Communications in Combinatorics and Optimization
Vol. 5 No. 1, 2020 pp.55-60
DOI: 10.22049/CCO.2019.26119.1077

CCO
Commun. Comb. Optim.

The topological ordering of covering nodes

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Received: 9 January 2018; Accepted: 19 August 2019
Published Online: 24 August 2019

Abstract: The topological ordering algorithm sorts nodes of a directed graph such that the order of the tail of each arc is lower than the order of its head. In this paper, we introduce the notion of covering between nodes of a directed graph. Then, we apply the topological ordering algorithm on graphs containing the covering nodes. We show that there exists a cut set with forward arcs in these graphs and the order of the covering nodes is successive.

Keywords: Directed graph, covering nodes, topological ordering algorithm

AMS Subject classification: 05C20, 05C85

1. Introduction

In graph theory, a graph is a set of nodes connected by arcs. Many systems in real world can be designed as graphs, for example, airlines, blood vessels in the body, telecommunications networks, roads and so on. Therefore, concept of graph and its properties has been entered in majority of sciences such as biology, sociology, economics, electrical engineering and so on [2, 3, 5]. A node may contain and pass on information. An arc between two nodes indicates the possibility of information exchange between those nodes.
Janssen et al. [4] limited themselves to position of a node in graph. This node is compared to any other node that set of its neighbours is a strict subset of the set of neighbours of primary node. Hence, the notion of covering was introduced. They expressed that a node covers node or equivalently is covered by if all neighbours of (if

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and are connected each other, then they do not consider) are also neighbours of (if and are connected each other, then they do not consider) and node has at least one extra neighbour. Also, some results were proved for the covering nodes, but the major assumption was that nodes want to avoid the situation of being covered. These graphs were named stable. Their results are used to design the communication networks.
The first definition of covering in terms of arc structure has been expressed in [7]. This relation was the generalization of relation defined for tournaments as introduced in [6]. Janssen et al. [4] presented an alternative definition in terms of the adjacency matrix for it. In these papers, notion of covering has been introduced on undirected graphs. In this paper, we generalized this concept on directed graphs. Then, some properties of the covering nodes such as the transitive relation are discussed. The transitive relation of covering nodes is valid for undirected graphs that it has been proved in [4], but we show that this relation is not established for directed graphs. Our major work is determination of the order of the covering nodes and by applying of the topological ordering algorithm [1]. An application of the topological ordering algorithm is to identify a directed cycle in the graph. A graph is acyclic if and only if it possesses a topological ordering of its nodes. We use notion of cut set to prove our results.

2. Basic facts

A directed graph (digraph) is a pair $G = (V, E)$, where V is a finite set of nodes and $E \subseteq \{(x, y) \mid x, y \in V\}$. An element $(u, v) \in E$ is called an arc from u to v. The nodes u and v are adjacent if $(u, v) \in E$ or $(v, u) \in E$. A digraph $G' = (V', E')$ is a subdigraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. The subdigraph of G obtained from G by deleting a set $E' \subseteq E$ is denoted by $G - E'$. The digraph $G = (V, E)$ is called simple if $(v, v) \notin E$ for each $v \in V$ and the number of arcs from u to v for each $u, v \in V$ be at most 1. A walk in directed graph G is a sequence of nodes $(v_1, v_2, \ldots, v_r)$ such that $(v_i, v_{i+1}) \in E$ or $(v_{i+1}, v_i) \in E$ for each $1 \leq i \leq r-1$. A directed walk is a walk $(v_1, v_2, \ldots, v_r)$ such that $(v_i, v_{i+1}) \in E$ for each $1 \leq i \leq r-1$. A path is a walk without any repetition of nodes and a directed path is a directed walk without any repetition of nodes. The path $(v_1, v_2, \ldots, v_r)$ is called cycle if $v_1 = v_r$. Also, a directed cycle is a directed path $(v_1, v_2, \ldots, v_r)$ with $v_1 = v_r$. A digraph is acyclic if it contains no directed cycle. The digraph G is connected if there exists at least a path between each two nodes. Otherwise, it is disconnected. A component of disconnected digraph is a maximal connected subdigraph. The adjacency matrix $M(G) = [m_{ij}]$ of the simple digraph $G = (V, E)$ is a $|V| \times |V|$ matrix such that m_{ij} is defined as follows:

$$m_{ij} = \begin{cases} 1 & (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

A cut is a partition of V into two subsets S and $\bar{S} = V - S$ such that each arc is having one endpoint in S and another in $\bar{S}$. This set of arcs is called cut set. An arc (u, v) is called forward arc of the cut if $u \in S$ and $v \in \bar{S}$. The topological ordering algorithm labels the nodes of directed graph G by distinct numbers from 1 to $|V|$ such that every arc joins a lower-labeled node to a higher-labeled node. The label of node u is represented as $order(u)$. Hence, for every arc $(u, v) \in E$, $order(u) < order(v)$.

3. Main results

We start with introducing two equivalent definitions of covering. Afterwards, some properties of the covering nodes are expressed.

Definition 1. Let $a, b \in V$ and $a \neq b$. The node a covers node b in G whenever

1. for each $x \in V \setminus \{a\}$, if $(b, x) \in E$ then $(a, x) \in E$ and if $(x, b) \in E$ then $(x, a) \in E$,
2. there exists a node $c \in V \backslash \{a, b\}$ such that $(a, c) \in E$ or $(c, a) \in E$ but $(b, c), (c, b) \notin E$.

We say that a and b are covering nodes if node a covers node b.
Let a covers b in G and let x_i, $i = 1, \ldots, r$, are nodes adjacent to both a and b. Also, let c_i, $i = 1, \ldots, k$, denote the nodes that are adjacent to a and are not adjacent to b. Now, we consider the rows and columns corresponding to a and b in $M(G)$. According to the condition 1 of Definition 1, we have the following relations:

$$\begin{cases} m_{ax_i} = m_{bx_i} = 0, 1 \\ m_{x_ia} = m_{x_ib} = 0, 1 \\ m_{ac_i} = 0, 1 \text{ and } m_{bc_i} = 0 \\ m_{c_ia} = 0, 1 \text{ and } m_{c_ib} = 0 \\ m_{au} = m_{bu} = 0 \text{ for each } u \in V \setminus \{x_1, \ldots, x_r, c_1, \ldots, c_k, a\} \\ m_{ua} = m_{ub} = 0 \text{ for each } u \in V \setminus \{x_1, \ldots, x_r, c_1, \ldots, c_k, a\}. \end{cases}$$

Therefore, for each $x \in V \setminus \{a\}$, we have:

$$m_{bx} \leq m_{ax} \quad \text{and} \quad m_{xb} \leq m_{xa}.$$

According to the condition 2 of Definition 1, there exists at least a node c_i in G such that $m_{c_ia} = 1$ or $m_{ac_i} = 1$. Therefore, the following definition is equivalent to Definition 1.

Definition 2. Let $a, b \in V$ and $a \neq b$. A node a covers node b in G, if

1. for each $x \in V \setminus \{a\}$, $m_{bx} \leq m_{ax}$ and $m_{xb} \leq m_{xa}$.
2. $(\sum_{x \in V} m_{ax} - \sum_{x \in V} m_{bx}) + (\sum_{x \in V} m_{xa} - \sum_{x \in V} m_{xb}) \geq 1$.

Proposition 1. *The transitive relation of covering nodes cannot be established for digraphs.*

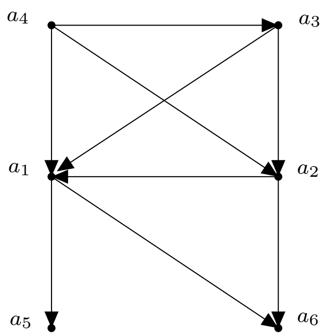

Figure 1. **A graph with covering nodes**

Proof. Let a covers b and b covers c. By the Definition 2, $m_{cb} \leq m_{ca}$ and $m_{ca} \leq m_{ba}$. Now, we have:

$$m_{cb} \leq m_{ba}. \tag{1}$$

Let a covers c. Therefore, the condition 1 of Definition 2 will be established for $x = b$, i.e., we have:

$$m_{cb} \leq m_{ab}. \tag{2}$$

Now, if $m_{cb} = 1$, the relation (1) implies $m_{ba} = 1$. Since G is a simple digraph, $m_{ab} = 0$. Therefore, the relation (2) is not valid and this is a contradiction. □

In Figure 1, a_1 covers a_2 and a_2 covers a_3. But a_1 does not cover a_3, since $(a_3, a_2) \in E$ does not imply $(a_1, a_2) \in E$.

Proposition 2. *Let G be an acyclic directed graph. Let $a, b \in V$ and node a covers node b such that $(a, b), (b, a) \notin E$. Then each directed path $(a, \alpha_1, \ldots, \alpha_k, x)$ from a to $x \in V \setminus \{a, b\}$ does not pass node b, that is, $\alpha_i \neq b$ for each $1 \leq i \leq k$.*

Proof. Let $p = (a, \alpha_1, \ldots, \alpha_k, x)$ is a directed path from node a to a node $x \in V \setminus \{a, b\}$. If $k = 0$, then our claim will be obvious. Let $k \neq 0$ and suppose to the contrary that the node b is in the path p. Consider the following two sets:

$$V_1 = \{u \in V \mid u \text{ is adjacent to the nodes } a \text{ and } b\}$$

$$V_2 = \{u \in V \mid u \text{ is adjacent to the node } a \text{ and is not adjacent to the node } b\}.$$

We have $\alpha_1 \in V_1$ or $\alpha_1 \in V_2$. In two cases, since b is on the path p, there is $\alpha_j \in V_1$ such that $(\alpha_j, b) \in E$. Since a covers b, $(\alpha_j, a) \in E$. Therefore, a directed cycle is produced and this is a contradiction. □

Theorem 1. *Let $G = (V, E)$ be an acyclic connected directed graph. Let $a, b \in V$ and a covers b. Then there exists a topological ordering such that $|order(a) - order(b)| = 1$.*

Proof. Let $\{(a, u), (b, v) \in E \mid u, v \in V\}$. We consider a cycle containing directed path (x_1, a, x_2) or (y_1, b, y_2), where $x_1, x_2, y_1, y_2 \in V$. Since G is an acyclic graph, this cycle is not a directed cycle and there exists at least a forward arc on this cycle. We select an arbitrary forward arc on the cycle and set this forward arc in A. Then, we delete this forward arc in G. We repeat this process and stop when there does not exist any cycle containing directed path (x_1, a, x_2) or (y_1, b, y_2). It is obvious that all arcs of A are forward. We claim that A is a cut set. Suppose, to the contrary, that $G - A$ is connected and consider node $x \in V$ such that $(a, x) \in A$. There exists a path P from a to x in the graph $G - A$. Now we can obtain a cycle C in G by adding arc (a, x) to the path P. Since G is an acyclic digraph, the cycle C has at least a forward arc and this is a contradiction. Therefore, A is a cut set with forward arcs and the digraph $G - A$ is unconnected with two components. Let $B = \{(u, a), (v, b) \in E \mid u, v \in V\}$. The graph $G - B$ is unconnected with at least three components. We consider two sets V_1 and V_2 as follows:

$$V_1 = \{x \in V \mid (x, a) \in E \,\text{or}\, (x, b) \in E\}$$
$$V_2 = \{x \in V \mid (a, x) \in E \,\text{or}\, (b, x) \in E\}\,.$$

We apply the topological ordering algorithm on component containing V_1. Set

$$\Delta = \max\{order(u) \mid u \in V_1\}\,.$$

Now, there are three cases as follows:

1. If $(a, b) \in E$, set $order(a) = \Delta + 1$ or $order(b) = \Delta + 2$,
2. If $(b, a) \in E$, set $order(b) = \Delta + 1$ or $order(a) = \Delta + 2$,
3. $(a, b), (b, a) \notin E$, we apply case 1 or case 2.

Finally, we apply the topological ordering algorithm on component containing V_2 with starting order $\Delta + 3$. Therefore, $|order(a) - order(b)| = 1$. □

4. Conclusions

This paper has introduced the notion of covering on directed graphs. As we have shown, some properties of the covering nodes on undirected graphs is not valid for directed graphs such as the transitive relation. In this paper, we focused on the topological order of the covering nodes. It was proven that the order of the covering nodes in the graph is successive.

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