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# On the super domination number of graphs

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**Abstract:** The open neighborhood of a vertex $v$ of a graph $G$ is the set $N(v)$ consisting of all vertices adjacent to $v$ in $G$. For $D \subseteq V(G)$, we define $\overline{D} = V(G) \setminus D$. A set $D \subseteq V(G)$ is called a super dominating set of $G$ if for every vertex $u \in \overline{D}$, there exists $v \in D$ such that $N(v) \cap \overline{D} = \{u\}$. The super domination number of $G$ is the minimum cardinality among all super dominating sets of $G$. In this paper, we obtain closed formulas and tight bounds for the super domination number of $G$ in terms of several invariants of $G$. We also obtain results on the super domination number of corona product graphs and Cartesian product graphs.

**Keywords:** Super domination number; Domination number; Cartesian product; Corona product

**AMS Subject classification:** 05C69, 05C70, 05C76

## 1. Introduction

Throughout the paper, let $G$ be a finite, simple, and undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v$ of $G$ is the set $N(v)$ consisting of all vertices adjacent to $v$ in $G$. For $D \subseteq V(G)$, we define $\overline{D} = V(G) \setminus D$. A set $D \subseteq V(G)$ is *a dominating set* of $G$ if every vertex in $\overline{D}$ has at least one neighbor in $D$, *i.e.*, $N(u) \cap D \neq \emptyset$ for every $u \in \overline{D}$. The *domination number* of $G$, denoted

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by $\gamma(G)$, is the minimum cardinality among all dominating sets of $G$. For topics on domination in graphs, we refer to [9, 10].
A set $D \subseteq V(G)$ is a *super dominating set* of $G$ if, for every vertex $u \in \overline{D}$, there exists $v \in D$ such that

$$N(v) \cap \overline{D} = \{u\}. \tag{1}$$

If $u$ and $v$ satisfy (1), then we say that $v$ is an *external private neighbor* of $u$ with respect to $\overline{D}$. The *super domination number* of $G$, denoted by $\gamma_{\rm sp}(G)$, is the minimum cardinality among all super dominating sets of $G$. A super dominating set of cardinality $\gamma_{\rm sp}(G)$ is called a $\gamma_{\rm sp}(G)$-set. The study of super domination in graphs was initiated in [14]. It was shown in [4] that determining the super domination number of a graph is an NP-hard problem. This suggests that computing the super domination number for special classes of graphs or obtaining good bounds on this graph parameter is worthy of investigation. For the super domination number of lexicographic product graphs and join graphs, see [4]. For the super domination number of rooted product graphs, see [13].
We recall some results on the super domination number of graphs. Let $K_n$, $K_{s,n-s}$, $P_n$ and $C_n$ denote the complete graph, the complete bi-partite graph, the path and the cycle of order $n \geq 2$, respectively. It was shown in [14] that $\gamma_{\rm sp}(K_n) = n-1$, $\gamma_{\rm sp}(K_{1,n-1}) = n-1$, and $\gamma_{\rm sp}(K_{s,n-s}) = n-2$ for $\min\{s, n-s\} \geq 2$. More generally, let $K_{a_1,a_2,\ldots,a_k}$ be a complete $k$-partite graph of order $n = \sum_{i=1}^{k} a_i \geq 2$. If at most one value $a_i$ is greater than one, then $K_{a_1,a_2,\ldots,a_k} \cong K_n$ or $K_{a_1,a_2,\ldots,a_k} \cong K_{n-a_i} + N_{a_i}$, where $N_{a_i}$ denotes the empty graph of order $a_i$ and $G+H$ denotes the join of graphs $G$ and $H$; thus $\gamma_{\rm sp}(K_{a_1,a_2,\ldots,a_k}) = n-1$ (see [4]). If there are at least two $a_i, a_j \geq 2$, it is easy to see that $\gamma_{\rm sp}(K_{a_1,a_2,\ldots,a_k}) = n-2$. In summary, we have the following

$$\gamma_{\rm sp}(K_{a_1,a_2,\ldots,a_k}) = \begin{cases} n-1 & \text{if at most one value } a_i \text{ is greater than one,} \\ n-2 & \text{otherwise.} \end{cases}$$

**Theorem 1.** [14] *For $n \geq 3$, $\gamma_{\rm sp}(P_n) = \lceil \frac{n}{2} \rceil$ and*

$$\gamma_{\rm sp}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \equiv 0, 3 \pmod 4, \\ \lceil \frac{n+1}{2} \rceil & \text{otherwise.} \end{cases}$$

**Theorem 2.** [14] *Let $G$ be a graph of order $n$. Then*

*(a) $\gamma_{\rm sp}(G) = n$ if and only if $G$ is an empty graph;*
*(b) $\gamma_{\rm sp}(G) \geq \lceil \frac{n}{2} \rceil$;*
*(c) $\gamma_{\rm sp}(G) = 1$ if and only if $G \cong K_1$ or $G \cong K_2$.*

Note that, for any graph $G$ of order $n$ without isolated vertices, Theorem 2(b) and the well-known bounds of $\gamma(G)$ (i.e., $1 \leq \gamma(G) \leq \lceil \frac{n}{2} \rceil$) imply

$$1 \leq \gamma(G) \leq \left\lceil \frac{n}{2} \right\rceil \leq \gamma_{\rm sp}(G) \leq n-1. \tag{2}$$

A characterization of connected graphs $G$ of order $n$ satisfying $\gamma_{\rm sp}(G) = \frac{n}{2}$ ($\gamma_{\rm sp}(G) = n-1$, respectively) was given in [14] ([4], respectively).
This paper is organized as follows. In section 2, we study the relationships between $\gamma_{\rm sp}(G)$ and several parameters of $G$, including the number of twin equivalence classes, domination number, secure domination number, matching number, 2-packing number, vertex cover number, etc. In section 3, we obtain a closed formula for the super domination number of corona product graphs. In section 4, we study the problem of finding the exact values or sharp bounds for the super domination number of Cartesian product graphs and express these in terms of invariants of the factor graphs.
For the remainder of the paper, definitions will be provided whenever needed.

## 2. Relationship between super domination number and other graph parameters

A *matching* in a graph $G$ is a set of pairwise non-adjacent edges of $G$. A maximum matching is a matching that contains the largest possible number of edges. The *matching number*, $\alpha'(G)$, of $G$ is the size of a maximum matching.

**Theorem 3.** *For any graph $G$ of order $n$,*

$$\gamma_{\rm sp}(G) \geq n - \alpha'(G).$$

*Proof.* Let $D$ be a $\gamma_{\rm sp}(G)$-set. Let $D^* \subseteq D$ with $|D^*| = |\overline{D}|$ such that, for every $u \in \overline{D}$, there exists $u^* \in D^*$ satisfying $N(u^*) \cap \overline{D} = \{u\}$. If we let $M = \{u^*u \in E(G) : u^* \in D^* \text{ and } u \in \overline{D}\}$, then $M$ is a matching of $G$, and thus $n - \gamma_{\rm sp}(G) = |\overline{D}| = |M| \leq \alpha'(G)$, as desired. □

We note that the bound of Theorem 3 is sharp. For example, if $G \cong K_1 + sK_2$, where $sH$ denotes $s$ disjoint copies of a graph $H$, then $\alpha'(G) = s$, $n = 2s+1$, and $\gamma_{\rm sp}(G) = s + 1 = n - \alpha'(G)$. For another example, $\gamma_{\rm sp}(C_n) = \lceil \frac{n}{2} \rceil = n - \alpha'(C_n)$ for $n \equiv 0, 3 \pmod 4$.
A *vertex cover* of $G$ is a set $X \subseteq V(G)$ such that each edge of $G$ is incident to at least one vertex of $X$. A minimum vertex cover is a vertex cover of smallest possible cardinality. The *vertex cover number*, $\beta(G)$, of $G$ is the cardinality of a minimum vertex cover of $G$. An *independent set* of $G$ is a set $X \subseteq V(G)$ such that no two vertices in $X$ are adjacent in $G$, and the *independence number*, $\alpha(G)$, of $G$ is the cardinality of a largest independent set of $G$. It is well known, due to Gallai, that $\alpha(G) + \beta(G) = |V(G)|$. It is also well known, due to König [12] and Egerváry [5], that $\alpha'(G) = \beta(G)$ for any bipartite graph $G$. So, Theorem 3 implies the following

**Corollary 1.** *For any bipartite graph $G$ of order $n$,*

$$\gamma_{\rm sp}(G) \geq n - \beta(G) = \alpha(G).$$

The bound of Corollary 1 is attained when $G \cong K_{1,n-1}$ or $G \cong Q_k$, the hypercube graph of order $2^k$. It is well known that $\beta(Q_k) = 2^{k-1}$ (see [8]), and we will show that $\gamma_{\rm sp}(Q_k) = 2^{k-1}$ in Section 4.
A set $S \subseteq V(G)$ is a *secure dominating set* of $G$ if $S$ is a dominating set of $G$ and, for every $v \in \overline{S}$, there exists $u \in N(v) \cap S$ such that $(S \setminus \{u\}) \cup \{v\}$ is also a dominating set of $G$. The *secure domination number* of $G$, denoted by $\gamma_s(G)$, is the minimum cardinality among all secure dominating sets of $G$. Secure domination was introduced by Cockayne et al. in [3].

**Theorem 4.** *For any graph $G$,*

$$\gamma_{\rm sp}(G) \geq \gamma_s(G).$$

*Proof.* Let $S$ be a $\gamma_{\rm sp}(G)$-set. For each $v \in \overline{S}$, let $v^* \in S$ such that $N(v^*) \cap \overline{S} = \{v\}$. Since both $S$ and $(S \setminus \{v^*\}) \cup \{v\}$ are dominating sets of $G$, $S$ is a secure dominating set of $G$. So, $\gamma_s(G) \leq |S| = \gamma_{\rm sp}(G)$. □

The bound of Theorem 4 is achieved whenever $\gamma_{\rm sp}(G) = \gamma(G)$, since $\gamma_{\rm sp}(G) \geq \gamma_s(G) \geq \gamma(G)$. For another example, $\gamma_{\rm sp}(K_{1,n-1}) = \gamma_s(K_{1,n-1}) = n - 1$.
The *closed neighborhood* of a vertex $v$ is the set $N[v] = N(v) \cup \{v\}$. We define a *twin equivalence relation* $\mathcal{R}$ on $V(G)$ as follows:

$$x\mathcal{R}y \iff x = y, \text{ or } N(x) = N(y) \text{ for } x \neq y, \text{ or } N[x] = N[y] \text{ for } x \neq y.$$

Notice that each twin equivalence class $U$ belongs to one of the following three types:

(E1) $U$ is a singleton;
(E2) $|U| > 1$ and $N(x) = N(y)$ for any distinct $x, y \in U$;
(E3) $|U| > 1$ and $N[x] = N[y]$ for any distinct $x, y \in U$.

The twin equivalence class of type (E1), (E2), and (E3) is called a *singleton*, a *false-twin equivalence class*, and a *true-twin equivalence class*, respectively. For example, for $r, s, t \geq 2$, $K_r + (K_s \cup K_t)$ has three true-twin equivalence classes of cardinality $r, s$ and $t$, $K_{r,s}$ has two false-twin equivalence classes of cardinality $r$ and $s$, $K_r + N_s$ has one true-twin equivalence class of cardinality $r$ and one false-twin equivalence class of cardinality $s$, and $K_1 + (K_r \cup N_s)$ has a singleton, a true-twin equivalence class of cardinality $r$ and a false-twin equivalence class of cardinality $s$.
The following straightforward lemma is useful in proving Theorem 5.

**Lemma 1.** *For any graph $G$, let $D$ be a $\gamma_{\rm sp}(G)$-set. Let $D^* \subseteq D$ with $|D^*| = |\overline{D}|$ such that, for every $u \in \overline{D}$, there exists $u^* \in D^*$ satisfying $N(u^*) \cap \overline{D} = \{u\}$. If $U \subseteq V(G)$ is a twin equivalence class, then $|U \cap \overline{D}| \leq 1$ and $|U \cap D^*| \leq 1$.*

**Theorem 5.** *Let $G$ be any graph of order $n$ with $t$ twin equivalence classes. Then*

$$\gamma_{\mathrm{sp}}(G) \geq n - t. \tag{3}$$

*Moreover, if $G$ is connected and $t \geq 3$, then*

$$\gamma_{\mathrm{sp}}(G) \geq n - t + 1. \tag{4}$$

*Proof.* Let $\{B_1, B_2, \dots, B_t\}$ be the partition of $V(G)$ induced by $\mathcal{R}$, and let $D$ be a $\gamma_{\mathrm{sp}}(G)$-set. By Lemma 1, $|B_i \cap D| \geq |B_i| - 1$ for each $i \in \{1, 2, \dots, t\}$. So, $\gamma_{\mathrm{sp}}(G) = |D| \geq (\sum_{i=1}^{t} |B_i|) - t = n - t$.
Now, suppose that $G$ is connected and $t \geq 3$; we will show that (4) holds. Assume, to the contrary, that $\gamma_{\mathrm{sp}}(G) \leq n-t$. By (3), $\gamma_{\mathrm{sp}}(G) = n-t$. Let $D^* \subseteq D$ with $|D^*| = |\overline{D}|$ such that, for every $u \in \overline{D}$, there exists $u^* \in D^*$ satisfying $N(u^*) \cap \overline{D} = \{u\}$. By Lemma 1, $|B_i \cap \overline{D}| = 1 = |B_i \cap D^*|$ for each $i \in \{1, 2, \dots, t\}$. So, there exist three twin equivalence classes, say $B_1, B_2, B_3$, such that every vertex in $B_1$ is adjacent to every vertex in $B_2 \cup B_3$. For each $i \in \{1, 2, 3\}$, let $x_i, y_i \in B_i$ such that $x_1, x_2, x_3 \in \overline{D}$ and $y_1, y_2, y_3 \in D^*$. Then $N(y_1) \cap \overline{D} \supseteq \{x_2, x_3\}$, which is a contradiction. So, (4) holds. $\square$

The equality of (3) holds when $G \cong K_n$ for $n \geq 2$, $G \cong K_{r,s}$ for $r, s \geq 2$, or $G \cong K_{p,q} \cup K_r$ for $p, q, r \geq 2$. For an example of $G$ satisfying the equality of (4), see Figure 1, where the solid vertices form a $\gamma_{\mathrm{sp}}(G)$-set; note that $G$ has a singleton and four false-twin equivalence classes.

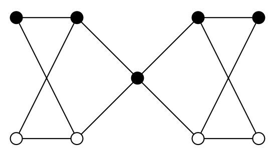

**Figure 1.** **A graph $G$ with five twin equivalence classes satisfying $\gamma_{\mathrm{sp}}(G) = |V(G)| - 4$.**

For a set $X \subseteq V(G)$, the *open neighborhood* of $X$ is $N(X) = \cup_{x \in X} N(x)$, and the *closed neighborhood* of $X$ is $N[X] = N(X) \cup X$. A set $S \subseteq V(G)$ is *open irredundant* if, for each $u \in S$,

$$N(u) \setminus N[S \setminus \{u\}] \neq \emptyset. \tag{5}$$

**Theorem 6.** [1] *If a graph $G$ has no isolated vertices, then $G$ has a minimum dominating set which is open irredundant.*

**Theorem 7.** *Let $G$ be a graph of order $n$ with no isolated vertices. Then*

$$\gamma_{\mathrm{sp}}(G) \leq n - \gamma(G).$$

*Proof.* By Theorem 6, there exists an open irredundant set $S \subseteq V(G)$ such that $|S| = \gamma(G)$. For each $u \in S$, by (5), there exists $v \in \overline{S}$ such that $N(v) \cap S = \{u\}$, which implies that $\overline{S}$ is a super dominating set of $G$. Therefore, $\gamma_{\rm sp}(G) \leq |\overline{S}| = n - \gamma(G)$. $\square$

Note that, for any graph $G$ of order $n$ with no isolated vertices, $\gamma(G) = \frac{n}{2}$ implies $\gamma_{\rm sp}(G) = \frac{n}{2}$ by Theorem 7 and (2). However, the converse is not true; for example, $\gamma_{\rm sp}(K_m \Box K_2) = m$ and $\gamma(K_m \Box K_2) = 2$ for $m \geq 3$, where $G \Box H$ denotes the Cartesian product of graphs $G$ and $H$.
A set $X \subseteq V(G)$ is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of distinct vertices $u, v \in X$. The 2-*packing number*, $\rho(G)$, of $G$ is the maximum cardinality among all 2-packings of $G$. It is well known that $\gamma(G) \geq \rho(G)$ for any graph $G$, and it was shown in [15] that $\gamma(T) = \rho(T)$ for any tree $T$. So, Theorem 7 implies the following

**Corollary 2.** *If $G$ is a graph of order $n$ with no isolated vertices, then*

$$\gamma_{\rm sp}(G) \leq n - \rho(G).$$

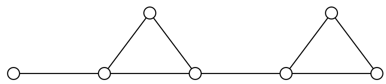

**Figure 2.** **A graph $G$ with $\gamma(G) = \rho(G) = 2$ and $\gamma_{\rm sp}(G) = |V(G)| - 2 = 5$.**

Note that both bounds of Theorem 7 and Corollary 2 are achieved for the graph in Figure 2, as well as for the corona product graphs $G \odot K_m$ and $G \odot N_m$, where $m \geq 1$ (see section 3).
For a simple graph $G$, the *degree* of $v \in V(G)$, denoted by deg$(v)$, is $|N(v)|$. For a $\gamma_{\rm sp}(G)$-set $D$ and for $u \in \overline{D}$, if $v \in D$ satisfies $N(v) \cap \overline{D} = \{u\}$, then $N[v] \setminus \{u\} \subseteq D$ and thus $\gamma_{\rm sp}(G) = |D| \geq \deg(v)$. On the other hand, it was shown in [17] that $\gamma(G) \geq \lceil \frac{n}{\Delta+1} \rceil$ for any graph $G$ of order $n$ with the maximum degree $\Delta$. So, these facts, together with Theorems 2(b) and 7, imply the following

**Proposition 1.** *For any non-empty graph $G$ of order $n$ with maximum degree $\Delta$ and minimum degree $\delta$,*

$$\max\left\{\left\lceil \frac{n}{2} \right\rceil, \delta\right\} \leq \gamma_{\rm sp}(G) \leq \left\lfloor \frac{n\Delta}{\Delta+1} \right\rfloor.$$

The lower bound of Proposition 1 holds for $K_n$, where $n \geq 2$. The upper bound of Proposition 1 holds for any connected graph $G$ of order $n$ with $\gamma_{\rm sp}(G) = n - 1$; then $\Delta = n - 1$ (see Theorem 5 of [4]). For another example with $\Delta < n - 1$, see the graph in Figure 2.
Corollary 1 and Theorem 7 imply the following.

**Theorem 8.** *Let $G$ be a bipartite graph. If $\gamma(G) = \beta(G)$, then*

$$\gamma_{\rm sp}(G) = \alpha(G).$$

If $G \cong K_{1,n-1}$ for $n \geq 2$, then $G$ is a bipartite graph satisfying $\gamma(G) = \beta(G) = 1$ and $\gamma_{\rm sp}(G) = \alpha(G) = n - 1$.
The *line graph*, $L(G)$, of a simple graph $G$ is the graph whose vertices are in one-to-one correspondence with the edges of $G$, and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ are adjacent.

**Theorem 9.** *For any connected graph $G$ of order $n$,*

$$\gamma_{\rm sp}(G) \leq n - \rho(L(G)).$$

*Proof.* Let $M$ be a 2-packing of $L(G)$ such that $|M| = \rho(L(G))$. Let $w_{i,j} \in V(L(G))$ correspond to $u_iu_j \in E(G)$, where $u_i, u_j \in V(G)$. Let $X, X' \subset V(G)$ be two disjoint sets of cardinality $|M|$ such that, for each $w_{i,j} \in M$, $|\{u_i, u_j\} \cap X| = 1 = |\{u_i, u_j\} \cap X'|$, i.e., for each vertex in $M \subseteq V(L(G))$, which corresponds to an edge in $G$, one endpoint of the edge belongs to $X \subset V(G)$ and the other endpoint of the edge belongs to $X' \subset V(G)$. Since $M$ is a 2-packing of $L(G)$, any two distinct vertices in $X$ are at distance at least three from each other in $G$. So, $V(G) \setminus X$ is a super dominating set of $G$ since, for each $v \in X$, there exists $v' \in X'$ such that $N(v') \cap X = \{v\}$. Thus, $\gamma_{\rm sp}(G) \leq |V(G) \setminus X| = n - |X| = n - |M| = n - \rho(L(G))$. □

The bound of Theorem 9 is achieved for both graphs in Figure 3, where Theorem 9 provides a better bound, compared to Corollary 2, for $G_1$ in Figure 3.

## 3. Super domination in corona product graphs

Let $G$ and $H$ be two graphs of order $n$ and $m$, respectively, and let $V(G) = \{u_1, u_2, \ldots, u_n\}$. The *corona product* $G \odot H$ is obtained from $G$ and $n$ copies of $H$, say $H_1, H_2, \ldots, H_n$, by drawing an edge from each vertex $u_i$ to every vertex of $H_i$ for each $i \in \{1, 2, \ldots, n\}$ (see [6]). Since $\gamma(G \odot H) = \rho(G \odot H) = n$, both bounds in Theorem 7 and Corollary 2 are achieved for $G \odot K_m$ and $G \odot N_m$, where $m \geq 1$.

**Theorem 10.** *Let $G$ be any graph of order $n$. For any non-empty graph $H$,*

$$\gamma_{\rm sp}(G \odot H) = n(\gamma_{\rm sp}(H) + 1),$$

*and $\gamma_{\rm sp}(G \odot N_m) = nm$ for $m \geq 1$.*

*Proof.* Let $V = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of $G$. For each $i \in \{1, 2, \ldots, n\}$, let $H_i$ be a non-empty graph of order $m$ with vertex set $W_i$ and let $Y_i \subset W_i$ be a $\gamma_{\mathrm{sp}}(H_i)$-set. Since $V \cup (\cup_{i=1}^{n} Y_i)$ is a super dominating set of $G \odot H$,

$$\gamma_{\mathrm{sp}}(G \odot H) \leq \left| V \cup \left( \bigcup_{i=1}^{n} Y_i \right) \right| = n + \sum_{i=1}^{n} |Y_i| = n(1 + \gamma_{\mathrm{sp}}(H)).$$

Next, we show that $\gamma_{\mathrm{sp}}(G \odot H) \geq n(1 + \gamma_{\mathrm{sp}}(H))$. Let $U$ be a $\gamma_{\mathrm{sp}}(G \odot H)$-set. If $u_i \notin U$ for some $i$, then $W_i \subseteq U$, which implies that $U' = (U \setminus W_i) \cup Y_i \cup \{u_i\}$ is a super dominating set of $G \odot H$ with $|U'| \leq |U|$ since $|Y_i| = \gamma_{\mathrm{sp}}(H) \leq m - 1$. So, we may assume that $V \subseteq U$. For each $i \in \{1, 2, \ldots, n\}$, if we let $U_i = U \cap W_i$, then $|U_i| \geq |Y_i| = \gamma_{\mathrm{sp}}(H)$. So,

$$\gamma_{\mathrm{sp}}(G \odot H) = |U| = |V| + \sum_{i=1}^{n} |U_i| \geq n(1 + \gamma_{\mathrm{sp}}(H)).$$

Therefore, the first equality holds.
For the second equality, since $\alpha'(G \odot N_m) = \gamma(G \odot N_m) = n$, Theorems 3 and 7 imply that $\gamma_{\mathrm{sp}}(G \odot N_m) = nm$ for $m \geq 1$. □

We note that an alternative proof for Theorem 10 can be derived from a formula obtained in [13] for the super domination number of rooted product graphs.

## 4. Super domination in Cartesian product graphs

The *Cartesian product* of two graphs $G$ and $H$, denoted by $G \Box H$, is the graph with the vertex set $V(G) \times V(H)$ such that $(u, v)$ is adjacent to $(u', v')$ in $G \Box H$ if and only if either $u = u'$ and $vv' \in E(H)$, or $v = v'$ and $uu' \in E(G)$. This operation is commutative in the sense that $G \Box H \cong H \Box G$, and it is also associative in the sense that $(F \Box G) \Box H \cong F \Box (G \Box H)$. A Cartesian product graph is connected if and only if both of its factors are connected, and it is a bipartite graph if and only if both of its factors are bipartite graphs. Cartesian product is a straightforward and natural construction, and it is in many respects the simplest graph product [7, 11]. For the structure and properties of Cartesian product graphs, we refer to [7, 11]. Examples of Cartesian product graphs are the *Hamming graph* $H_{n,m}$ (the Cartesian product of $n$ copies of $K_m$), the *hypercube* $Q_n \cong H_{n,2}$, the *grid graph* $P_n \Box P_m$, the *cylinder graph* $C_n \Box P_m$, and the *torus graph* $C_n \Box C_m$.
Next, we introduce some notations that will be used in stating our results. We denote by $\mathcal{S}(G)$ the collection of all $\gamma_{\mathrm{sp}}(G)$-sets. For any $S \in \mathcal{S}(G)$, let $\mathcal{P}(S)$ be the collection of all subsets $S^* \subseteq S$ of cardinality $|S^*| = |\overline{S}|$ such that, for each $u \in \overline{S}$, there exists $u^* \in S^*$ satisfying $N(u^*) \cap \overline{S} = \{u\}$. We define $\lambda(G)$ as follows:

$$\lambda(G) = \max_{S \in \mathcal{S}(G), S^* \in \mathcal{P}(S)} \{|X| : X \subseteq S \text{ and } N(X) \cap (\overline{S} \cup S^*) = \emptyset\}.$$

For example, if $G_1 \cong K_1 + (2K_2 \cup 2K_1)$ in Figure 3, then $\gamma_{\mathrm{sp}}(G_1) = 5$, $S_1 = \{u_1, u_3, u_5, u_6, u_7\} \in \mathcal{S}(G_1)$, $\mathcal{P}(S_1) = \{\{u_1, u_3\}\}$ and $\lambda(G_1) = 2$; if $G_2 \cong K_1 + (K_2 \cup K_1)$ in Figure 3, then $\gamma_{\mathrm{sp}}(G_2) = 3$, $S_2 = \{w_1, w_3, w_4\} \in \mathcal{S}(G_2)$, $\mathcal{P}(S_2) = \{\{w_1\}, \{w_4\}\}$ and $\lambda(G_2) = 1$.

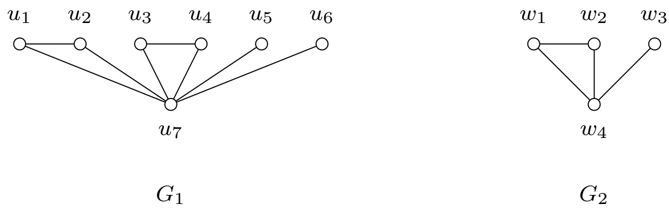


**Figure 3.** **Two graphs $G_1$ and $G_2$ with $\lambda(G_1) = 2$ and $\lambda(G_2) = 1$.**

If $G$ has $n$ vertices and $\deg(v) = n - 1$, then $v$ is a *universal vertex* of $G$. It is readily seen that the following remark holds.

**Remark 1.** Let $v$ be a universal vertex of a graph $G$ of order $n$ and let $S \in \mathcal{S}(G)$. If $v \in \overline{S} \cup S^*$ for some $S^* \in \mathcal{P}(S)$, then $\gamma_{\mathrm{sp}}(G) = n - 1$.

Note that, for the graph $G_2$ in Figure 3, $w_4$ is a universal vertex of $G_2$, $S = \{w_1, w_3, w_4\} \in \mathcal{S}(G_2)$, $w_4 \in \overline{S} \cup S^*$ for some $S^* = \{w_4\} \in \mathcal{P}(S)$, and $\gamma_{\mathrm{sp}}(G_2) = |V(G_2)| - 1 = 3$.

**Theorem 11.** *For any graphs $G$ and $H$ of order $n \geq 2$ and $m \geq 2$, respectively,*

$$\left\lceil \frac{nm}{2} \right\rceil \leq \gamma_{\mathrm{sp}}(G \Box H) \leq m\gamma_{\mathrm{sp}}(G) - \lambda(G)(m - \gamma_{\mathrm{sp}}(H)).$$

*Proof.* The lower bound follows from Theorem 2(b). So, we prove the upper bound. Let $S$ be a $\gamma_{\mathrm{sp}}(G)$-set, $S^* \in \mathcal{P}(S)$, and $X \subseteq S$ such that $|X| = \lambda(G)$ and $N(X) \cap (\overline{S} \cup S^*) = \emptyset$. We claim that, for any $\gamma_{\mathrm{sp}}(H)$-set $S'$, the set

$$W = V(G \Box H) \setminus \big( \big(\overline{S} \times V(H)\big) \cup \big(X \times \overline{S'}\big) \big)$$

is a super dominating set of $G \Box H$. To see this, we fix $(x, y) \in \overline{W}$. Then $x \in \overline{S}$ or $x \in X$. We consider two cases.
**Case 1:** $x \in \overline{S}$**.** In this case, there exists $x^* \in S^*$ such that $N(x^*) \cap \overline{S} = \{x\}$. Since $\{x^*\} \times N(y) \subseteq W$, $(N(x^*) \times \{y\}) \cap \overline{W} = (N(x^*) \cap \overline{S}) \times \{y\} = \{(x, y)\}$ and

$$N((x^*, y)) = (\{x^*\} \times N(y)) \cup (N(x^*) \times \{y\}),$$

we conclude that $N((x^*, y)) \cap \overline{W} = \{(x, y)\}$.

**Case 2:** $x \in X$. In this case, $N(x) \cap (\overline{S} \cup S^*) = \emptyset$ and $y \in \overline{S'}$. Since $S'$ is a super dominating set of $H$, there exists $y' \in S'$ such that $N(y') \cap \overline{S'} = \{y\}$. Note that, if there exists $w \in N(x) \cap X$, then $S \setminus \{w\}$ is a super dominating set of $G$, which contradicts to the assumption that $S$ is a $\gamma_{\rm sp}(G)$-set. So. $X$ is an independent set of $G$. Thus,

$$N((x,y')) \cap \overline{W} = N((x,y')) \cap (X \times \overline{S'}) = \{x\} \times (N(y') \cap \overline{S'}) = \{(x,y)\}.$$

Therefore, $W$ is a super dominating set of $G \square H$, which implies that

$$\gamma_{\rm sp}(G \square H) \leq |W| = nm - m|\overline{S}| - |X \times \overline{S'}| = m\gamma_{\rm sp}(G) - \lambda(G)(m - \gamma_{\rm sp}(H)),$$

as desired. □

As a direct consequence of Theorem 11, we have the following

**Corollary 3.** *For any graphs $G$ and $H$ of order $n \geq 2$ and $m \geq 2$, respectively,*

$$\gamma_{\rm sp}(G \square H) \leq \min\{m\gamma_{\rm sp}(G), n\gamma_{\rm sp}(H)\}.$$

The following result is a direct consequence of Theorem 11 and Corollary 3.

**Theorem 12.** *Let $G$ and $H$ be two graphs of order $n \geq 2$ and $m \geq 2$, respectively. If $\gamma_{\rm sp}(G) = \frac{n}{2}$ or $\gamma_{\rm sp}(H) = \frac{m}{2}$, then*

$$\gamma_{\rm sp}(G \square H) = \frac{nm}{2}.$$

Theorem 12 implies that, for any graph $G$ of order $n \geq 2$, $\gamma_{\rm sp}(G \square K_2) = n$. So, for $n \geq 2$, $\gamma_{\rm sp}(K_n \square K_2) = n = \min\{2(n-1), n\}$, which shows the sharpness of the bound in Corollary 3. Since the hypercube graph $Q_k$ is defined as $Q_1 = K_2$ and $Q_k = Q_{k-1} \square K_2$ for $k \geq 2$, Theorem 12 implies $\gamma_{\rm sp}(Q_k) = 2^{k-1}$ for $k \geq 1$.
From Theorems 1 and 2(b), and Corollary 3, we obtain the following result.

**Theorem 13.** *Let $n \geq 3$. For any graph $H$ of order $m \geq 2$, the followings hold.*

- *If $n \equiv 0 \pmod 2$, then $\gamma_{\rm sp}(P_n \square H) = \frac{nm}{2}$.*
- *If $n \equiv 1 \pmod 2$, then $\frac{nm}{2} \leq \gamma_{\rm sp}(P_n \square H) \leq \frac{(n+1)m}{2}$.*
- *If $n \equiv 0 \pmod 4$, then $\gamma_{\rm sp}(C_n \square H) = \frac{nm}{2}$.*
- *If $n \equiv 1, 3 \pmod 4$, then $\frac{nm}{2} \leq \gamma_{\rm sp}(C_n \square H) \leq \frac{(n+1)m}{2}$.*
- *If $n \equiv 2 \pmod 4$, then $\frac{nm}{2} \leq \gamma_{\rm sp}(C_n \square H) \leq \frac{(n+2)m}{2}$.*

The most famous open problem involving Cartesian product graphs on the topic of domination is known as Vizing's conjecture [16], which states that $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for any graphs $G$ and $H$. For partial results on Vizing's conjecture, see [2, 7]. Now, we state a Vizing-like conjecture on super domination.

**Conjecture 2.** (Vizing-like conjecture) For any graphs $G$ and $H$,

$$\gamma_{\rm sp}(G\Box H) \geq \gamma_{\rm sp}(G)\gamma_{\rm sp}(H).$$

For an example satisfying Conjecture 2, let $G$ and $H$ be two graphs of order $n \geq 2$ and $m \geq 2$, respectively, such that $\gamma_{\rm sp}(G) = \frac{n}{2}$ or $\gamma_{\rm sp}(H) = \frac{m}{2}$; then $\gamma_{\rm sp}(G\Box H) \geq \gamma_{\rm sp}(G)\gamma_{\rm sp}(H)$ by Theorem 12.

We denote by $I(G)$ the number of vertices of degree one in any graph $G$. To obtain another corollary of Theorem 11, we need the following lemma.

**Lemma 2.** *For a graph $G$, let $S \in \mathcal{S}(G)$. If there exists a universal vertex $v$ of $G$ such that $v \notin \overline{S} \cup S^*$ for some $S^* \in \mathcal{P}(S)$, then $\lambda(G) \geq I(G)$.*

*Proof.* Let $v$ be a universal vertex of $G$ of order $n$. If $I(G) = 0$, then we are done. So, let $I(G) > 0$.

First, suppose that $\gamma_{\rm sp}(G) = n - 1$. If $G \cong K_{1,n-1}$, then $v \in \overline{S} \cup S^*$ for any $S \in \mathcal{S}(G)$ and $S^* \in \mathcal{P}(S)$. If $G \not\cong K_{1,n-1}$, then, for any pair of adjacent vertices $x, y \in V(G) \setminus \{v\}$, we have $S = V(G) \setminus \{x\} \in \mathcal{S}(G)$ and $S^* = \{y\} \in \mathcal{P}(S)$. So, for any $w \in I(G)$, $N(w) \cap (\overline{S} \cup S^*) = \emptyset$, and thus $\lambda(G) \geq I(G)$.

Next, suppose that $\gamma_{\rm sp}(G) \leq n - 2$. By Remark 1, $v \notin \overline{S} \cup S^*$ for any $S \in \mathcal{S}(G)$ and $S^* \in \mathcal{P}(S)$. So, for any $u \in I(G)$, $N(u) \cap (\overline{S} \cup S^*) = \emptyset$, and thus $\lambda(G) \geq I(G)$. □

Theorem 11 and Lemma 2 imply the following

**Proposition 2.** *For a graph $G$, let $S \in \mathcal{S}(G)$. If there exists a universal vertex $v$ of $G$ such that $v \notin \overline{S} \cup S^*$ for some $S^* \in \mathcal{P}(S)$, then for any graph $H$ of order $m \geq 2$,*

$$\gamma_{\rm sp}(G\Box H) \leq m\gamma_{\rm sp}(G) - I(G)(m - \gamma_{\rm sp}(H)).$$

For the sharpness of the bound in Proposition 2, let $G \cong K_1 + (K_2 \cup K_1)$ and $H \cong K_m$ for $m \geq 3$. We leave it to the reader to verify that $\gamma_{\rm sp}(G\Box H) = 3m - 1 = m\gamma_{\rm sp}(G) - I(G)(m - \gamma_{\rm sp}(H))$.

Remark 1 and Proposition 2 imply the following

**Corollary 4.** *Let $G$ be a graph of order $n$ with maximum degree $n-1$. If $\gamma_{\rm sp}(G) \leq n-2$, then for any graph $H$ of order $m$,*

$$\gamma_{\rm sp}(G\Box H) \leq m\gamma_{\rm sp}(G) - I(G)(m - \gamma_{\rm sp}(H)).$$

Next, we provide a sharp bound of $\gamma_{\mathrm{sp}}(G\Box H)$ in terms of the orders of $G$ and $H$.

**Theorem 14.** *Let $G$ and $H$ be non-empty graphs of order $n \geq 2$ and $m \geq 2$, respectively. Then*

$$\gamma_{\mathrm{sp}}(G\Box H) \leq nm - n - m + 4.$$

*Moreover,*

$$\gamma_{\mathrm{sp}}(K_n\Box K_m) = \begin{cases} m & \text{if } n = 2 \text{ and } m \geq 2, \\ 2m & \text{if } n = 3 \text{ and } m \geq 3, \\ nm - n - m + 4 & \text{if } n \geq 4 \text{ and } m \geq 4. \end{cases}$$

*Proof.* Let $x_1, x_2 \in V(G)$ and $y_1, y_2 \in V(H)$ such that $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. Let $X \subseteq V(G\Box H)$ such that

$$\overline{X} = ((V(G) \setminus \{x_1, x_2\}) \times \{y_1\}) \cup (\{x_1\} \times (V(H) \setminus \{y_1, y_2\}))$$

with $|\overline{X}| = n + m - 4$. Then $X$ is a super dominating set of $G\Box H$, since, for each $(x, y_1) \in \overline{X}$, there exists $(x, y_2) \in X$ such that $N((x, y_2)) \cap \overline{X} = \{(x, y_1)\}$ and, for each $(x_1, y) \in \overline{X}$, there exists $(x_2, y) \in X$ such that $N((x_2, y)) \cap \overline{X} = \{(x_1, y)\}$. So,

$$\gamma_{\mathrm{sp}}(G\Box H) \leq |X| = nm - |\overline{X}| = nm - n - m + 4. \tag{6}$$

Next, we determine $\gamma_{\mathrm{sp}}(K_n\Box K_m)$ for $n, m \geq 2$. Let $W$ be a $\gamma_{\mathrm{sp}}(K_n\Box K_m)$-set. If $(x, y) \in \overline{W}$ and $(a, b) \in W$ such that $N((a, b)) \cap \overline{W} = \{(x, y)\}$, then $x = a$ or $y = b$; if $x = a$, then $\overline{W} \cap (V(K_n) \times \{b\}) = \emptyset$; if $y = b$, then $\overline{W} \cap (\{a\} \times V(K_m)) = \emptyset$. If $(x, y), (x', y) \in \overline{W}$ for $x \neq x'$, then $\overline{W} \cap (\{x, x'\} \times V(K_m)) = \{(x, y), (x', y)\}$, since, for each $y' \in V(K_m) \setminus \{y\}$, we have $N((x, y)) \subseteq N[(x', y)] \cup N[(x, y')]$ and $N((x', y)) \subseteq N[(x, y)] \cup N[(x', y')]$. Similarly, if $(x, y), (x, y') \in \overline{W}$ for $y \neq y'$, then $\overline{W} \cap (V(K_n) \times \{y, y'\}) = \{(x, y), (x, y')\}$. So,

$$|\overline{W}| \leq \max\{n, m, n + m - 4\}. \tag{7}$$

If $n = 2$ and $m \geq 2$, then $|\overline{W}| \leq m$ by (7), and thus $|W| \geq m$. If $V(K_2) = \{x, x'\}$, then $\{x\} \times V(K_m)$ forms a super dominating set of $K_2\Box K_m$, and hence $|W| \leq m$. Thus, $\gamma_{\mathrm{sp}}(K_2\Box K_m) = m$ for $m \geq 2$. If $n = 3$ and $m \geq 3$, then $|W| \geq 2m$ by (7). If $V(K_3) = \{x_1, x_2, x_3\}$, then $\{x_1, x_2\} \times V(K_m)$ forms a super dominating set of $K_3\Box K_m$, and hence $|W| \leq 2m$. Thus, $\gamma_{\mathrm{sp}}(K_3\Box K_m) = 2m$ for $m \geq 3$. If $n \geq 4$ and $m \geq 4$, then $\gamma_{\mathrm{sp}}(K_n\Box K_m) \geq nm - n - m + 4$ by (7), and thus $\gamma_{\mathrm{sp}}(K_n\Box K_m) = nm - n - m + 4$ by (6). □

We recall the following result on the independence number of Cartesian product graphs.

**Theorem 15.** [16] *For any graphs $G$ and $H$ of order $n$ and $m$, respectively,*

$$\alpha(G\Box H) \geq \alpha(G)\alpha(H) + \min\{n-\alpha(G), m-\alpha(H)\}.$$

From Corollary 1 and Theorem 15, we have the following result.

**Theorem 16.** *For any pair of bipartite graphs $G$ and $H$,*

$$\gamma_{\rm sp}(G\Box H) \geq \alpha(G)\alpha(H) + \min\{\beta(G), \beta(H)\}.$$

**Theorem 17.** *For $r, s \geq 1$,*

$$\gamma_{\rm sp}(K_{1,r}\Box K_{1,s}) = rs + 1.$$

*Proof.* Let $s \geq r \geq 1$. By Theorem 16, $\gamma_{\rm sp}(K_{1,r}\Box K_{1,s}) \geq rs+1$. Next, we show that $\gamma_{\rm sp}(K_{1,r}\Box K_{1,s}) \leq rs+1$. First, let $r=1$. If $V(K_{1,1}) = \{x, x'\}$, then $\{x\}\times V(K_{1,s})$ forms a super dominating set of $K_{1,1}\Box K_{1,s}$, and thus $\gamma_{\rm sp}(K_{1,1}\Box K_{1,s}) \leq s+1$. Now, let $r \geq 2$. Let $V(K_{1,r}) = \{u_0, u_1, u_2, \ldots, u_r\}$ and $V(K_{1,s}) = \{v_0, v_1, v_2, \ldots, v_s\}$, where $u_0$ and $v_0$, respectively, is the universal vertex of $K_{1,r}$ and $K_{1,s}$. Let $X \subseteq V(K_{1,r}\Box K_{1,s})$ such that

$$\overline{X} = ((V(K_{1,r}) \setminus \{u_0\}) \times \{v_0\}) \cup (\{u_r\} \times (V(K_{1,s}) \setminus \{v_0, v_s\})) \cup \{(u_{r-1}, v_s)\}$$

with $|\overline{X}| = r+s$. Then, for each $(u_i, v_0) \in \overline{X} \setminus \{(u_r, v_0)\}$, we have $(u_i, v_1) \in X$ with $N((u_i, v_1)) \cap \overline{X} = \{(u_i, v_0)\}$, and, for each $(u_r, v_j) \in \overline{X} \setminus \{(u_r, v_0)\}$, we have $(u_0, v_j) \in X$ and $N((u_0, v_j)) \cap \overline{X} = \{(u_r, v_j)\}$. Also, note that $(u_0, v_s) \in X$ such that $N((u_0, v_s)) \cap \overline{X} = \{(u_{r-1}, v_s)\}$, and $(u_r, v_s) \in X$ such that $N((u_r, v_s)) \cap \overline{X} = \{(u_r, v_0)\}$. So, $X$ is a super dominating set of $K_{1,r}\Box K_{1,s}$, and thus $\gamma_{\rm sp}(K_{1,r}\Box K_{1,s}) \leq |X| = (r+1)(s+1) - (r+s) = rs+1$. Therefore, $\gamma_{\rm sp}(K_{1,r}\Box K_{1,s}) = rs+1$. □

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