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The 2-dimension of a tree

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Abstract: Let x and y be two distinct vertices in a connected graph G. The x, y-location of a vertex w is the ordered pair of distances from w to x and y, that is, the ordered pair $(d(x, w), d(y, w))$. A set of vertices W in G is x, y-located if any two vertices in W have distinct x, y-locations. A set W of vertices in G is 2-located if it is x, y-located, for some distinct vertices x and y. The 2-dimension of G is the order of a largest set that is 2-located in G. Note that this notion is related to the metric dimension of a graph, but not identical to it. We study in depth the trees T that have a 2-locating set, that is, have 2-dimension equal to the order of T. Using these results, we have a nice characterization of the 2-dimension of arbitrary trees.

Keywords: resolvability, location number, 2-dimension, tree, 2-locating set

AMS Subject classification: 05C05, 05C75, 05C12

1. Introduction

The metric dimension of a connected graph has been the focus of research since the middle of the 1970's. In 1975 Slater [4] introduced his ideas of locating sets and location number of a graph. A year later, Harary and Melter [2] introduced the same

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ideas with a different terminology. In their terminology Slater's locating set was a resolving set, and Slater's location number was the metric dimension of a graph. The terminology of Harary and Melter has become the standard in the literature. In our view the terminology of location, introduced by Slater, is more apt for our problems, so we will basically follow his terminology, with a few exceptions. In [3] two new resolvability parameters are introduced. For this paper we only need some specific notions. We give the basics, for details see the next section.

Let $G = (V, E)$ be a connected graph, and let $S = \{s_1, s_2, \ldots, s_k\}$ be an ordered set of vertices in G. The set S can distinguish two vertices v and w, if there exists a vertex s_i in S such that $d(s_i, v) \neq d(s_i, w)$. A set W is S-located (or resolved by S) if any two vertices in W can be distinguished by S. The order of a smallest set that can locate/resolve all vertices of G is the location number/metric dimension of G. In [3] a slightly different perspective is chosen. There the order k of the locating set is fixed, and we search for a largest set W in G that can be located by a set of order k. In [3] this parameter is called the k-dimension of G. This new parameter opens up a whole new set of questions and problems.

Our focus in this paper is to determine, in a structural way, the 2-dimension of a tree. In Section 3, we determine the structure of trees that have location number 2. We use these results to determine the structure of all maximal sets in arbitrary trees that are 2-located, see Section 4. Thus we obtain a structural result on the 2-dimension of a tree.

In his seminal paper "Leaves of trees" [4], Peter Slater already obtained an algorithmic result how to find the location number/metric dimension of a tree. But these results do not yet provide insight in the structural aspects. The aim of this Note is to fill this gap in the case of location number 2 for trees.

2. Setting the stage

Let $G = (V, E)$ be a connected, simple graph without loops, with vertex set V and edge set E. The *order* of G is the number $|V|$. Let u and v be two vertices in V. The degree $deg_G(u)$ of u is the number of neighbors of u. The *distance* $d_G(u, v)$ between u and v is the length of a shortest u,v-path. When no confusion arises, we will delete the subscript G in deg_G and d_G.

Let $T = (V, E)$ be a tree, and let w be a vertex in T and v a neighbor of w. The subtree of T consisting of w and all vertices of T that can be reached via a path along v is called the *branch* at w along v, and is denoted T_{wv}. Clearly, there are as many as $deg(w)$ branches at w. As usual, a *leaf* of T is a vertex of degree 1 in T

A *sensor* s is a vertex that can measure distances in G. So s 'knows' $d(s, v)$, for any vertex v in G. Let $S = \{s_1, s_2, \ldots, s_k\}$ be an *ordered set* of sensors. The S*-location* of a vertex v of G is the vector $(d(s_1, v), d(s_2, v), \ldots, d(s_k, v))$. In general, vertices might have the same S-location. Take for instance the complete graph K_n, and a sequence of sensors S with $k < n - 1$. Then the $n - k$ vertices that are not in S all have the all-1 vector of length k as S-location. A vertex outside S has no 0's in its

S-vector. Note that the vertices in S all have different S-locations, since the S-vector of a vertex s_i in S has a 0 in the i-th position, and nowhere else.

We want to put sensors on G so that the ordered set of sensors S can determine the location of each vertex v in G in the following sense. For any two vertices u and v, their S-location is different. So each vertex in G is uniquely determined by its distances to the sensors in S. Such a set S is called a *locating set*. It is called a *resolving set* of G in the terminology of Harary and Melter [2]. Because the language of location, as introduced by Slater [4], in our view is much more apt for the problem, we will use the location terminology. If S is a locating set of order k, then we will call S also a *k-locating set*. Note that V is always a locating set for G, and $V - u$ is a locating set, for any vertex u of G. So one of the main problems is to determine minimal and minimum locating sets. The size of a minimum locating set is the *location number* $\ell(G)$ of G. In the Harary-Melter terminology this is the *metric dimension* $dim(G)$ of G.

The only graphs that have a 1-locating set are the paths, and in this case each of the end points of the path forms a 1-locating set. So the location number of a graph G is 1 if and only if G is a path, see [1]. Take any subpath of a path that contains an end point. Then the vertices of this subpath also form a locating set. So the path P_n of order n has locating sets of orders 1 through $n - 1$.

A locating set S for a connected graph G has the property that each vertex has a unique S-location. Using a slightly different perspective, one may also ask the following question. Given a subset S of the vertex set V of a connected graph G, one searches for a subset W of V such that all vertices in W have distinct S-vectors. We say that S *locates* W, and W is called an *S-located set*. If $W = V$, then S is a locating set for G. Otherwise, one would like to know the order of a maximum S-located set in G. From this perspective the next step is: if the order of such a set is small, can we find another set S' with $|S'| = |S|$, such that S' can locate a larger set than S. This perspective leads to the following definition, which was introduced in [3]. Note that, in this definition, we follow the Harary-Melter terminology of dimension, where $dim(G)$ is the metric dimension (location number) of G.

Definition 1. Let G be a connected graph, and let k be a positive integer. The *k-th dimension* $dim_k(G)$ of G is the maximum order of a set $W \subseteq V$ such that there exists a set of order k that locates W.

In [3] the parameter $dim_k(G)$ is called the *k-th upper dimension* within the broader context that is studied there. In Section 3, we study trees with location number 2. We continue in Section 4 with studying the second dimension $dim_2(T)$ of an arbitrary tree T.

3. Trees with a 2-locating set

In this section we study trees that have a minimum locating set of order 2, which we usually write as $S = \{x, y\}$, with $x = s_1$ and $y = s_2$. We call such a set a *2-locating set*. Note that such a tree with location number 2 might have several 2-locating sets. So there is the question whether there are trees that have a unique 2-locating set, see Proposition 1.

Before we turn to the trees with location number 2, we consider the paths. A path of order n is denoted by P_n. The vertices of degree 1 in a path are called the *end points* of the path. All other vertices on the path are *internal vertices* of the path. In [1] it is shown that the paths are precisely the graphs with location number 1, where each of the end points of the path forms a 1-locating set. But a path of positive length also has 2-locating sets. For instance, the two end points form a 2-locating set, but any two adjacent vertices form a 2-locating set as well.

Let $T = (V, E)$ be a tree with location number 2. We fix a 2-locating set $S = \{x, y\}$ of T. Let Q be the path between x and y. We call Q the *spine* of T with respect to S, and the internal vertices of Q the *spinal vertices* of T. By convention Q will always be a path going from *left* to *right*, x being the left most vertex and y being the right most vertex of Q. We set $q = d(x, y)$, i.e. q is the length of Q.

Claim 1. *$1 \leq deg(x) \leq 2$ and $1 \leq deg(y) \leq 2$.*

Proof. Clearly, x has a neighbor on Q. Suppose to the contrary that x has two other neighbors v and w. Then we have $d(x, v) = d(x, w) = 1$ and $d(y, v) = d(y, w) = d(y, x) + 1 = q + 1$. So v and w have the same S-location. This is impossible.
Similarly, we have $1 \leq deg(y) \leq 2$. □

If x has a second neighbor besides the one on Q, then we can extend Q beyond x to the left. Similarly, if $deg(y) = 2$, then we can extend Q to the right. Let P be a maximal path in T containing Q. Then P consists of Q plus possibly vertices to the left of x and to the right of y. Let z_x be the left most vertex of P and z_y be the right most vertex of P, so that P is the path between z_x and z_y. Then z_x and z_y are leaves of T. We denote by P_x the subpath of P between z_x and x, and by P_y the subpath of P between y and z_y. Note that P_x might be of length 0, that is, $z_x = x$. Similarly, P_y might be of length 0, in which case $z_y = y$.

Claim 2. *Any vertex on P_x or P_y has degree at most 2.*

Proof. Let P_x have length at least 2, for otherwise nothing has to be proved. Let w be any internal vertex of P_x. Then w has two neighbors u and v on P_x, with, say, u between w and x, and v farther from x than w. Suppose that w has another neighbor a besides u and v. Then we have $d(a, x) = d(v, x)$, and $d(a, y) = d(v, y)$, which is forbidden. □

This claim has an immediate but important consequence, which we state in the next claim.

Claim 3. *There is a unique maximal path P containing the spine Q.*

Claim 4. $d(x, y) \geq 2$.

Proof. Assume that x and y are adjacent. Then, by Claims 1 and 2, we have $T = P$, so T is just a path. But a path has location number 1, contradicting the fact that T has location number 2. □

This implies that T has at least one spinal vertex, that is, an internal vertex of Q.

Claim 5. *Let w be a spinal vertex. Then $2 \leq deg(w) \leq 3$.*

Proof. Since w, being a spinal vertex, is an internal vertex of Q, it has two neighbors on Q. If w has two other neighbors outside Q, then these have the same S-location. So w has at most one neighbor outside Q. □

Claim 6. *At least one spinal vertex has degree 3.*

Proof. If no spinal vertex has degree 3, then T is the path P, so that it has location number 1. Contradiction. □

Claim 7. *Let w be a spinal vertex of degree 3, and let v be the neighbor of w outside Q. Then the branch T_{wv} is a path.*

Proof. Let u be any vertex of the branch T_{wv} distinct from w. We need to prove that u has degree 1 or 2. Assume that $deg(u) \geq 3$, and let r and t be two distinct neighbors of u different from the one on the path between w and u. Then r and t have the same S-location. Impossible. □

What we have proved so far is that a tree with a 2-locating set, that does not have location number 1, has maximum degree 3, and all vertices of degree 3 are internal vertices of a path. Moreover, a 2-locating set consists of two vertices x and y such that all vertices of degree 3 are internal vertices of the path Q between x and y.
The converse is true as well.

Claim 8. *Let $T = (V, E)$ be a tree with maximum degree 3 such that all vertices of degree 3 are internal vertices of a path R in T. Let P be a maximal path containing R. Take a vertex x of P on one side of the vertices of degree 3 and a vertex y of P on the other side of the vertices of degree 3. Then $S = \{x, y\}$ is a 2-locating set.*

Proof. We may assume that P goes from left to right, and that x is left from y. Let $Q = x \to x_1 \to x_2 \to \ \dots \to x_{k-1} \to y$ be the path between x and y. Then Q is a subpath of P and contains all vertices of degree 3 in T as internal vertices. Take two vertices u and v. If both are on P, then, clearly, they have different S-locations. So, we may assume that v is not on P. Let v be on the path Q_i branching off at x_i from Q. Write $p = d(x_i, v)$. Then the S-location of v is $(i + p, k - i + p)$.
First we observe that any other vertex on Q or Q_i has a different S-location. For, take any such vertex w, then it has a different distance to x and/or to y than v.
Now consider u. Assume that u has the same S-location as v. If u is on P, then it must be on $P - Q$. If u is on the left from x, say at distance j from x, then $d(y, u) = k + j$. Since u and v have the same S-location, we have $j = i + p$ and $k + j = k - i + p$. This implies that $k + i + p = k - i + p$, which is impossible, since $i > 0$. Similarly, if u is on the right from y, say at distance r from y, then $d(x, u) = k + r$. Since u and v have the same S-location, we have $k + r = i + p$ and $r = k - i + p$. This implies that $i - k + p = k - i + p$, which is impossible, since $0 < i < k$. Hence u is not on P.
So let u be on a path Q_j branching off from Q at x_j, with $x_j \neq x_i$, and write $q = d(u, x_j)$. Then the S-location of u is $(j + q, k - j + q)$. Without loss of generality, we may assume that $0 < j < i$. For u to have the same S-location as v, we need that $i + p = j + q$ and $k - i + p = k - j + q$. This gives $0 < i - j = q - p = p - q$, which is impossible.
Thus we may conclude that any two distinct vertices have different S-locations. □

A simple corollary of this last claim is that, given any 2-locating set $S = \{x, y\}$ with the paths P_x and P_y as defined above, any vertex s_x on P_x together with any vertex s_y on P_y forms a 2-locating set $S' = \{s_x, s_y\}$.
All these claims together yield a proof of the following theorem.

Theorem 1. *Let T be a tree. Then the location number of T is 2 if and only if the maximum degree of T is 3 and all vertices of degree 3 lie on a path.*

In Figures 1 and 2 we exhibit two trees with a 2-locating set $S = \{x, y\}$. The first one has diameter 7, so an odd diameter, the second one has diameter 8, so an even diameter. Both have the property that, among the trees with a 2-locating set with diameter 7, respectively 8, they are of maximum order. This follows from Theorem 2 below.
We can construct such trees for any odd, and any even diameter D. We call such a tree a *pipe-organ* of diameter D. Let Q be a path between the vertices x and y, with $Q = x \to x_1 \to x_2 \to \dots \to x_k \to y_k \to y_{k-1} \to \dots \to y_1 \to y$ in case the diameter is odd, viz. $2k + 1$, and $Q = x \to x_1 \to x_2 \to \dots \to x_k \to z \to y_k \to y_{k-1} \to \dots \to y_1 \to y$ in case the diameter is even, viz. $2k + 2$. At x_i as well as y_i we append a path of length i, for $i = 1, 2, \dots, k$, and in the case of even diameter, we also append a path of length $k + 1$ at z. For $k = 3$, this construction yields precisely the pipe organs in Figures 1 and 2.

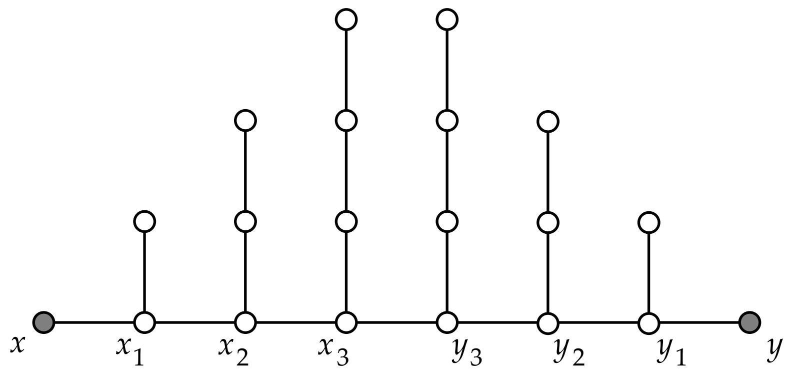

Figure 1. **An odd pipe organ**

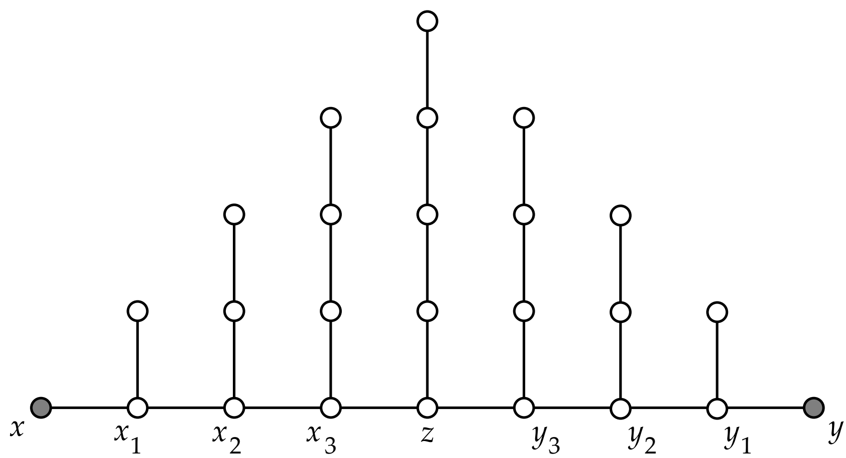

Figure 2. **An even pipe organ**

The order of a pipe organ of diameter D is easily computed. If $D = 2\ell + 1$ is odd, then it is $(\ell + 1)(\ell + 2)$. And if $D = 2\ell$ is even, then the order if $(\ell + 1)^2$.
We will show that the pipe-organ of diameter D is precisely the tree of maximum order with diameter D and location number 2. For our purposes we introduce the following terminology. A D-tree is a tree of maximum order among the trees with a 2-locating set and diameter D.

Claim 9. *Let T be a tree of diameter D, with a 2-locating set $S = \{x, y\}$. If x is not a leaf of T, then T is not a D-tree.*

Proof. Note that, due to Claim 1, we have $deg(x) = 2$. Due to Claim 2, the path P_x has length at least 1. Let Q be the spinal path between x and y, and let z be the neighbor of x on P_x. Due to Claim 8, the set $\{z, y\}$ is also a 2-locating set. Now adding a new vertex w adjacent to x creates a tree T' with one more vertex than T, which still has diameter D and a 2-locating set $\{z, y\}$. So T is not a D-tree. □

Claim 10. *Let T be a D-tree with 2-locating set $S = \{x, y\}$. Then x and y are leaves of T.*

Proof. This follows immediately from the previous claim. □

Claim 11. *Let T be a D-tree with 2-locating set $S = \{x, y\}$, and let Q be the spinal path between x and y. Then all internal vertices of Q have degree 3.*

Proof. Assume to the contrary that p is an internal vertex of Q of degree 2. If we add a new vertex w adjacent to p, this new tree T' still has $\{x, y\}$ as 2-locating set, and the diameter still remains D. This contradicts the fact that T is a D-tree. □

Theorem 2. *Let T be a tree of order n with a 2-locating set. Then T is a D-tree if and only if T is a pipe organ of diameter D.*

Proof. First let T be a D-tree, and let $\{x, y\}$ be a 2-locating set in T. By Claim 10, x and y are leaves of T. Let Q be the path between x and y. By Claim 11, all internal vertices of Q have degree 3. Due to Theorem 1, no other vertex can have degree 3.
Next we prove that Q has length D. Suppose to the contrary that Q is of length less than D. If x were not the end point of a longest path, then we could add a new vertex x' adjacent to x without changing the diameter, while the resulting tree would still have the 2-locating set $S = \{x, y\}$. For, the S-location of x' would be $(1, k + 1)$, and the only other vertex at distance 1 from x would be the neighbor of x on Q, which has S-location $(1, k - 1)$.
So x is the end point of a longest path R_x. Let t_x be the other end point of R_x. Since Q is not a longest path, t_x is different from y. Let R_x and Q share the subpath between x and a vertex w_x of degree 3 on Q. Since R_x is a longest path, we have $d(w_x, x) \geq d(w_x, y)$. Similarly, there is a longest path R_y of length D that has y as one end point, and t_y as the other end point, with $t_y \neq x$. Let R_y and Q share the path between w_y and y. Then we have $d(w_y, x) \geq d(w_y, y)$. Hence w_y cannot be to the right of w_x on Q. So w_y must be either equal to w_x or to the left of w_x on Q.
Suppose that w_y is to the left of w_x on Q. We depict this situation in Figure 3. Since R_y is a longest path and Q is not, we have $d(t_y, w_y) > d(x, w_y)$. Similarly, we have $d(t_x, w_x) > d(y, w_x)$. So the path between t_x and t_y is longer than the longest paths R_x and R_y, which is impossible. Hence it follows that $w_x = w_y = w$ and $t_x = t_y = t$. Thus we have the situation as in Figure 4.

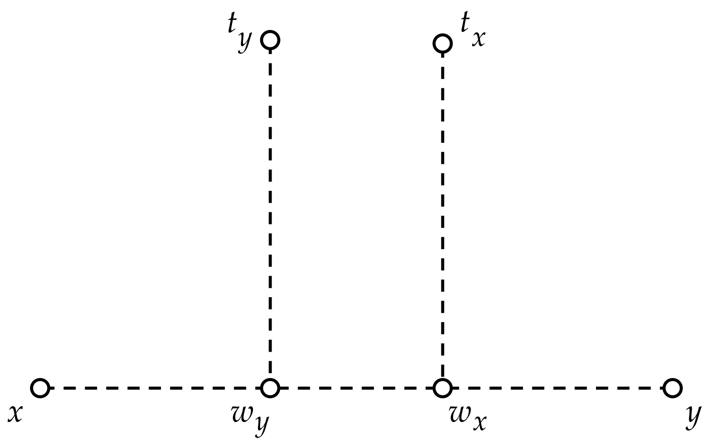

Figure 3. **The case** $t_x \neq t_y$

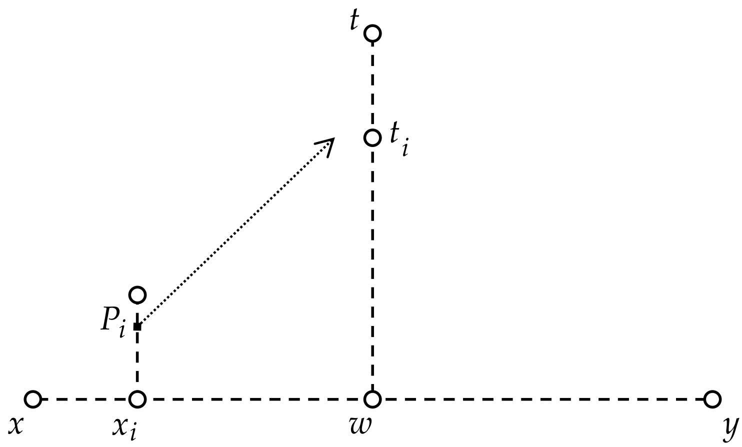

Figure 4. **The case** $t = t_x = t_y$

Now the path R_x between t and x is a longest path, and the path R_y between t and y is a longest path, but the path Q between x and y is not a longest path. These facts imply that $d(w, x) = d(w, y) < d(t, w)$. Since Q is the spinal path, the vertices of degree 3 all lie on Q, and all internal vertices of the path between t and w are of degree 2. Let the path between x and w be $x \to x_1 \to x_2 \to \ldots \to x_k \to w$, and that between t and w be $t \to t_1 \to t_2 \to \ldots \to t_\ell \to w$. Then we have $\ell > k$. Since $x_1, x_2, \ldots, x_k$ are internal vertices of Q, they might be of degree 3. If x_i has degree 3,

then there is a path P_i pending at x_i. The length of this path cannot exceed $d(x_i, x)$, for otherwise we would have a path that is longer than R_x. Now we move P_i from x_i to t_i, by which we mean that we delete all vertices of P_i except x_i and add a new path of the same length as P_i at vertex t_i. We apply this construction for $i = 1, 2, \ldots, k$. It is clear that we do not create a longer longest path. Moreover, now all vertices of degree 3 lie on the path R_y, and thus $\{t, y\}$ is a 2-locating set. Now t_{k+1} is a vertex of degree 2 on the spinal path with respect to the 2-locating set $\{t, y\}$. Hence we can add a new vertex adjacent to t_{k+1}, thus creating a larger tree of diameter D with a 2-locating set. This contradicts the fact that T is a D-tree.
So Q has length D. This puts bounds on the lengths of the paths that branch off from internal vertices of Q. For T to be a tree of diameter D of maximum order, each of these branches should have the maximum length possible. So T is indeed a pipe organ. The order of T is $(\ell + 1)(\ell + 2)$ in case of an odd diameter $D = 2\ell + 1$, and it is $(\ell + 1)^2$ in case of an even diameter $D = 2\ell$.
Conversely, let T be a pipe organ of diameter D, as in Figure 1 or 2. Clearly, T has a 2-locating set. In case $D = 2\ell + 1$, the order of T is $(\ell + 1)(\ell + 2)$. In case $D = 2\ell$, the order of T is $(\ell + 1)^2$. According to the first part of the proof, the order of T equals the order of a D-tree. So T is a tree of maximum order among the trees of diameter D with location number 2. Therefore T is a D-tree. □

The pipe organs are the trees of maximum order with a given diameter and a 2-locating set. But this does not mean that any tree with diameter D and a 2-locating set is a subtree of a pipe organ. A simple examples suffices to show this. Take a path on five vertices $v_1 \to v_2 \to v_3 \to v_4 \to v_5$. Add a pendant vertex at v_2 and v_4, and add a long pendant path of length k at v_3. Then the diameter is $k + 2$, and the spinal vertices are v_2, v_3, v_4. Clearly, this is not a subtree of a pipe organ as soon as $k > 2$.
Above we asked the question: is there a tree T with a unique 2-locating set? The answer is no, unless $T = K_2$. In a way, we are cheating here, because K_2 is a path and thus has location number 1. But it also has a 2-locating set.

Proposition 1. *A tree T has a unique 2-locating set if and only if $T = K_2$.*

Proof. Recall that K_2 is a path, and so has location number 1, but the two vertices still form a 2-locating set, which is, of course, the unique 2-locating set.
Conversely, let T be a tree with at least 3 vertices. Assume that $\{x, y\}$ is the unique 2-locating set in T. Let Q be the path between x and y. If, say, x is not a leaf, then it has a neighbor z not on Q. But then $\{z, y\}$ is also a 2-locating set. So both x and y are leaves. Hence, T not being K_2, they cannot be adjacent. Let w be the neighbor of x. Note that w is on Q. If w has degree 2, then $\{w, y\}$ is also a 2-locating set. If w has degree 3, then let t be the neighbor of w not on Q. Now the path between t and y contains all vertices of degree 3 in T, so also $\{t, y\}$ is a 2-locating set, by Theorem 1. □

The pipe organ of diameter at least 3 has exactly four 2-locating sets. It turns out that there are no trees with exactly two 2-locating sets, and there are precisely two trees with three 2-locating sets.

Proposition 2. *Let T be a tree with a 2-locating set, with $T \neq K_2$. Then T has at least three 2-locating sets. T has precisely three 2-locating sets if and only if $T = P_3$ or $T = K_{1,3}$.*

Proof. Since T has a 2-locating set, T has either location number 1 or location number 2. First let T have location number 1, so that T is a path P_n. Then we have $n \geq 3$. The two end points of T form a 2-locating set, and any two adjacent vertices form such a set. Hence T has at least n 2-locating sets. If $n = 3$, then T has no other 2-locating sets, so T has precisely three 2-locating sets. Otherwise, T has at least four 2-locating sets.
Next let T have location number 2. Then T has maximum degree 3, and all vertices of degree 3 lie on a path. Let R be the smallest path containing all vertices of degree 3. Now there are at least three spinal paths, that is, paths containing R and with end points being vertices of degree 1 or 2 in T. The two end points of a spinal path form a 2-locating set. So we have at least three 2-locating sets. If $T = K_{1,3}$, then there are precisely three 2-locating sets, viz. the three pairs of leaves.
Let T be a tree of maximum degree 3 that is not $K_{1,3}$. If T has only one vertex of degree 3, then clearly T has a spinal path P of length at least 3. If T has more than one vertex of degree 3, then again T has a spinal path P of length at least 3. Let x and y be the end points of P, and let x_1 be the neighbor of x on P and y_1 the neighbor of y on P. Since P has length at least 3, we have $x_1 \neq y_1$. Now, if x_1 has a neighbor x' outside P, then both x and x' are candidates for a 2-locating set. Otherwise x and x_1 are both candidates for a 2-locating set. Similarly, we find two candidates y and y', or y and y_1 on the side of y with respect to P for a 2-locating set. Thus we find at least four 2-locating sets. This shows that $K_{1,3}$ is the only tree of maximum degree 3 with exactly three 2-locating sets. □

Of course there are many more questions to be answered concerning the number of 2-locating sets. For example, given the order of T, what is the maximum number of 2-locating sets that is possible? For now, we leave such questions as open problems.
The next step is to study the second dimension of arbitrary trees. This we will do in the next section.

4. The 2-dimension of a tree

Recall that the 2-*dimension* $dim_2(G)$ of a graph G is the maximum number of vertices that can be located (resolved) by a set of order 2, that is, by a pair of vertices. In this section we study the 2-dimension of trees.
An experience that most people share is that after a while some of the teeth of a comb break off. But the resulting object is still called a comb. This is the reason for

the choice of the following terminology. A *comb* is a tree of maximum degree 3 such that all vertices of degree 3 lie on a path. The smallest path containing all vertices of degree 3 is the *spine* of the comb. Any path between vertices of degree at most 2 containing the spine will be called a *spinal path*. The two end points of any spinal path form a 2-locating set for the comb. It follows from the results in the previous section that the combs are precisely the trees T with $dim_2(T) = n$, with n the order of the tree.

Let us now get to the 2-dimension of an arbitrary tree. First we want to determine what kinds of sets in a tree are located by two sensors. We call a set W (x, y)*-located* by a pair of distinct vertices x and y if there are no two vertices u and v in W with $d(u, x) = d(v, x)$ and $d(u, y) = d(v, y)$, that is, the two sensors x and y are able to distinguish between any two vertices in W. As usual, we call an (x, y)-located set W *maximal* if adding any vertex to W results in a set that is not (x, y)-located, and we call such a set *maximum* if it is an (x, y)-located set of maximum order.

Theorem 3. *Let T be a tree, and let x and y be two distinct vertices of T. Then any maximal (x, y)-located set is a maximum (x, y)-located set.*

Proof. Take two distinct vertices x and y, and let P be the path between x and y in T. If we delete all the edges of P, then the resulting graph consists of components being subtrees of T, each of which contains a unique vertex of P. Let T_w be the component containing w, for each w on P. We can view T_w as a rooted tree with root w. Let k_w be the length of a longest path in T_w with w as one of its end points. Denote by $N_i(w)$ the set of vertices in T_w at distance i from w. Let a and b be two distinct vertices in $N_i(w)$. Then we have $d(a, x) = d(a, w) + d(w, x) = i + d(w, x)$, and $d(a, y) = d(a, w) + d(w, y) = i + d(w, y)$. Similarly, we have $d(b, x) = i + d(w, x)$ and $d(b, y) = i + d(w, y)$. So a and b cannot be both in an (x, y)-located set. Obviously, a vertex a in $N_i(w)$ and a vertex y in $N_j(w)$, with $i \neq j$ have different $\{x, y\}$-location. So let R be a subset of the vertices of T_w. Then R is (x, y)-located if and only if $0 \leq |N_i(w) \cap R| \leq 1$, for $0 \leq i \leq k_w$.

If a is a vertex in T_w and b is a vertex in $T_{w'}$, with $w \neq w'$, then a and b cannot have the same $\{x, y\}$-location. This implies that a set W is a maximal (x, y)-located set if and only if $|N_i(w) \cap W| = 1$, for $0 \leq i \leq k_w$, for any w on P. So a maximal (x, y)-located set is also a maximum (x, y)-located set. □

Corollary 1. *For any two vertices x and y in a tree T, there exists a maximum (x, y)-located set which induces a comb in T.*

Proof. In the proof of Theorem 3, we can take a path in T_w starting in w of maximum length, for any w on P. This produces a comb, the vertex set of which is a maximum (x, y)-located set. □

Theorem 4. *Let T be a tree. Then the 2-dimension of T is the maximum order of comb in T.*

Proof. Take two distinct vertices x and y in T. By Corollary 1, we can find a comb, the vertex set of which is a maximum (x, y)-located set. If we maximize this over all pairs (x, y), then we get a maximum 2-located set, the order of which is the 2-dimension of T. So the 2-dimension of T is at most the maximum order of a comb. On the other hand, let Q be a comb of maximum order in T. Take two vertices x and y of Q such that the vertices of degree 3 in Q all lie on the path between x and y. We have seen in the previous section that the vertex set of Q is (x, y)-located, so the 2-dimension of T is at least the order of Q. Hence the 2-dimension of T is exactly the order of a maximum comb in T. □

5. Concluding remarks and open problems

Our aim in this paper is to initiate the study of the k-dimension of a graph, by providing complete and structural results in the case of the 2-dimension of a tree. To accomplish this, we obtained a structural characterization of trees with location number/metric dimension 2. Along the way, we came across questions and problems that we have left open so far. For instance, there is the question of the number of 2-locating sets in a tree. We gave some very first answers for this question.
Another intriguing question is the following: how to determine the 2-dimension of an arbitrary connected graph G. And what can we say about the structure of maximum 2-located sets? A simple example shows already that the situation is quite different from that in the tree case. Take the complete graph K_n with $n \geq 3$. Any maximum 2-located set is a triangle. So it is 2-connected. The trees show that a maximum 2-located set need not be connected, but we can always find a connected one. Is this also true for arbitrary connected graphs?
Finally, what can be said if we go to the next level, and ask about the 3-dimension of a tree?

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