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# Some new bounds on the general sum–connectivity index

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**Abstract:** Let $G = (V, E)$ be a simple connected graph with $n$ vertices, $m$ edges and sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(v_i)$, where $v_i \in V$. With $i \sim j$ we denote adjacency of vertices $v_i$ and $v_j$. The general sum–connectivity index of graph is defined as $\chi_\alpha(G) = \sum_{i\sim j}(d_i + d_j)^\alpha$, where $\alpha$ is an arbitrary real number. In this paper we determine relations between $\chi_{\alpha+\beta}(G)$ and $\chi_{\alpha+\beta-1}(G)$, where $\alpha$ and $\beta$ are arbitrary real numbers, and obtain new bounds for $\chi_\alpha(G)$. Also, by the appropriate choice of parameters $\alpha$ and $\beta$, we obtain a number of old/new inequalities for different vertex–degree–based topological indices.

**Keywords:** Topological indices, vertex degree, sum-connectivity index

**AMS Subject classification:** 05C50, 15A18

## 1. Introduction

Let $G = (V, E)$, be a simple connected graph with $n$ vertices and $m$ edges, where $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$. Let $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta$, $d_i = d(v_i)$, and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$ be sequences of vertex and edge degrees, respectively. With $i \sim j$ we denote adjacency of vertices $v_i$ and $v_j$. Let $e = \{i, j\} \in E$ be an arbitrary edge of $G$. The degree of an edge $e$ is defined as $d(e) = d_i + d_j - 2$. In addition, we use the following notation: $\Delta_e = d(e_1) + 2 \geq d(e_2) + 2 \geq \cdots \geq d(e_m) + 2 = \delta_e$.

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A line graph $L(G)$ of a graph G, is a graph such that each vertex of $L(G)$ represents an edge of $G$ and two vertices of $L(G)$ are adjacent if and only if their corresponding edges are adjacent in $G$.

A graph invariant, or topological index, is a numeric quantity associated with a graph which characterize the topology of graph and is invariant under graph automorphism. Here we list some vertex–degree–based graph invariants that are of interest for our work.

Historically, the first vertex-degree-based (VDB) structure descriptors were the graph invariants that are nowadays called Zagreb indices. The first Zagreb index, $M_1$, is defined as [13]

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \,.$$

In [5] it was shown that the first Zagreb index can also be expressed as

$$M_1(G) = \sum_{i\sim j}(d_i + d_j)\,.$$

It can be easily verified that the following is valid

$$M_1(G) = \sum_{i=1}^{m}(d(e_i) + 2)\,,$$

that is, it can be considered as edge-degree-based topological index as well.

The sum–connectivity index, $SC(G)$, proposed in [28], is defined as

$$SC(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i + d_j}}.$$

Generalization of $SC(G)$ and $M_1(G)$ was introduced in [29] and named general sum–connectivity index. It is defined as

$$\chi_\alpha(G) = \sum_{i\sim j}(d_i + d_j)^\alpha, \quad \chi_0(G) = m,$$

where $\alpha$ is an arbitrary real number. In [16] it was shown that $\chi_\alpha(G)$ satisfies the expression

$$\chi_\alpha(G) = \sum_{i=1}^{m}(d(e_i) + 2)^\alpha.$$

In what follows me mention some particular indices of this kind that are of interest for the present work.

- For $\alpha = 2$, $\chi_2(G) = HM(G)$, the hyper-Zagreb index is obtained [26].

- For $\alpha = 1/2$, $\chi_{1/2}(G) = RSC(G)$, the reciprocal sum–connectivity index could be obtained.
- For $\alpha = -1$, $2\chi_{-1}(G) = H(G)$, the harmonic index is obtained [7].
- For $\alpha = -2$, $\chi_{-2}(G) = RHM(G)$, the reciprocal hyper–Zagreb index could be obtained.

One can easily observe that for the hyper–Zagreb index holds

$$HM(G) = F(G) + 2M_2(G),$$

where

$$F(G) = \sum_{i=1}^{n} d_i^3 \quad \text{and} \quad M_2(G) = \sum_{i\sim j} d_i d_j$$

are the forgotten index [9] and the second Zagreb index [12], respectively. Details on the mathematical theory of Zagreb indices can be found in [1, 3, 10, 11, 20, 23].
In this paper we establish relations between $\chi_{\alpha+\beta}(G)$ and $\chi_{\alpha+\beta-1}(G)$, where $\alpha$ and $\beta$ are arbitrary real numbers, and obtain new bounds for $\chi_\alpha(G)$. Also, by the appropriate choice of parameters $\alpha$ and $\beta$, a number of new/old inequalities that reveal relationships between above mentioned topological indices are obtained. More on these and some other results of this type can be found, for example, in [2, 4, 6, 14, 16, 24, 25].

## 2. Preliminaries

In this section, we recall some discrete analytical inequalities for real number sequences that will be used subsequently.
Let $p = (p_i)$, $i = 1, 2, \dots, m$, be nonnegative real number sequence and $a = (a_i)$, $i = 1, 2, \dots, m$, positive real number sequence. Then for any real $\alpha$, such that $\alpha \geq 1$ or $\alpha \leq 0$, holds (see e.g. [18])

$$\left(\sum_{i=1}^{m} p_i\right)^{\alpha-1} \sum_{i=1}^{m} p_i a_i^\alpha \geq \left(\sum_{i=1}^{m} p_i a_i\right)^{\alpha}. \tag{1}$$

If $0 \leq \alpha \leq 1$, then the sense of (1) reverses. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or for some $t$, $1 \leq t \leq m-1$, holds $p_1 = p_2 = \cdots = p_t = 0$, $p_{t+1} = p_{t+2} = \cdots = p_m$ and $a_{t+1} = a_{t+2} = \cdots = a_m$.

Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \dots, m$, be positive real number sequences. In [21] it was proven that for any $r \geq 0$ holds

$$\sum_{i=1}^{m} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=1}^{m} x_i\right)^{r+1}}{\left(\sum_{i=1}^{m} a_i\right)^{r}}. \tag{2}$$

Equality holds if and only if $r=0$ or $\frac{x_1}{a_1}=\frac{x_2}{a_2}=\cdots=\frac{x_m}{a_m}$.

## 3. Main results

In the following theorem we establish relationship between $\chi_{\alpha+\beta}(G)$ and $\chi_{\alpha+\beta-1}(G)$, where $\alpha$ and $\beta$ are arbitrary real numbers.

**Theorem 1.** *Let $G$ be a graph with $m \geq 3$ edges such that $\Delta_e \neq \delta_e$, and $\beta$ be an arbitrary real number. Then for any real $\alpha$, $\alpha \leq 0$ or $\alpha \geq 1$, holds*

$$\begin{aligned} &\delta_e\chi_{\alpha+\beta-1}(G)+\frac{(\chi_{\beta+1}(G)-\delta_e\chi_\beta(G))^\alpha}{(\chi_\beta(G)-\delta_e\chi_{\beta-1}(G))^{\alpha-1}} \leq \chi_{\alpha+\beta}(G) \\ &\leq \Delta_e\chi_{\alpha+\beta-1}(G)-\frac{(\Delta_e\chi_\beta(G)-\chi_{\beta+1}(G))^\alpha}{(\Delta_e\chi_{\beta-1}(G)-\chi_\beta(G))^{\alpha-1}}. \end{aligned} \tag{3}$$

*If $0 \leq \alpha \leq 1$, then the opposite inequalities hold. Equalities hold if and only if either $\alpha=0$, $\alpha=1$, or $d(e_i)+2 \in \{\delta_e, \Delta_e\}$ for every $i=2,3,\ldots,m-1$.*

*Proof.* For real numbers $\alpha$ and $\beta$ we have that

$$\chi_{\alpha+\beta}(G)-\delta_e\chi_{\alpha+\beta-1}(G)=\sum_{i=1}^m (d(e_i)+2-\delta_e)\,(d(e_i)+2)^{\alpha+\beta-1} \tag{4}$$

and

$$\Delta_e\chi_{\alpha+\beta-1}(G)-\chi_{\alpha+\beta}(G)=\sum_{i=1}^m (\Delta_e-d(e_i)-2)\,(d(e_i)+2)^{\alpha+\beta-1}. \tag{5}$$

For $r=\alpha$, $p_i=(d(e_i)+2-\delta_e)\,(d(e_i)+2)^{\beta-1}$, $a_i=d(e_i)+2$, $i=1,2,\ldots,m$, the inequality (1) becomes

$$\begin{aligned} &\left(\sum_{i=1}^m (d(e_i)+2-\delta_e)\,(d(e_i)+2)^{\beta-1}\right)^{\alpha-1}\sum_{i=1}^m (d(e_i)+2-\delta_e)\,(d(e_i)+2)^{\alpha+\beta-1} \\ &\geq \left(\sum_{i=1}^m (d(e_i)+2-\delta_e)\,(d(e_i)+2)^{\beta}\right)^{\alpha}. \end{aligned}$$

Based on the conditions given in the statement of Theorem 1 we have that $\Delta_e \neq \delta_e$, i.e. $L(G)$ is not a regular graph. Accordingly, from the above follows

$$\sum_{i=1}^m (d(e_i)+2-\delta_e)\,(d(e_i)+2)^{\alpha+\beta-1} \geq \frac{(\chi_{\beta+1}(G)-\delta_e\chi_\beta(G))^\alpha}{(\chi_\beta(G)-\delta_e\chi_{\beta-1}(G))^{\alpha-1}}. \tag{6}$$

From this inequality and (4) we get left side of (3).

For $r = \alpha$, $p_i = (\Delta_e - d(e_i) - 2)\,(d(e_i)+2)^{\beta-1}$, $a_i = d(e_i)+2$, $i = 1,2,\dots,m$, the inequality (1) transforms into

$$\left(\sum_{i=1}^{m}(\Delta_e - d(e_i) - 2)\,(d(e_i)+2)^{\beta-1}\right)^{\alpha-1}\sum_{i=1}^{m}(\Delta_e - d(e_i) - 2)\,(d(e_i)+2)^{\alpha+\beta-1}$$
$$\geq \left(\sum_{i=1}^{m}(\Delta_e - d(e_i) - 2)\,(d(e_i)+2)^{\beta}\right)^{\alpha}.$$

Again, from the conditions given in the statement of Theorem 1 we have that $\Delta_e \neq \delta_e$, that is $L(G)$ is not a regular graph. Therefore we obtain

$$\sum_{i=1}^{m}(\Delta_e - d(e_i) - 2)\,(d(e_i)+2)^{\alpha+\beta-1} \geq \frac{(\Delta_e\chi_\beta(G) - \chi_{\beta+1}(G))^{\alpha}}{(\Delta_e\chi_{\beta-1}(G) - \chi_\beta(G))^{\alpha-1}}. \qquad (7)$$

According to the above and (5) we get right side of (3).
By a similar procedure we get that the opposite inequalities are valid in (3) when $0 \leq \alpha \leq 1$.
Equalities in (6) and (7) hold if and only if either $\alpha = 0$, $\alpha = 1$, or for some $t$, $1 \leq t \leq m-2$, holds $d(e_1)+2 = \cdots = d(e_t)+2 > d(e_{t+1})+2 = \cdots = d(e_{m-1})+2$. This implies that equalities in (3) are attained if and only if either $\alpha = 0$, $\alpha = 1$, or $d(e_i)+2 \in \{\delta_e, \Delta_e\}$ for every $i = 2,3,\dots,m-1$, $\Delta_e \neq \delta_e$. □

In the following corollary of Theorem 1 we determine lower bound for $\chi_{\alpha+\beta}(G)$.

**Corollary 1.** *Let $G$ be a simple connected graph with $m \geq 3$ edges such that $\Delta_e \neq \delta_e$, and $\beta$ is an arbitrary real number. Then for any real number $\alpha$, $\alpha \geq 1$ or $\alpha \leq 0$, holds*

$$\chi_{\alpha+\beta}(G) \geq \frac{1}{\Delta_e - \delta_e}\left(\frac{\Delta_e(\chi_{\beta+1}(G) - \delta_e\chi_\beta(G))^{\alpha}}{(\chi_\beta(G) - \delta_e\chi_{\beta-1}(G))^{\alpha-1}} + \frac{\delta_e(\Delta_e\chi_\beta(G) - \chi_{\beta+1}(G))^{\alpha}}{(\Delta_e\chi_{\beta-1}(G) - \chi_\beta(G))^{\alpha-1}}\right). \qquad (8)$$

*If $0 \leq \alpha \leq 1$, then the opposite inequality holds. Equality holds if and only if either $\alpha = 0$, $\alpha = 1$, or $d(e_i)+2 \in \{\delta_e, \Delta_e\}$ for every $i = 2,3,\dots,m-1$.*

*Proof.* Let $\alpha$ be an arbitrary real number such that $\alpha \geq 1$ or $\alpha \leq 0$. According to (4) and (5) and inequalities (6) and (7) we have that

$$\chi_{\alpha+\beta}(G) - \delta_e\chi_{\alpha+\beta-1}(G) \geq \frac{(\chi_{\beta+1}(G) - \delta_e\chi_\beta(G))^{\alpha}}{(\chi_\beta(G) - \delta_e\chi_{\beta-1}(G))^{\alpha-1}}$$

and

$$\Delta_e\chi_{\alpha+\beta-1}(G) - \chi_{\alpha+\beta}(G) \geq \frac{(\Delta_e\chi_\beta(G) - \chi_{\beta+1}(G))^{\alpha}}{(\Delta_e\chi_{\beta-1}(G) - \chi_\beta(G))^{\alpha-1}}.$$

From the previous inequalities follow

$$(\Delta_e - \delta_e)\chi_{\alpha+\beta}(G) \geq \frac{\Delta_e(\chi_{\beta+1}(G) - \delta_e\chi_\beta(G))^\alpha}{(\chi_\beta(G) - \delta_e\chi_{\beta-1}(G))^{\alpha-1}} + \frac{\delta_e(\Delta_e\chi_\beta(G) - \chi_{\beta+1}(G))^\alpha}{(\Delta_e\chi_{\beta-1}(G) - \chi_\beta(G))^{\alpha-1}}.$$

Since $\Delta_e \neq \delta_e$, from the preceding inequality we obtain (8).
In a similar way we prove that opposite inequality holds in (8) when $0 \leq \alpha \leq 1$. □

For some particular values of parameters $\alpha$ and $\beta$ the following corollaries are obtained.

**Corollary 2.** *Let $G$ be a simple connected graph with $m \geq 2$ edges such that $\Delta_e \neq \delta_e$. Then for any real number $\alpha$, $\alpha \geq 1$ or $\alpha \leq 0$, holds*

$$\delta_e\chi_{\alpha-1}(G) + \frac{(M_1(G) - m\delta_e)^\alpha}{(m - \frac{\delta_e}{2}H(G))^{\alpha-1}} \leq \chi_\alpha(G) \leq \Delta_e\chi_{\alpha-1}(G) - \frac{(m\Delta_e - M_1(G))^\alpha}{(\frac{\Delta_e}{2}H(G) - m)^{\alpha-1}}.$$

*If $0 \leq \alpha \leq 1$, then the opposite inequalities are valid. Equalities hold if and only if $\alpha = 0$, $\alpha = 1$, or $d(e_i) + 2 \in \{\delta_e, \Delta_e\}$ for every $i = 2, 3, \ldots, m-1$.*
*For any real number $\alpha$, $\alpha \geq 2$ or $\alpha \leq 1$, holds*

$$\begin{aligned}\delta_e\chi_{\alpha-1}(G) + \frac{(HM(G) - \delta_e M_1(G))^{\alpha-1}}{(M_1(G) - m\delta_e)^{\alpha-2}} &\leq \chi_\alpha(G)\\ \leq \Delta_e\chi_{\alpha-1}(G) - \frac{(\Delta_e M_1(G) - HM(G))^{\alpha-1}}{(m\Delta_e - M_1(G))^{\alpha-2}}&.\end{aligned}$$

*If $1 \leq \alpha \leq 2$, then the opposite inequalities hold. Equalities hold if and only if $\alpha = 1$, or $\alpha = 2$, or $d(e_i) + 2 \in \{\delta_e, \Delta_e\}$ for every $i = 2, 3, \ldots, m-1$.*

**Corollary 3.** *Let $G$ be a simple connected graph with $m \geq 2$ edges such that $\Delta_e \neq \delta_e$. Then for every real number $\alpha$, $\alpha \geq 1$ or $\alpha \leq 0$, holds*

$$\chi_\alpha(G) \geq \frac{1}{\Delta_e - \delta_e}\left(\frac{\Delta_e(M_1(G) - m\delta_e)^\alpha}{(m - \frac{\delta_e}{2}H(G))^{\alpha-1}} + \frac{\delta_e(m\Delta_e - M_1(G))^\alpha}{(\frac{\Delta_e}{2}H(G) - m)^{\alpha-1}}\right).$$

*When $0 \leq \alpha \leq 1$, then the opposite inequality holds. Equality holds if and only if either $\alpha = 0$, $\alpha = 1$, or $d(e_i) + 2 \in \{\delta_e, \Delta_e\}$ for every $i = 2, 3, \ldots, m-1$.*
*For $\alpha \geq 2$ or $\alpha \leq 1$ we have*

$$\chi_\alpha(G) \geq \frac{1}{\Delta_e - \delta_e}\left(\frac{\Delta_e(HM(G) - \delta_e M_1(G))^{\alpha-1}}{(M_1(G) - m\delta_e)^{\alpha-2}} + \frac{\delta_e(\Delta_e M_1(G) - HM(G))^{\alpha-1}}{(m\Delta_e - M_1(G))^{\alpha-2}}\right).$$

*When $1 \leq \alpha \leq 2$, then the opposite inequality holds. Equality holds if and only if either $\alpha = 1$, $\alpha = 2$, or $d(e_i) + 2 \in \{\delta_e, \Delta_e\}$ for every $i = 2, 3, \ldots, m-1$.*

**Corollary 4.** *Let $G$ be a simple connected graph with $m \geq 2$ edges. Then for any real number $\alpha \geq 1$ holds*

$$\chi_\alpha(G) \geq \delta_e \chi_{\alpha-1}(G) + \left(m - \frac{\delta_e}{2} H(G)\right) \left(\frac{2m}{H(G)}\right)^\alpha .$$

*Equality is attained if and only if $\alpha = 1$ or $L(G)$ is a regular graph.*
*For any real number $\alpha \geq 2$ holds*

$$\chi_\alpha(G) \geq \delta_e \chi_{\alpha-1}(G) + (M_1(G) - m\delta_e) \left(\frac{M_1(G)}{m}\right)^{\alpha-1} .$$

*Equality is attained if and only if $\alpha = 2$ or $L(G)$ is a regular graph.*

**Corollary 5.** *Let $G$ be a simple connected graph with $m \geq 2$ edges. Then*

$$F(G) \geq \delta_e M_1(G) + \frac{(M_1(G) - m\delta_e)^2}{m - \frac{\delta_e}{2} H(G)} - 2M_2(G) \quad (\Delta_e \neq \delta_e),$$

$$F(G) \geq \frac{1}{2} \left(\delta_e M_1(G) + \frac{(M_1(G) - m\delta_e)^2}{m - \frac{\delta_e}{2} H(G)}\right) \quad (\Delta_e \neq \delta_e),$$

$$F(G) \geq \delta_e M_1(G) + \left(m - \frac{\delta_e}{2} H(G)\right) \frac{4m^2}{H(G)^2} - 2M_2(G),$$

$$F(G) \geq \frac{1}{\Delta_e - \delta_e} \left(\frac{\Delta_e (M_1(G) - m\delta_e)^2}{m - \frac{\delta_e}{2} H(G)} + \frac{\delta_e (m\Delta_e - M_1(G))^2}{\frac{\Delta_e}{2} H(G) - m}\right) - 2M_2(G),$$

$$F(G) \geq \frac{1}{2(\Delta_e - \delta_e)} \left(\frac{\Delta_e (M_1(G) - m\delta_e)^2}{m - \frac{\delta_e}{2} H(G)} + \frac{\delta_e (m\Delta_e - M_1(G))^2}{\frac{\Delta_e}{2} H(G) - m}\right),$$

$$F(G) \leq \Delta_e M_1(G) - \frac{(m\Delta_e - M_1(G))^2}{\frac{\Delta_e}{2} H(G) - m} - 2M_2(G),$$

$$M_2(G) \leq \frac{1}{4} \left(\Delta_e M_1(G) - \frac{(m\Delta_e - M_1(G))^2}{\frac{\Delta_e}{2} H(G) - m}\right).$$

In the next theorem we determine a relation between $\chi_{2\alpha}(G)$ and $\chi_\alpha(G)$.

**Theorem 2.** *Let $G$ be a simple connected graph with $m \geq 3$ edges. Then for any real $\alpha$ holds*

$$m\chi_{2\alpha}(G) - \chi_\alpha(G)^2 \geq \frac{m}{2} \left(\Delta_e^\alpha - \delta_e^\alpha\right)^2 .$$

*Equality holds if and only if $\alpha = 0$ or $(d(e_2) + 2)^\alpha = (d(e_3) + 2)^\alpha = \cdots = (d(e_{m-1}) + 2)^\alpha = \frac{\Delta_e^\alpha + \delta_e^\alpha}{2}$.*

*Proof.* According to the Lagrange's identity (see e.g. [19]) we have that

$$
\begin{aligned}
& m\chi_{2\alpha}(G) - \chi_\alpha(G)^2 = m\sum_{i=1}^{m}(d(e_i)+2)^{2\alpha} - \left(\sum_{i=1}^{m}(d(e_i)+2)^{\alpha}\right)^2 \\
& = \sum_{1\le i<j\le m} \left((d(e_i)+2)^\alpha - (d(e_j)+2)^\alpha\right)^2 \\
& \ge \sum_{i=2}^{m-1} \left((\Delta_e^\alpha - (d(e_i)+2)^\alpha)^2 + \left((d(e_i)+2)^\alpha - \delta_e^\alpha)\right)^2 + (\Delta_e^\alpha - \delta_e^\alpha)^2\right. \\
& \ge \frac{1}{2}\sum_{i=2}^{m-1} (\Delta_e^\alpha - \delta_e^\alpha)^2 + (\Delta_e^\alpha - \delta_e^\alpha)^2 = \frac{m}{2}\left(\Delta_e^\alpha - \delta_e^\alpha\right)^2,
\end{aligned}
$$

which completes the proof. □

**Corollary 6.** *Let $G$ be a simple connected graph with $m \ge 2$ edges. Then*

$$
F(G) \ge \frac{M_1(G)^2}{m} - 2M_2(G) + \frac{1}{2}\left(\Delta_e - \delta_e\right)^2, \tag{9}
$$

$$
\frac{m}{2}H(G) - SC(G)^2 \ge \frac{m}{2}\left(\frac{1}{\sqrt{\Delta_e}} - \frac{1}{\sqrt{\delta_e}}\right)^2, \tag{10}
$$

$$
mM_1(G) - RSC(G)^2 \ge \frac{m}{2}\left(\sqrt{\Delta_e} - \sqrt{\delta_e}\right)^2,
$$

$$
mRHM(G) - \frac{1}{4}H(G)^2 \ge \frac{m}{2}\left(\frac{1}{\Delta_e} - \frac{1}{\delta_e}\right)^2.
$$

**Remark 1.** The inequality (9) was proven in [17]. It is stronger than

$$
F(G) \ge \frac{M_1(G)^2}{m} - 2M_2(G),
$$

which was proven in [9] (see also [8]).
The inequality (10) was proven in [14]. It is stronger than

$$
SC(G) \le \sqrt{\frac{mH(G)}{2}}, \tag{11}
$$

proven in [15].
In [30] it was proven that

$$
SC(G) \le \sqrt{\frac{mR(G)}{2}}.
$$

Since $H(G) \le R(G)$ ( see [27]), the inequality (11), and consequently (10), is stronger than the above one.

Since $M_1(G) \geq \frac{4m^2}{n}$, according to (9) we have that

$$F(G) + 2M_2(G) \geq \frac{16m^3}{n^2} + \frac{1}{2}(\Delta_e - \delta_e)^2 \,,$$

which is stronger than

$$F(G) + 2M_2(G) \geq \frac{16m^3}{n^2} \,,$$

that was proven in [22].

In the following theorem we establish a relationship between $\chi_{2\alpha-\beta}(G)$, $\chi_\alpha(G)$ and $\chi_\beta(G)$, for arbitrary real numbers $\alpha$ and $\beta$.

**Theorem 3.** *Let $G$ be a simple connected graph with $m \geq 3$ edges. Then for any real numbers $\alpha$ and $\beta$ hold*

$$\left(\chi_{2\alpha-\beta}(G) - \Delta_e^{2\alpha-\beta} - \delta_e^{2\alpha-\beta}\right)\left(\chi_\beta(G) - \Delta_e^\beta - \delta_e^\beta\right) \geq \left(\chi_\alpha(G) - \Delta_e^\alpha - \delta_e^\alpha\right)^2 \,, \tag{12}$$

*with equality if and only if $\alpha = \beta$ or $d(e_2) + 2 = d(e_3) + 2 = \cdots = d(e_{m-1}) + 2$.*

*Proof.* The inequality (2) can be considered as

$$\sum_{i=2}^{m-1} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=2}^{m-1} x_i\right)^{r+1}}{\left(\sum_{i=2}^{m-1} a_i\right)^r}.$$

For $r = 1$, $x_i = (d(e_i) + 2)^\alpha$, $a_i = (d(e_i) + 2)^\beta$, $i = 2, 3, \ldots, m-1$, where $\alpha$ and $\beta$ are arbitrary real numbers, the above inequality becomes

$$\sum_{i=2}^{m-1} \frac{((d(e_i) + 2)^\alpha)^2}{(d(e_i) + 2)^\beta} \geq \frac{\left(\sum_{i=2}^{m-1} (d(e_i) + 2)^\alpha\right)^2}{\sum_{i=2}^{m-1} (d(e_i) + 2)^\beta}, \tag{13}$$

that is

$$\sum_{i=2}^{m-1} (d(e_i) + 2)^{2\alpha-\beta} \geq \frac{\left(\sum_{i=2}^{m-1} (d(e_i) + 2)^\alpha\right)^2}{\sum_{i=2}^{m-1} (d(e_i) + 2)^\beta},$$

i.e.

$$\chi_{2\alpha-\beta}(G) - \Delta_e^{2\alpha-\beta} - \delta_e^{2\alpha-\beta} \geq \frac{(\chi_\alpha(G) - \Delta_e^\alpha - \delta_e^\alpha)^2}{\chi_\beta(G) - \Delta_e^\beta - \delta_e^\beta},$$

wherefrom (12) is obtained.
Equality in (13) is attained if and only if $(d(e_2)+2)^{\alpha-\beta} = (d(e_3)+2)^{\alpha-\beta} = \cdots = (d(e_{m-1})+2)^{\alpha-\beta}$, which implies that equality in (12) holds if and only if $\alpha = \beta$ or $d(e_2)+2 = d(e_3)+2 = \cdots = d(e_{m-1})+2$. □

**Theorem 4.** *Let $G$ be a simple connected graph with $m \geq 2$ edges. Then for any real numbers $\alpha$ and $\beta$ hold*

$$\left(\chi_{2\alpha-\beta}(G) - \Delta_e^{2\alpha-\beta}\right)\left(\chi_\beta(G) - \Delta_e^\beta\right) \geq (\chi_\alpha(G) - \Delta_e^\alpha)^2 . \tag{14}$$

*Equality holds if and only if $\alpha = \beta$ or $d(e_2)+2 = d(e_3)+2 = \cdots = d(e_m)+2 = \delta_e$.*

*Proof.* The inequality (2) can be considered as

$$\sum_{i=2}^{m} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum_{i=2}^{m} x_i\right)^{r+1}}{\left(\sum_{i=2}^{m} a_i\right)^{r}} .$$

For $r = 1$, $x_i = (d(e_i)+2)^\alpha$, $a_i = (d(e_i)+2)^\beta$, $i = 2,3,\ldots,m$, where $\alpha$ and $\beta$ are arbitrary real numbers, the above inequality becomes

$$\sum_{i=2}^{m} \frac{((d(e_i)+2)^\alpha)^2}{(d(e_i)+2)^\beta} \geq \frac{\left(\sum_{i=2}^{m}(d(e_i)+2)^\alpha\right)^2}{\sum_{i=2}^{m}(d(e_i)+2)^\beta}, \tag{15}$$

that is

$$\sum_{i=2}^{m}(d(e_i)+2)^{2\alpha-\beta} \geq \frac{\left(\sum_{i=2}^{m}(d(e_i)+2)^\alpha\right)^2}{\sum_{i=2}^{m}(d(e_i)+2)^\beta},$$

i.e.

$$\chi_{2\alpha-\beta}(G) - \Delta_e^{2\alpha-\beta} \geq \frac{(\chi_\alpha(G) - \Delta_e^\alpha)^2}{\chi_\beta(G) - \Delta_e^\beta},$$

from which (14) is obtained.
Equality in (15) is attained if and only if $(d(e_1)+2)^{\alpha-\beta} = (d(e_2)+2)^{\alpha-\beta} = \cdots = (d(e_{m-1}+2)^{\alpha-\beta}$, which implies that equality in (14) holds if and only if $\alpha = \beta$ or $d(e_1)+2 = d(e_2)+2 = \cdots = d(e_{m-1})+2$. □

**Theorem 5.** *Let $G$ be a simple connected graph with $m \geq 1$ edges. Then for any real numbers $\alpha$ and $\beta$ hold*

$$\chi_\alpha(G) \leq \sqrt{\chi_\beta(G)\chi_{2\alpha-\beta}(G)}. \tag{16}$$

*Equality holds if and only if $\alpha = \beta$ or $L(G)$ is regular.*

*Proof.* For $r = 1$, $x_i = (d(e_i)+2)^\alpha$, $a_i = (d(e_i)+2)^\beta$, $i = 1, 2, \ldots, m$, the inequality (2) transforms into

$$\sum_{i=1}^{m} \frac{((d(e_i)+2)^\alpha)^2}{(d(e_i)+2)^\beta} \geq \frac{\left(\sum_{i=1}^{m}(d(e_i)+2)^\alpha\right)^2}{\sum_{i=1}^{m}(d(e_i)+2)^\beta}, \tag{17}$$

that is

$$\chi_{2\alpha-\beta}(G) \geq \frac{\chi_\alpha(G)^2}{\chi_\beta(G)},$$

from which (16) is obtained.
Equality in (17), and consequently in (16), holds if and only if $\alpha = \beta$ or $d(e_1)+2 = d(e_2)+2 = \cdots = d(e_m)+2$, that is if and only if $\alpha = \beta$ or $L(G)$ is regular. □

**Corollary 7.** *Let $G$ be a simple connected graph with $m \geq 1$ edges. Then for any real $\alpha$ we have*

$$\chi_\alpha(G) \leq \sqrt{m\chi_{2\alpha}(G)} \tag{18}$$

*and*

$$\chi_\alpha(G) \leq \sqrt{M_1(G)\chi_{2\alpha-1}(G)}.$$

The inequality (18) was proven in [25].

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