of weight 0 and thus $\gamma_{stR}^{NN}(K_{m,n}) \leq 0$.
Now we show that $\gamma_{stR}^{NN}(K_{m,n}) \geq 0$. Let g be a $\gamma_{stR}^{NN}(K_{m,n})$-function. It follows that

$$\gamma_{stR}^{NN}(K_{m,n}) = \omega(g) = \sum_{1 \leq i \leq m} g(x_i) + \sum_{1 \leq j \leq n} g(y_j) = f(N(x_1)) + f(N(y_1)) \geq 0,$$

and this completes the proof. □

3. Bounds on $\gamma_{stR}^{NN}(G)$

In this section we start with some simple upper bounds on the nonnegative signed total Roman domination number of a graph. Furthermore, we show that $\gamma_{stR}^{NN}(G) \geq \frac{3}{4}(\sqrt{8n+1} + 1) - n$ and $\gamma_{stR}^{NN}(G) \geq (10n - 12m)/5$. In addition, if G is a bipartite graph of order n, then we prove that $\gamma_{stR}^{NN}(G) \geq \frac{3}{2}(\sqrt{4n+1} - 1) - n$.

Proposition 7. If G is a connected graph of order $n \geq 2$, then

$$\gamma_{stR}^{NN}(G) \leq n,$$

with equality if and only if $G = K_2$.

Proof. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v) = 1$ for each vertex $v \in V(G)$. Then the function f is an NNSTRDF on G of weight n and thus $\gamma_{stR}^{NN}(G) \leq n$. By Proposition 1, if $G = K_2$, then $\gamma_{stR}^{NN}(G) = 2 = n$.

Conversely, assume that $\gamma_{stR}^{NN}(G) = n$. If the diameter, $\text{diam}(G) = 1$, then G is the complete graph, and Proposition 2 implies the desired result. Let now $\text{diam}(G) \geq 2$, and let $u_1u_2\dots u_p$ be a diametral path. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(u_1) = -1$, $f(u_2) = 2$ and $f(x) = 1$ otherwise. Since $p \geq 3$, it is easy to verify that f is an NNSTRDF on G of weight $n - 1$, a contradiction. □

Corollary 1. Let G be a graph of order $n \geq 2$ with $\delta(G) \geq 1$. Then $\gamma_{stR}^{NN}(G) = n$ if and only if G consists of $\frac{n}{2}$ complete graphs K_2.

Theorem 1. If G is a graph of order $n \geq 2$ with $\delta(G) \geq 2$, then

$$\gamma_{stR}^{NN}(G) \leq n + 1 - 2\lfloor \frac{\delta(G)}{2} \rfloor.$$

Proof. Define $t = \lfloor \frac{\delta(G)}{2} \rfloor$. Let $v \in V(D)$ be a vertex of maximum degree, and let $A = \{u_1, u_2, \dots, u_t\}$ be a set of t neighbors of v. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v) = 2$, $f(u_i) = -1$ for $1 \leq i \leq t$ and $f(w) = 1$ for $w \in V(G) - (A \cup \{v\})$. If $x \in V(G) - (A \cup \{v\})$, then

$$f(N(x)) \geq -t + (\delta(G) - t) = \delta(G) - 2t = \delta(G) - 2\lfloor \frac{\delta(G)}{2} \rfloor \geq 0.$$

If $x \in A$, then

$$f(N(x)) \geq -(t-1) + 2 + (\delta(G) - t) = \delta(G) + 3 - 2t = \delta(G) + 3 - 2\lfloor \frac{\delta(G)}{2} \rfloor \geq 0.$$

Now if $x = v$, then

$$f(N(x)) = -t + (\Delta(G) - t) = \Delta(G) - 2t = \Delta(G) - 2\lfloor \frac{\delta(G)}{2} \rfloor \geq 0.$$

Therefore f is an NNSTRDF on G of weight $2 - t + (n - t - 1) = n + 1 - 2t$ and thus $\gamma_{stR}^{NN}(G) \leq n + 1 - 2t = n + 1 - 2\lfloor \frac{\delta(G)}{2} \rfloor$. □

Proposition 2 shows that Theorem 1 is sharp when n is odd.
In [8], the following proposition for the signed total Roman k-domination function is proved when $k \geq 1$.

Proposition 8. *[8]* Let $k \geq 1$ be an integer. Assume that $f = (V_{-1}, V_1, V_2)$ is an STRkDF on a graph G of order n. If $\delta \geq k$, then

1. $(\Delta + \delta)\omega(f) \geq (\delta + 2k - \Delta)n + (\delta - \Delta)|V_2|$.
2. $\omega(f) \geq \frac{(\delta+2k-2\Delta)n}{2\Delta+\delta} + |V_2|$.

It is a simple matter to verify that Proposition 8 remains valid for $k = 0$. Hence we have the following useful result.

Proposition 9. If $f = (V_{-1}, V_1, V_2)$ is an NNSTRDF on a graph G of order $n \geq 2$ and minimum degree $\delta \geq 1$, then

1. $(\Delta + \delta)\omega(f) \geq (\delta - \Delta)n + (\delta - \Delta)|V_2|$.
2. $\omega(f) \geq \frac{(\delta-2\Delta)n}{2\Delta+\delta} + |V_2|$.

As an application of the 1. inequality in Proposition 9, we obtain a lower bound on the nonnegative signed total Roman domination number for regular graphs.

Corollary 2. If G is an r-regular graph with $r \geq 1$, then $\gamma_{stR}^{NN}(G) \geq 0$.

Propositions 6 demonstrates that Corollary 2 is sharp when $m = n$ and $m \geq 5$.

Corollary 3. If G is a graph with $1 \leq \delta < \Delta$, then

$$\gamma_{stR}^{NN}(G) \geq \frac{2n(\delta - \Delta)}{2\Delta + \delta}$$

Proof. Multiplying both sides of the inequality 2. in Proposition 9 by $\Delta - \delta$ and adding the resulting inequality to the inequality 1. in Proposition 9, we obtain

$$\gamma_{stR}^{NN}(G) \geq \frac{(-4\Delta^2 + 4\Delta\delta)n}{2\Delta(2\Delta + \delta)} = \frac{2n(\delta - \Delta)}{2\Delta + \delta}.$$

□

Example 1. *Let $x_1, x_2, \ldots, x_{2p-2}$ be the leaves of the star $K_{1,2p-2}$ with $p \geq 3$. If we add the edges $x_1x_2, x_2x_3, \ldots, x_{2p-3}x_{2p-2}, x_{2p-2}x_1$ to the star $K_{1,2p-2}$, then denote the resulting graph by H. Now let $H_1, H_2, \ldots, H_p$ be p copies of H with the central vertices $v_1, v_2, \ldots, v_p$. Define the graph G as the disjoint union of $H_1, H_2, \ldots, H_p$ such that all central vertices are pairwise adjacent. Then $\delta(G) = 3$, $\Delta(G) = 3(p-1)$ and $n(G) = p(2p-1)$. Define the function $f : V(G) \to \{-1, 1, 2\}$ by $f(v_i) = 2$ for $1 \leq i \leq p$ and $f(x) = -1$ otherwise. It is easy to verify that $\sum_{x \in N(u)} f(x) = 0$ for every vertex $u \in V(G)$. Therefore f is an NNSTRDF on G of weight*

$$\omega(f) = -2p(p-2) = \frac{2n(G)(\delta(G) - \Delta(G))}{2\Delta(G) + \delta(G)}.$$

Example 1 shows that Corollary 3 is sharp.

Theorem 2. Let G be a graph of order $n \geq 2$ with $\delta(G) \geq 1$. Then

$$\gamma_{stR}^{NN}(G) \geq \delta(G) + 3 - n.$$

Proof. Let f be a $\gamma_{stR}^{NN}(G)$-function. If $f(x) = 1$ for each vertex $x \in V(G)$, then $\gamma_{stR}^{NN}(G) = n \geq \delta(G)+3-n$. Now assume that there exists a vertex w with $f(w) = -1$. Then w has a neighbor v with $f(v) = 2$. Therefore we obtain the desired bound as follows.

$$\begin{aligned}\gamma_{stR}^{NN}(G) &= \sum_{x \in V(G)} f(x) = f(v) + \sum_{x \in N(v)} f(x) + \sum_{x \in V(G)-N[v]} f(x) \\ &\geq 2 + 0 - (n - d(v) - 1) = 3 + d(v) - n \geq \delta(G) + 3 - n.\end{aligned}$$

□

Proposition 2 shows that Theorem 2 is sharp.

Corollary 4. Let G be an r-regular graph of order n with $r \geq 1$. If $r = n-2$, then $\gamma_{stR}^{NN}(G) \geq 1$.

Corollary 4 is an improvement of Corollary 2 for the special case that G is $(n-2)$-regular. Combining Corollary 4 with Theorem 1, we arrive at the next result.

Corollary 5. Let G be an r-regular graph of order n with $r \geq 1$. If $r = n-2$ and n is even, then $1 \leq \gamma_{stR}^{NN}(G) \leq 3$, and if $r = n-2$ and n is odd, then $1 \leq \gamma_{stR}^{NN}(G) \leq 4$.

We call a set $S \subseteq V(G)$ a 2-packing of the graph G if $N[u] \cap N[v] = \emptyset$ for any two distinct vertices of $u, v \in S$. The maximum cardinality of a 2-packing is the 2-packing number of G, denoted by $\rho(G)$.

Theorem 3. If G is a graph of order $n \geq 2$ with $\delta(G) \geq 1$, then

$$\gamma_{stR}^{NN}(G) \geq \delta(G) \cdot \rho(G) - n.$$

Proof. Let $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ be a 2-packing of G, and let f be a $\gamma_{stR}^{NN}(G)$-function. If we define the set $A = \bigcup_{i=1}^{\rho(G)} N(v_i)$ then, since $\{v_1, v_2, \ldots, v_{\rho(G)}\}$ is a 2-packing of G, we have

$$|A| = \sum_{i=1}^{\rho(G)} d(v_i) \geq \delta(G) \cdot \rho(G).$$

It follows that

$$\begin{aligned}\gamma_{stR}^{NN}(G) = \sum_{u \in V(G)} f(u) &= \sum_{i=1}^{\rho(G)} f(N(v_i)) + \sum_{u \in V(G)-A} f(u) \\ &\geq \sum_{u \in V(G)-A} f(u) \geq -n + |A| \\ &\geq \delta(G) \cdot \rho(G) - n.\end{aligned}$$

□

Corollary 6. If G is a graph of order $n \geq 2$ with $\delta(G) \geq 1$, then

$$\gamma_{stR}^{NN}(G) \geq \delta(G)(1 + \lfloor \frac{\text{diam(G)}}{3} \rfloor) - n.$$

Proof. Let $d = \text{diam(G)} = 3t + r$ with integers $t \geq 0$ and $0 \leq r \leq 2$, and let $\{v_1, v_2, \ldots, v_d\}$ be a diametral path. Then $A = \{v_0, v_3, \ldots, v_{3t}\}$ is a 2-packing of G such that $|A| = 1 + \lfloor \frac{\text{diam(G)}}{3} \rfloor$. Since $\rho(G) \geq |A|$, Theorem 3 implies that

$$\gamma_{stR}^{NN}(G) \geq \delta(G) \cdot \rho(G) - n \geq \delta(G)(1 + \lfloor \frac{\text{diam(G)}}{3} \rfloor) - n.$$

□

Example 2. *Let B be isomorphic to the complete graph K_{p^2} with vertex set $\{x_1, x_2, \ldots, x_{p^2}\}$, and let $A_1, A_2, \ldots, A_p$ be isomorphic to the complete graph K_{2p+1} with $p \geq 2$. Now let H be the disjoint union of $A_1, A_2, \ldots, A_p$ and B such that each vertex of A_i is adjacent to each vertex of $\{x_{(i-1)p+1}, x_{(i-1)p+2}, \ldots, x_{ip}\}$ for $1 \leq i \leq p$. Then $\delta(H) = 3p$, $\rho(H) = p$ and $n(H) = 3p^2 + p$. Define the function $f : V(H) \to \{-1, 1, 2\}$ by $f(x_i) = 2$ for $1 \leq i \leq p^2$ and $f(x) = -1$ otherwise. It is easy to verify that $\sum_{x \in N(u)} f(x) \geq 0$ for every vertex $u \in V(H)$. Therefore f is an NNSTRDF on H of weight*

$$\omega(f) = -p = \delta(H) \cdot \rho(H) - n.$$

Example 2 shows that the Theorem 3 is sharp.
Now we determine a lower bound on the nonnegative signed total Roman domination number of a graph. For this purpose, we define a family of graphs as follows. For $k \geq 2$, let $\mathcal{F}_k = \{F_k \mid k \geq 2\}$ be a family of graph as follows. Let X be the vertex set of the complete graph K_k, and let F_k be the graph obtained from K_k by adding $2k-2$ new vertices to each vertex of the complete graph such that for each new vertex x, $1 \leq d(x) \leq 3$ and for every $u \in X$, $d(u) = 3(k-1)$. We note that F_k has order $n = k(2k-1) = 2k^2 - k$. Let $\mathcal{F} = \bigcup_{k \geq 2} \mathcal{F}_k$.

Theorem 4. If G is a graph of order $n \geq 2$ with $\delta(G) \geq 1$, then

$$\gamma_{stR}^{NN}(G) \geq \frac{3}{4}(\sqrt{8n+1} + 1) - n,$$

with equality if and only if $G \in \mathcal{F}$.

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stR}^{NN}(G)$-function. If $V_{-1} = \emptyset$, then $\gamma_{stR}^{NN}(G) = n \geq \frac{3}{4}(\sqrt{8n+1} + 1) - n$. Hence, we may assume that $V_{-1} \neq \emptyset$. Since each vertex in V_{-1} has at least one neighbor in V_2, it follows from the Pigeonhole Principle that at least one vertex v of V_2 has at least $\frac{|V_{-1}|}{|V_2|} = \frac{n_{-1}}{n_2}$ neighbors in V_{-1}. Therefore, $0 \leq f(N(v)) \leq 2(n_2 - 1) + n_1 - \frac{n_{-1}}{n_2}$, and so $2n_2^2 + n_1 n_2 - 2n_2 - n_{-1} \geq 0$. Since

$n = n_2 + n_1 + n_{-1}$, we have equivalently that $2n_2^2 + n_1n_2 - n_2 + n_1 - n \geq 0$. Since $n_2 \geq 1$ and n_1 is a non-negative integer, we observe that $n_1^2 \geq n_1$, and thus

$$\frac{8}{9}n_1^2 + \frac{5}{3}n_1n_2 - \frac{5}{3}n_1 \geq \frac{8}{9}n_1 + \frac{5}{3}n_1 - \frac{5}{3}n_1 = \frac{8}{9}n_1 \geq 0.$$

Therefore

$$\begin{aligned}
&2(n_2 + \frac{2}{3}n_1 - \frac{1}{4})^2 - \frac{1}{8} - n = 2n_2^2 + \frac{8}{9}n_1^2 + \frac{8}{3}n_1n_2 - n_2 - \frac{2}{3}n_1 - n \\
&\geq (2n_2^2 + n_1n_2 - n_2 + n_1 - n) + (\frac{8}{9}n_1^2 + \frac{5}{3}n_1n_2 - \frac{5}{3}n_1) \\
&\geq 2n_2^2 + n_1n_2 - n_2 + n_1 - n \geq 0
\end{aligned}$$

or equivalently, $3n_2 + 2n_1 \geq \frac{3}{4}(\sqrt{8n+1} + 1)$. Thus

$$\gamma_{stR}^{NN}(G) = 3n_2 + 2n_1 - n \geq \frac{3}{4}(\sqrt{8n+1} + 1) - n$$

which establishes the desired lower bound.
Suppose that $\gamma_{stR}^{NN}(G) = \frac{3}{4}(\sqrt{8n+1} + 1) - n$. Then all the above inequalities must be equalities. In particular, $n_1 = 0$ and $2n_2^2 - 2n_2 = n_{-1}$. Furthermore, each vertex of V_{-1} is adjacent to exactly one vertex of V_2 and therefore has degree one, two or three in G, while each vertex of V_2 is adjacent to all other $n_2 - 1$ vertices of V_2 and to $2n_2 - 2$ vertices of V_{-1}. Therefore, $G \in \mathcal{F}$.
On the other hand, suppose that $G \in \mathcal{F}$. Then $G \in \mathcal{F}_k$ and $G = F_k$ such that $k \geq 2$. Assigning to every vertex of K_k the value 2, and to all other vertices the value -1, we produce an NNTSRDF f of weight

$$f(V) = \sum_{v \in V} f(v) = 2k - k(2k-2) = -2k^2 + 4k = \frac{3}{4}(\sqrt{8n+1} + 1) - n.$$

Therefore,

$$\gamma_{stR}^{NN}(G) \leq f(V) = \frac{3}{4}(\sqrt{8n+1} + 1) - n.$$

Consequently,

$$\gamma_{stR}^{NN}(G) = \frac{3}{4}(\sqrt{8n+1} + 1) - n.$$

□

Theorem 5. If G is a connected graph of order $n \geq 2$ and size m, then

$$\gamma_{stR}^{NN}(G) \geq \frac{1}{5}(10n - 12m).$$

Proof. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stR}^{NN}(G)$-function, $|V_i| = n_i$, $m(G[V_i]) = m_i$ for $i \in \{-1, 1, 2\}$ and $|V_1 \cup V_2| = n_{12}$ and $m(G[V_1 \cup V_2]) = m_{12}$. If $V_{-1} = \emptyset$, then $\gamma_{stR}^{NN}(G) = n \geq \frac{10n-12m}{5}$. Now we assume that $V_{-1} \neq \emptyset$. Since each vertex of V_{-1} is adjacent to at least one vertex of V_2, we have

$$\sum_{v \in V_2} |[v, V_{-1}]| = |[V_{-1}, V_2]| \geq n_{-1}.$$

Furthermore, for each $v \in V_2$, we observe that $0 \leq f(N(v)) = 2|[v, V_2]| + |[v, V_1]| - |[v, V_{-1}]|$ and thus $|[v, V_{-1}]| \leq 2|[v, V_2]| + |[v, V_1]|$. We deduce that

$$\begin{aligned} n_{-1} &\leq \sum_{v \in V_2} |(v, V_{-1}] \leq \sum_{v \in V_2} (2|[v, V_2]| + |[v, V_1]|) \\ &= 4m_2 + |[V_1, V_2]| = 4m_{12} - 4m_1 - 3|[V_1, V_2]| \end{aligned}$$

and thus $m_{12} \geq (n_{-1} + 4m_1 + 3|[V_1, V_2]|)/4$. This inequality and $n_{-1} \leq |[V_{-1}, V_2]|$ lead to

$$\begin{aligned} m &\geq m_{12} + |[V_{-1}, V_2]| + |[V_1, V_{-1}]| \\ &\geq \frac{1}{4}(n_{-1} + 4m_1 + 3|[V_1, V_2]|) + n_{-1} + |[V_1, V_{-1}]| \\ &= \frac{1}{4}(5n_{-1} + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) \\ &= \frac{1}{4}(5n - 5n_{12} + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|). \end{aligned}$$

It follows that

$$n_{12} \geq \frac{1}{5}(5n - 4m + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|)$$

and so

$$\begin{aligned} \gamma_{stR}^{NN}(G) &= 2n_2 + n_1 - n_{-1} = 3n_2 + 2n_1 - n = 3n_{12} - n - n_1 \\ &\geq \frac{3}{5}(5n - 4m + 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]|) - n - n_1 \\ &= \frac{1}{5}(10n - 12m) + \frac{3}{5}(4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]| - \frac{5}{3}n_1). \end{aligned}$$

Let

$$\mu(n_1) = 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]| - \frac{5}{3}n_1.$$

It suffices to show that $\mu(n_1) \geq 0$, because then $\gamma_{stR}^{NN}(G) \geq \frac{1}{5}(10n - 12m)$, which establish the desired lower bound. If $n_1 = 0$, then $\mu(n_1) = 0$. Now we assume that

$n_1 \geq 1$. Let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ of order $h_1, h_2, \ldots, h_t$. Since G is connected, each component H_i contains a vertex adjacent to a vertex of V_2 or to a vertex of V_{-1} for $1 \leq i \leq t$. This implies

$$\begin{aligned} m_1 + |[V_1, V_2]| + |[V_1, V_{-1}]| &\geq (h_1 - 1) + (h_2 - 1) + \ldots + (h_t - 1) + t \\ &= h_1 + h_2 + \ldots + h_t = n_1. \end{aligned}$$

This leads to

$$\begin{aligned} \mu(n_1) &= 4m_1 + 3|[V_1, V_2]| + 4|[V_1, V_{-1}]| - \frac{5}{3}n_1 \\ &> 3m_1 + 3|[V_1, V_2]| + 3|[V_1, V_{-1}]| - 3n_1 \geq 0, \end{aligned}$$

and the proof is complete. □

Corollary 7. If T is a tree of order $n \geq 2$, then

$$\gamma_{stR}^{NN}(T) \geq \frac{12 - 2n}{5}.$$

Our next example demonstrates that the lower bounds in Theorem 5 and Corollary 7 are sharp.

Example 3. *For $k \geq 2$, let F_k be the graph obtained from a connected graph F of order k by adding $2d_F(v)$ pendant edges to each vertex v of F. Then*

$$n(F_k) = n(F) + \sum_{v \in V(F)} 2d_F(v) = n(F) + 4m(F)$$

and

$$m(F_k) = m(F) + \sum_{v \in V(F)} 2d_F(v) = 5m(F).$$

Assigning to every vertex in $V(F)$ the weight 2 and to every vertex in $V(F_k) - V(F)$ the weight -1 produces an NNSTRDF f of weight

$$\omega(f) = 2n(F) - \sum_{v \in V(F)} 2d_F(v) = 2n(F) - 4m(F) = \frac{10n(F_k) - 12m(F_k)}{5}.$$

Using Theorem 5, we obtain $\gamma_{stR}^{NN}(F_k) = \frac{10n(F_k) - 12m(F_k)}{5}$.

Theorem 6. If G is a bipartite graph of order $n \geq 3$ with $\delta(G) \geq 1$, then

$$\gamma_{stR}^{NN}(G) \geq \frac{3}{2}(\sqrt{4n+1} - 1) - n.$$

Proof. Let X and Y be the partite sets of the bipartite graph G. Let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stR}^{NN}(G)$-function and let X_{-1}, X_1, and X_2 be the set of vertices in X that are assigned the value -1, 1 and 2, respectively under f. Let Y_{-1}, Y_1, and Y_2 be defined analogously. Let $|X_{-1}| = s$, $|X_1| = s_1$, $|X_2| = s_2$, $|Y_{-1}| = t$, $|Y_1| = t_1$, $|Y_2| = t_2$. Thus, $n_{-1} = s + t$, $n_1 = s_1 + t_1$ and $n_2 = s_2 + t_2$. If $n_{-1} = 0$, then $\gamma_{stR}^{NN}(G) = n \geq \frac{3}{2}(\sqrt{4n+1} - 1) - n$, since $n \geq 3$. Thus assume, without loss of generality, that $s \geq 1$ and therefore $t_2 \geq 1$. We First show that

$$s \leq t_2(2s_2 + s_1), \qquad t \leq s_2(2t_2 + t_1). \tag{1}$$

For each vertex $y \in Y_2$, we have that $2d_{X_2}(y) + d_{X_1}(y) - d_{X_{-1}}(y) = f(N(y)) \geq 0$, and so $d_{X_{-1}}(y) \leq 2d_{X_2}(y) + d_{X_1}(y) \leq 2s_2 + s_1$. By the definition of an NNSTRDF, each vertex in X_{-1} is adjacent to at least one vertex in Y_2, and so

$$\begin{aligned} s = |X_{-1}| \leq |[X_{-1}, Y_2]| &= \sum_{y \in Y_2} d_{X_{-1}}(y) \\ &\leq \sum_{y \in Y_2} (2s_2 + s_1) \\ &\leq t_2(2s_2 + s_1). \end{aligned}$$

Analogously, we have that $t \leq s_2(2t_2 + t_1)$. Now we show that

$$s_1 + s_2 + t_1 + t_2 \geq \sqrt{n + \frac{1}{4}} + \frac{1}{3}(s_1 + t_1) - \frac{1}{2}. \tag{2}$$

Since s_1 and t_1 are non-negative integers, we observe that $s_1^2 \geq s_1$ and $t_1^2 \geq t_1$. Thus

$$\frac{4}{9}s_1^2 + \frac{2}{3}s_1 \geq s_1, \qquad \frac{4}{9}t_1^2 + \frac{2}{3}t_1 \geq t_1. \tag{3}$$

We note that for integers s and t, we have $s^2 + t^2 \geq 2st$, with equality if and only if $s = t$. Hence by simple algebra and by inequalities (1) and (3), we have that

$$\begin{aligned} &(\frac{2}{3}s_1 + s_2 + \frac{2}{3}t_1 + t_2 + \frac{1}{2})^2 \\ &\geq s_2^2 + t_2^2 + 2s_2t_2 + \frac{4}{3}s_2t_1 + \frac{4}{3}s_1t_2 + s_2 + t_2 + \frac{4}{9}s_1^2 + \frac{2}{3}s_1 + \frac{4}{9}t_1^2 + \frac{2}{3}t_1 + \frac{1}{4} \\ &\geq 4s_2t_2 + s_2t_1 + s_1t_2 + s_2 + t_2 + s_1 + t_1 + \frac{1}{4} \\ &\geq s + t + s_2 + t_2 + s_1 + t_1 + \frac{1}{4} \\ &= n + \frac{1}{4}. \end{aligned}$$

The desired inequality now follows by taking squaring roots on both sides and rearranging terms. We now return to the proof of Theorem 6. By inequality (2), we have

$$\begin{aligned}
\gamma_{stR}^{NN}(G) &= 2n_2 + n_1 - n_{-1} \\
&= 3n_2 + 2n_1 - n \\
&= 3(n_2 + n_1) - n_1 - n \\
&= 3(s_2 + t_2 + s_1 + t_1) - (s_1 + t_1) - n \\
&\geq 3(\sqrt{n + \frac{1}{4}} + \frac{1}{3}(s_1 + t_1) - \frac{1}{2}) - (s_1 + t_1) - n \\
&= 3\sqrt{n + \frac{1}{4}} - \frac{3}{2} - n \\
&= \frac{3}{2}(\sqrt{4n+1} - 1) - n
\end{aligned}$$

which establishes the desired lower bound. □

Our next example demonstrates that the lower bounds in Theorem 6 is sharp.

Example 4. *For $k \geq 2$, let B_k be the bipartite graph obtained from the complete bipartite graph $K_{k,k}$ by adding $2k$ pendant edges to each vertex of $K_{k,k}$. Then $n(B_k) = 4k^2 + 2k$. Assigning to every vertex in $K_{k,k}$ the weight 2 and to all other vertices the weight -1 produces an NNSTRDF f of weight*

$$\omega(f) = 4k - 4k^2 = \frac{3}{2}(\sqrt{4n+1} - 1) - n.$$

Using Theorem 6, we obtain $\gamma_{stR}^{NN}(B_k) = \frac{3}{2}(\sqrt{4n+1} - 1) - n$.

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